2.094

FINITE ELEMENT ANALYSIS OF SOLIDS AND FLUIDS

SPRING 2008

Homework 5 - Solution

Instructor:	Prof. K. J. Bathe	Assigned:	03/06/2008
		Due:	03/13/2008

Problem 1 (10 points):

$$\tau_{ij} = \kappa \varepsilon_v \delta_{ij} + 2G\varepsilon'_{ij} \quad \text{(a)}$$

$$\tau_{ij} = C_{ijrs}\varepsilon_{rs} \quad \text{(b)}$$

$$\underline{\tau} = \underline{C}\,\underline{\varepsilon} \quad \text{(c)}$$

Let's start from equation (b). Using $\gamma_{ij} = \varepsilon_{ij} + \varepsilon_{ji} (i \neq j)$ and $C_{ijrs} = \lambda \delta_{ij}\delta_{rs} + \mu(\delta_{ir}\delta_{js} + \delta_{is}\delta_{jr})$,

$$\begin{aligned}\tau_{11} &= C_{1111}\varepsilon_{11} + C_{1122}\varepsilon_{22} + C_{1133}\varepsilon_{33} + C_{1112}\gamma_{12} + C_{1123}\gamma_{23} + C_{1131}\gamma_{31} \\ &= (\lambda + 2\mu)\varepsilon_{11} + \lambda\varepsilon_{22} + \lambda\varepsilon_{33}\end{aligned}$$

$$\tau_{22} = \lambda\varepsilon_{11} + (\lambda + 2\mu)\varepsilon_{22} + \lambda\varepsilon_{33}$$

$$\tau_{33} = \lambda\varepsilon_{11} + \lambda\varepsilon_{22} + (\lambda + 2\mu)\varepsilon_{33}$$

$$\tau_{12} = C_{1211}\varepsilon_{11} + C_{1222}\varepsilon_{22} + C_{1233}\varepsilon_{33} + C_{1212}\gamma_{12} + C_{1223}\gamma_{23} + C_{1231}\gamma_{31} = \mu\gamma_{12}$$

$$\tau_{23} = \mu\gamma_{23}$$

$$\tau_{31} = \mu\gamma_{31}$$

Therefore,

$$\underline{C} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}$$

Substituting $\lambda = \dfrac{E\nu}{(1+\nu)(1-2\nu)}$ and $\mu = \dfrac{E}{2(1+\nu)}$, we obtain $\underline{C}$ in Table 4.3.

Hence equation (c) is equivalent to equation (b).

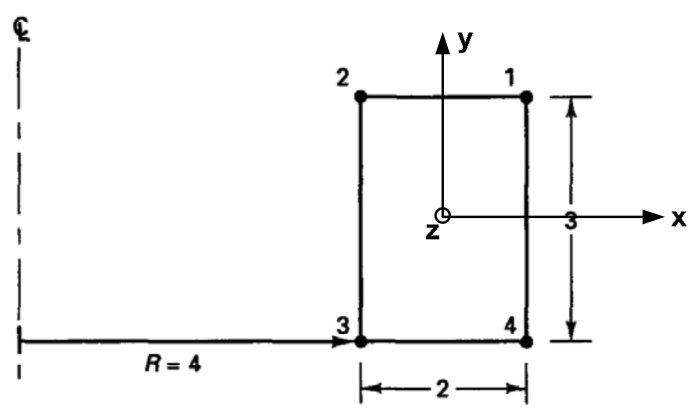
Now derive equation (a) from equation (b).

