2.094

Finite Element Analysis of Solids and Fluids

Spring 2008

Homework 7 - Solution

		Assigned:	04/03/2008
Instructor:	Prof. K. J. Bathe	Due:	04/10/2008

Problem 1 (20 points):

(a)

$$ {}_{t}^{0}\underline{X} = {}_{t}^{0}\underline{R}\,{}_{t}^{0}\underline{U} = \begin{bmatrix} \cos 45° & -\sin 45° \\ \sin 45° & \cos 45° \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{4} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{2} & -\frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix} $$

$$ {}_{0}^{t}\underline{X} = \left({}_{t}^{0}\underline{X} \right)^{-1} = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ -\frac{2\sqrt{2}}{3} & \frac{2\sqrt{2}}{3} \end{bmatrix} $$

(b)

$$ {}_{0}^{t}\underline{\varepsilon} = \frac{1}{2}\left({}_{0}^{t}\underline{X}^{T}\,{}_{0}^{t}\underline{X} - \underline{I} \right) = \begin{bmatrix} \frac{17}{18} & \frac{5}{9} \\ \frac{5}{9} & \frac{17}{18} \end{bmatrix} $$

$$ {}_{0}^{t}\underline{S} = \begin{bmatrix} 11 & 7 & 0 \\ 7 & 11 & 0 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} \frac{17}{18} \\ \frac{17}{18} \\ \frac{5}{9} \end{bmatrix} = \begin{bmatrix} 17 \\ 17 \\ 5 \end{bmatrix} $$

$$ \frac{{}^{t}\rho}{{}^{0}\rho} = \frac{{}^{0}V}{{}^{t}V} = \frac{(1.5)(1)(thickness)}{(2)(2)(thickness)} = \frac{3}{8} $$

Therefore,

$$ {}^{t}\underline{\tau} = \frac{{}^{t}\rho}{{}^{0}\rho}\, {}^{t}_{0}\underline{X}\, {}^{t}_{0}\underline{S}\, {}^{t}_{0}\underline{X}^{T} = \begin{bmatrix} 33 & 0 \\ 0 & 8 \end{bmatrix} $$

$$ {}^{t}\tau_{11} = 33, \quad {}^{t}\tau_{22} = 8, \quad {}^{t}\tau_{12} = 0 $$

Hence the Cauchy stress ${}^{t}\underline{\tau}$ given by the program is not correct.

We can identify the program error by noting that $\frac{33+8}{2} = 20.5$, $20.5 + 12.5 = 33$ and $20.5 - 12.5 = 8$. Hence a rotation of $45°$ was wrongly applied. Therefore,

$$ {}^{t}\underline{\tau}\Big|_{program} = \underline{R}\, {}^{t}\underline{\tau}\Big|_{above} \underline{R}^{T} \text{ where } \underline{R} = \begin{bmatrix} \cos 45° & -\sin 45° \\ \sin 45° & \cos 45° \end{bmatrix} $$

Problem 2 (10 points):

Since $H, h \ll b$, we only consider the displacement u_r in the x_1-direction with the plane stress assumption.

Total Lagrangian formulation

$$ {}^{t}f^{B} = {}^{t}\rho\, {}^{t}r\omega^{2} $$

$$ {}^{0}t = H\left(\frac{r-b}{a-b}\right) + h\left(\frac{r-a}{b-a}\right), \; {}^{t}t = H\left(\frac{r-{}^{t}b}{{}^{t}a-{}^{t}b}\right) + h\left(\frac{r-{}^{t}a}{{}^{t}b-{}^{t}a}\right) \text{ (Thickness)} $$

$$ {}_{0}e_{rr} = \frac{\partial u_r}{\partial\, {}^{0}r} + \frac{\partial\, {}^{t}u_r}{\partial\, {}^{0}r}\frac{\partial u_r}{\partial\, {}^{0}r}, \; {}_{0}\eta_{rr} = \frac{1}{2}\left(\frac{\partial u_r}{\partial\, {}^{0}r}\right)^{2} $$

$$ {}_{0}e_{\theta\theta} = \frac{u_r}{{}^{0}r} + \frac{{}^{t}u_r u_r}{{}^{0}r^{2}}, \; {}_{0}\eta_{\theta\theta} = \frac{1}{2}\left(\frac{u_r}{{}^{0}r}\right)^{2} $$

Therefore,

$$\int_a^b {}_0C_{ijrs}\,{}_0e_{rs}\,\delta\,{}_0e_{ij}\,{}^0r\,{}^0t\,dr + \int_a^b {}_0^tS_{ij}\,\delta\,{}_0\eta_{ij}\,{}^0r\,{}^0t\,dr$$
$$= \int_{{}^{t+\Delta t}a}^{{}^{t+\Delta t}b} {}^{t+\Delta t}\rho\,{}^{t+\Delta t}r\,\omega^2\,{}^{t+\Delta t}r\,{}^{t+\Delta t}t\,\delta u_r\,dr - \int_a^b {}_0^tS_{ij}\,\delta\,{}_0e_{ij}\,{}^0r\,{}^0t\,dr$$