$$\begin{aligned}
\tau_{ij} &= C_{ijrs}\varepsilon_{rs} = C_{ijrs}\left(\varepsilon'_{rs} + \frac{\varepsilon_v}{3}\delta_{rs}\right) = \left\{\lambda\delta_{ij}\delta_{rs} + \mu\left(\delta_{ir}\delta_{js} + \delta_{is}\delta_{jr}\right)\right\}\left(\varepsilon'_{rs} + \frac{\varepsilon_v}{3}\delta_{rs}\right) \\
&= \lambda\delta_{ij}\delta_{rs}\varepsilon'_{rs} + \lambda\delta_{ij}\delta_{rs}\frac{\varepsilon_v}{3}\delta_{rs} + \mu\delta_{ir}\delta_{js}\varepsilon'_{rs} + \mu\delta_{is}\delta_{jr}\varepsilon'_{rs} + \mu\delta_{ir}\delta_{js}\frac{\varepsilon_v}{3}\delta_{rs} + \mu\delta_{is}\delta_{jr}\frac{\varepsilon_v}{3}\delta_{rs} \\
&= \lambda\delta_{ij}\varepsilon'_{rr} + \lambda\delta_{ij}\frac{\varepsilon_v}{3}\delta_{rr} + \mu\varepsilon'_{ij} + \mu\varepsilon'_{ji} + \mu\frac{\varepsilon_v}{3}\delta_{ij} + \mu\frac{\varepsilon_v}{3}\delta_{ji} \\
&= \lambda\varepsilon_v\delta_{ij} + 2\mu\varepsilon'_{ij} + \frac{2\mu}{3}\varepsilon_v\delta_{ij} \\
&= \left(\lambda + \frac{2\mu}{3}\right)\varepsilon_v\delta_{ij} + 2\mu\varepsilon'_{ij} \\
&= \kappa\varepsilon_v\delta_{ij} + 2G\varepsilon'_{ij}
\end{aligned}$$

Here we used

$$\varepsilon'_{rr} = \varepsilon'_{11} + \varepsilon'_{22} + \varepsilon'_{33} = 0 \text{ and } \delta_{rr} = \delta_{11} + \delta_{22} + \delta_{33} = 3$$

Problem 2 (10 points):



$$h_1 = \frac{1}{4}(1+x)\left(1+\frac{2}{3}y\right),\ h_2 = \frac{1}{4}(1-x)\left(1+\frac{2}{3}y\right),\ h_3 = \frac{1}{4}(1-x)\left(1-\frac{2}{3}y\right),\ h_4 = \frac{1}{4}(1+x)\left(1-\frac{2}{3}y\right)$$

Define $\hat{\underline{u}}^T = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 & v_1 & v_2 & v_3 & v_4 \end{bmatrix}$, then, $\underline{H} = \begin{bmatrix} h_1 & h_2 & h_3 & h_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_1 & h_2 & h_3 & h_4 \end{bmatrix}$

The deviatoric strains are

$$\underline{\varepsilon'} = \begin{bmatrix} \varepsilon_{xx} - \frac{1}{3}\varepsilon_v \\ \varepsilon_{yy} - \frac{1}{3}\varepsilon_v \\ \gamma_{xy} \\ \varepsilon_{zz} - \frac{1}{3}\varepsilon_v \end{bmatrix} = \begin{bmatrix} \frac{2}{3}\varepsilon_{xx} - \frac{1}{3}\varepsilon_{yy} - \frac{1}{3}\varepsilon_{zz} \\ -\frac{1}{3}\varepsilon_{xx} + \frac{2}{3}\varepsilon_{yy} - \frac{1}{3}\varepsilon_{zz} \\ \gamma_{xy} \\ -\frac{1}{3}\varepsilon_{xx} - \frac{1}{3}\varepsilon_{yy} + \frac{2}{3}\varepsilon_{zz} \end{bmatrix} = \underline{B}_D \hat{\underline{u}}$$

where

$$\varepsilon_{xx} = \frac{\partial u}{\partial x},\ \varepsilon_{yy} = \frac{\partial v}{\partial y},\ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},\ \text{and}\ \varepsilon_{zz} = \frac{u}{x+5}$$

Therefore,

$$\underline{B}_D = \begin{bmatrix} \frac{2}{3}h_{1,x} - \frac{1}{3}\frac{h_1}{x+5} & \frac{2}{3}h_{2,x} - \frac{1}{3}\frac{h_2}{x+5} & \frac{2}{3}h_{3,x} - \frac{1}{3}\frac{h_3}{x+5} & \frac{2}{3}h_{4,x} - \frac{1}{3}\frac{h_4}{x+5} & -\frac{1}{3}h_{1,y} & -\frac{1}{3}h_{2,y} & -\frac{1}{3}h_{3,y} & -\frac{1}{3}h_{4,y} \\ -\frac{1}{3}h_{1,x} - \frac{1}{3}\frac{h_1}{x+5} & -\frac{1}{3}h_{2,x} - \frac{1}{3}\frac{h_2}{x+5} & -\frac{1}{3}h_{3,x} - \frac{1}{3}\frac{h_3}{x+5} & -\frac{1}{3}h_{4,x} - \frac{1}{3}\frac{h_4}{x+5} & \frac{2}{3}h_{1,y} & \frac{2}{3}h_{2,y} & \frac{2}{3}h_{3,y} & \frac{2}{3}h_{4,y} \\ h_{1,y} & h_{2,y} & h_{3,y} & h_{4,y} & h_{1,x} & h_{2,x} & h_{3,x} & h_{4,x} \\ -\frac{1}{3}h_{1,x} + \frac{2}{3}\frac{h_1}{x+5} & -\frac{1}{3}h_{2,x} + \frac{2}{3}\frac{h_2}{x+5} & -\frac{1}{3}h_{3,x} + \frac{2}{3}\frac{h_3}{x+5} & -\frac{1}{3}h_{4,x} + \frac{2}{3}\frac{h_4}{x+5} & -\frac{1}{3}h_{1,y} & -\frac{1}{3}h_{2,y} & -\frac{1}{3}h_{3,y} & -\frac{1}{3}h_{4,y} \end{bmatrix}$$

The volumetric strain is

$$\varepsilon_v = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \underline{B}_v \hat{\underline{u}}$$

$$\underline{B}_V = \begin{bmatrix} h_{1,x} + \frac{h_1}{x+5} & h_{2,x} + \frac{h_2}{x+5} & h_{3,x} + \frac{h_3}{x+5} & h_{4,x} + \frac{h_4}{x+5} & h_{1,y} & h_{2,y} & h_{3,y} & h_{4,y} \end{bmatrix}$$

The pressure is

$$\underline{p} = \underline{H}_p \hat{\underline{p}}$$

where

$$\underline{\hat{p}} = \left[p_0 \right]$$

Therefore

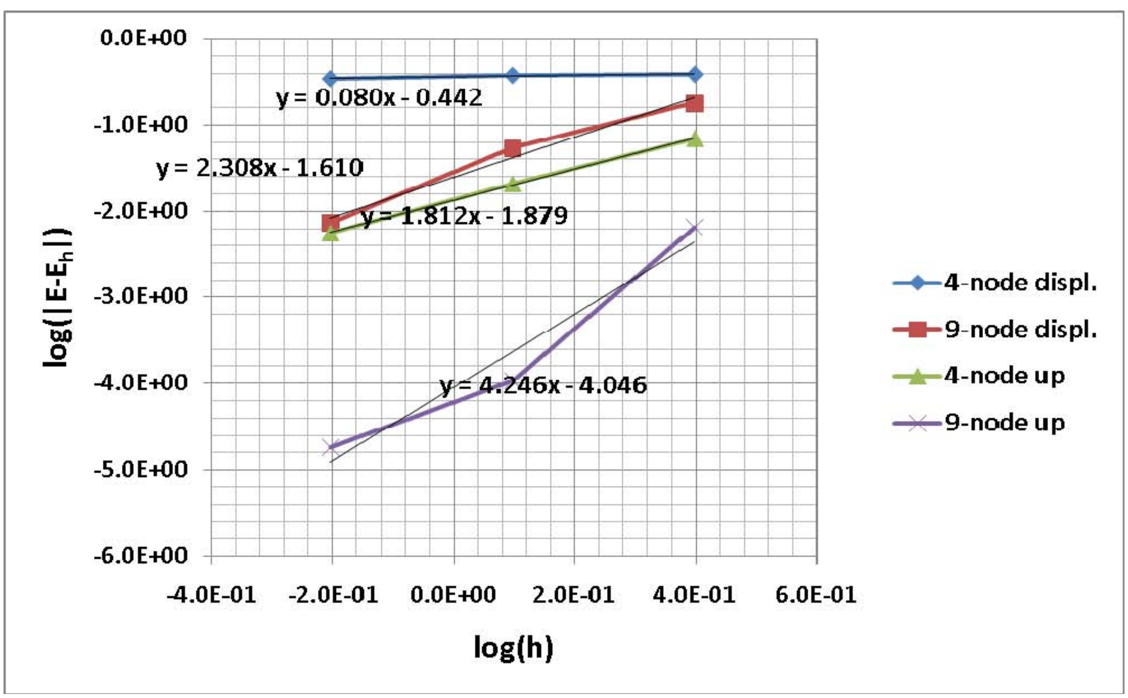
$$\underline{H}_p = [1]$$

And,

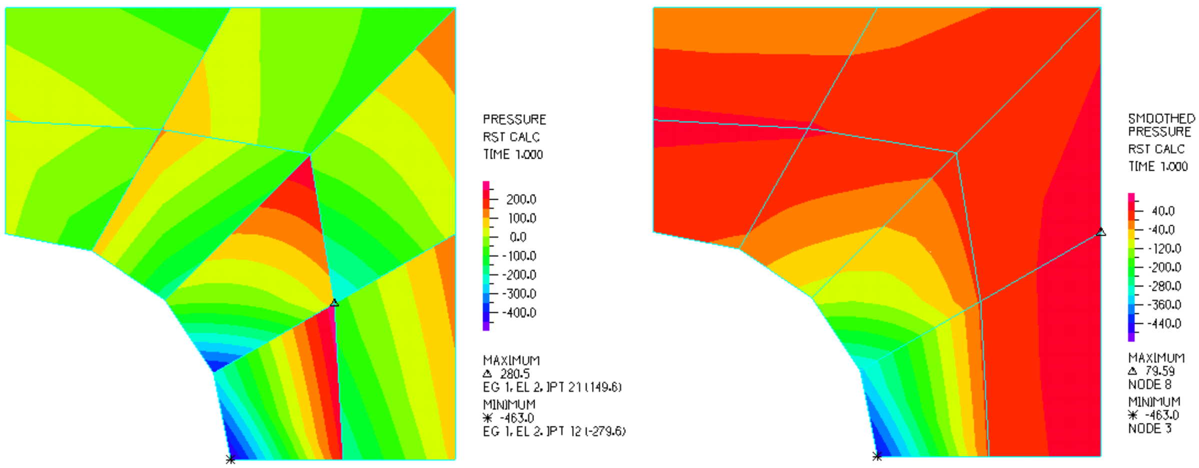
$$\underline{C}' = \begin{bmatrix} 2G & 0 & 0 & 0 \\ 0 & 2G & 0 & 0 \\ 0 & 0 & G & 0 \\ 0 & 0 & 0 & 2G \end{bmatrix}$$

Problem 3 (20 points):

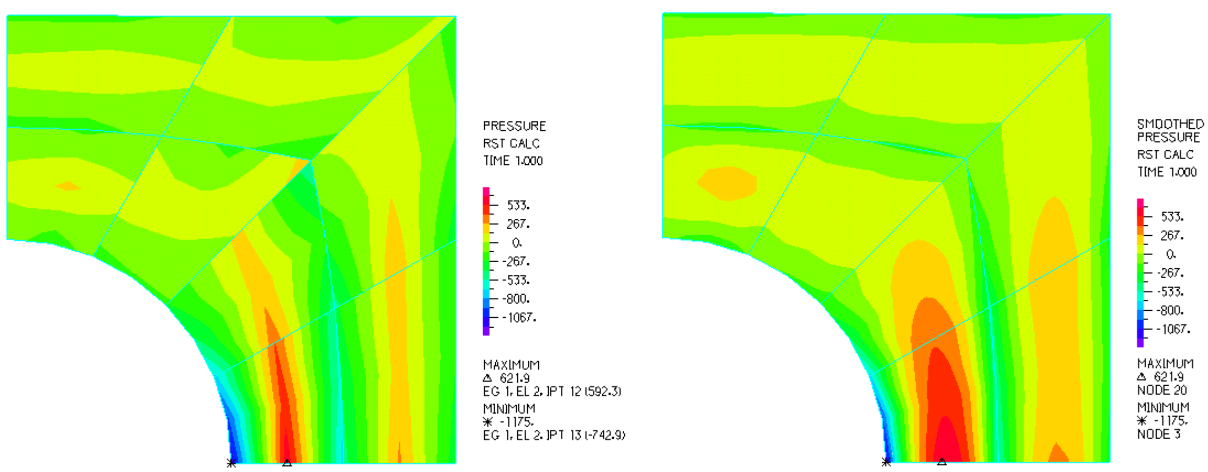
We can see in convergence curves that the displacement-based elements are bad when a material is incompressible. However, when the mixed (u/p) elements are used, we can obtain almost optimal convergence rates. Also, the curves of the mixed elements are shifted much down from those of the displacement-based elements, which means that the solutions obtained using the mixed elements are much more accurate. (Note that even the 4/1 u/p element is better than the 9-node displacement-based element in this problem.)



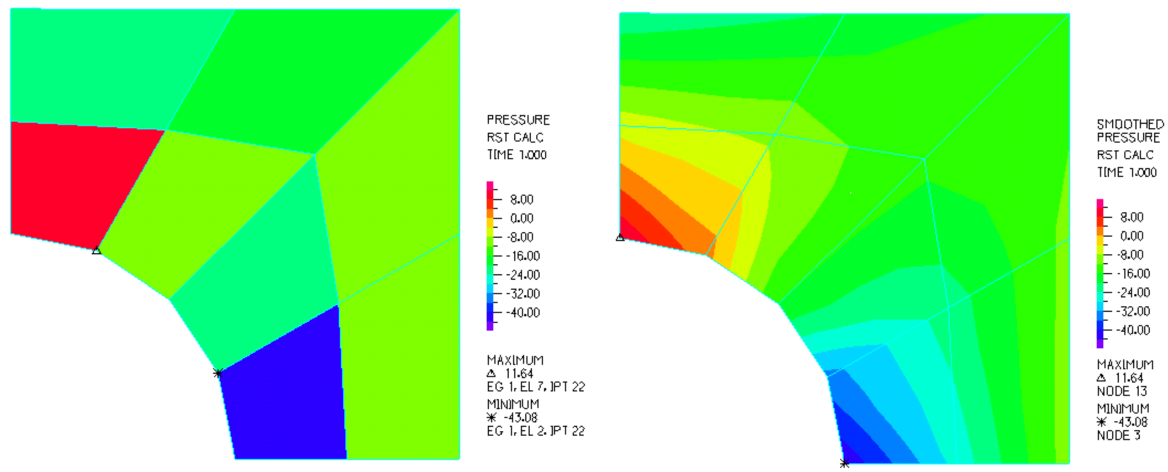
In addition, the mixed elements predict better pressure distributions with reasonable magnitudes.



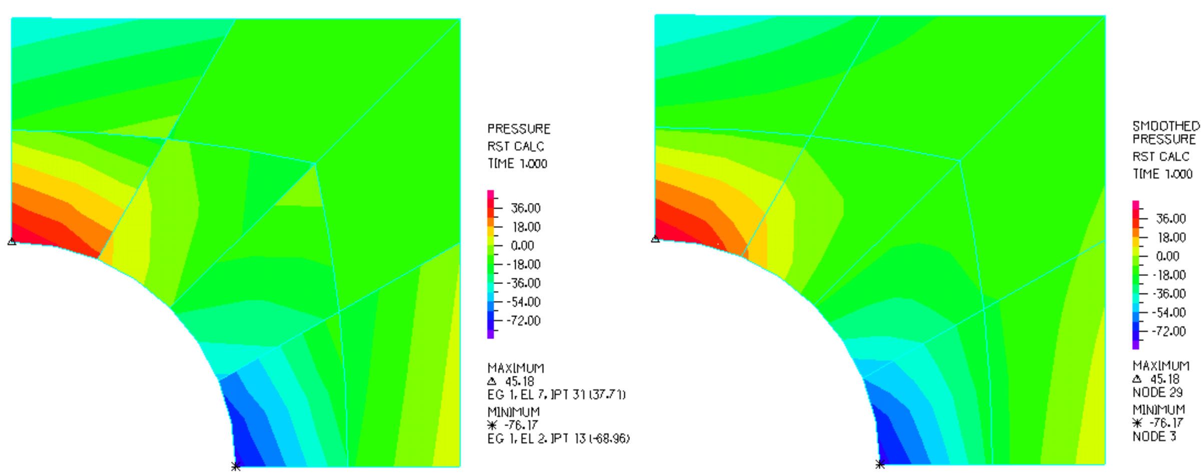
(4-node displacement-based element)



(9-node displacement-based element)



(4-node u/p element)



(9-node u/p element)

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