2.094

FINITE ELEMENT ANALYSIS OF SOLIDS AND FLUIDS

SPRING 2008

Homework 3 - Solution

		Assigned:	02/21/2008
Instructor:	Prof. K. J. Bathe	Due:	02/28/2008

Problem 1 (10 points):

From the plane strain condition,

$$\varepsilon_{zz} = \frac{1}{E}\left\{\tau_{zz} - \nu(\tau_{xx} + \tau_{yy})\right\} = 0$$

$$\therefore \tau_{zz} = \nu(\tau_{xx} + \tau_{yy})$$

The strains are

$$\varepsilon_{xx} = \frac{1}{E}\left\{\tau_{xx} - \nu(\tau_{yy} + \tau_{zz})\right\} = \frac{1}{E}\left\{(1-\nu^2)\tau_{xx} - \nu(1+\nu)\tau_{yy}\right\} = 4.7667\times10^{-7}$$

$$\varepsilon_{yy} = \frac{1}{E}\left\{\tau_{yy} - \nu(\tau_{xx} + \tau_{zz})\right\} = \frac{1}{E}\left\{(1-\nu^2)\tau_{yy} - \nu(1+\nu)\tau_{xx}\right\} = 4.3333\times10^{-8}$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy} = \frac{2(1+\nu)}{E}\tau_{xy} = 8.6667\times10^{-7}$$

The displacements in the four-node element can be written as

$$u(x,y) = a_0 + a_1 x + a_2 y + a_3 xy$$

$$v(x,y) = b_0 + b_1 x + b_2 y + b_3 xy$$

(*Note that you can also use the following form of displacements as in the class.

$$u(x,y) = h_1(x,y)u_1 + h_2(x,y)u_2 + h_3(x,y)u_3 + h_4(x,y)u_4$$

$$v(x,y) = h_1(x,y)v_1 + h_2(x,y)v_2 + h_3(x,y)v_3 + h_4(x,y)v_4$$

But, then equations are slightly complicate to solve. Please see attached sample solution which uses this method)

Then,

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = a_1 + a_3 y = 4.7667 \times 10^{-7}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = b_2 + b_3 x = 4.3333 \times 10^{-8}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = a_2 + a_3 x + b_1 + b_3 y = 8.6667 \times 10^{-7}$$

Since these equations hold for all x and y,

$$a_3 = b_3 = 0$$

$$a_1 = 4.7667 \times 10^{-7}$$

$$b_2 = 4.3333 \times 10^{-8}$$

$$a_2 + b_1 = 8.6667 \times 10^{-7}$$

Therefore,

$$u(x, y) = a_0 + 4.7667 \times 10^{-7} x + a_2 y$$

$$v(x, y) = b_0 + b_1 x + 4.3333 \times 10^{-8} y$$

The boundary conditions with setting the node 3 to the origin of the coordinate system are

$$u(0,0) = a_0 = 0$$

$$v(0,0) = b_0 = 0$$

$$v(5,0) = b_0 + 5b_1 = 0$$

Finally,

$$u(x, y) = 4.7667 \times 10^{-7} x + 8.6667 \times 10^{-7} y$$

$$v(x, y) = 4.3333 \times 10^{-8} y$$

The displacements at each node are

	u (in.)	v (in.)
Node 1 (x=5, y=8)	9.3167×10^{-6}	3.4666×10^{-7}
Node 2 (x=0, y=8)	6.9334×10^{-6}	3.4666×10^{-7}
Node 3 (x=0, y=0)	0	0
Node 4 (x=5, y=0)	2.3833×10^{-6}	0

Problem 2 (20 points):

(a) The stresses can be obtained from a bilinear interpolation.

$$\tau_{xx}^{(4)} = \frac{1}{4}(1+x)(1+y)(298.5) + \frac{1}{4}(1-x)(1+y)(624.0) + \frac{1}{4}(1-x)(1-y)(1.146) + \frac{1}{4}(1+x)(1-y)(-324.3)$$
$$= 149.8365 - 162.7365x + 311.4135y - 0.0135xy$$

$$\tau_{yy}^{(4)} = \frac{1}{4}(1+x)(1+y)(-914.9) + \frac{1}{4}(1-x)(1+y)(169.9) + \frac{1}{4}(1-x)(1-y)(-16.96) + \frac{1}{4}(1+x)(1-y)(-1102)$$
$$= -465.9900 - 542.4600x + 93.4900y + 0.0600xy$$

$$\tau_{xy}^{(4)} = \frac{1}{4}(1+x)(1+y)(-370.8) + \frac{1}{4}(1-x)(1+y)(-588.8) + \frac{1}{4}(1-x)(1-y)(-209.1) + \frac{1}{4}(1+x)(1-y)(8.865)$$
$$= -289.9587 + 108.9912x - 189.8412y + 0.0087xy$$

Note that the coefficients of 'xy' in each stress must be zero because of our displacement assumption. But here we have them due to rounding. (The strains do not have 'xy' terms because they are derivatives of the displacements.)

$$B^{(4)} = \frac{1}{4}\begin{bmatrix} (1+y) & -(1+y) & -(1-y) & (1-y) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1+x) & (1-x) & -(1-x) & -(1+x) \\ (1+x) & (1-x) & -(1-x) & -(1+x) & (1+y) & -(1+y) & -(1-y) & (1-y) \end{bmatrix}$$

$$F^{(4)} = \int_{-1}^{1}\int_{-1}^{1} B^{(4)T}\begin{bmatrix} \tau_{xx}^{(4)} \\ \tau_{yy}^{(4)} \\ \tau_{xy}^{(4)} \end{bmatrix}(0.1)\,dxdy$$
$$= \begin{bmatrix} 0.00127 \\ -57.99301 \\ 28.02571 \\ 29.96603 \\ -100.0049 \\ 6.806910 \\ 51.18483 \\ 42.01317 \end{bmatrix}$$

(b) Consider element 4.

a. Horizontal equilibrium:

$0-57.99+28.03+29.97 \cong 0$

b. Vertical equilibrium:

$-100+6.81+51.18+42.01=0$

c. Moment equilibrium about its local node 3:

$-100\times 2+57.99\times 2+42.01\times 2=0$

Problem 3 (10 points):

For the element A,

$$\{\boldsymbol{u_1}, \boldsymbol{v_1}, \boldsymbol{u_4}, \boldsymbol{v_4}\}^T = \{\boldsymbol{U_1}, \boldsymbol{U_2}, \boldsymbol{U_3}, \boldsymbol{U_4}\}^T$$

The corresponding components of the stiffness matrix are

$$\widehat{\boldsymbol{K}}_A = \begin{bmatrix} a_{11} & a_{12} & a_{17} & a_{18} \\ a_{21} & a_{22} & a_{27} & a_{28} \\ a_{71} & a_{72} & a_{77} & a_{78} \\ a_{81} & a_{82} & a_{87} & a_{88} \end{bmatrix} \begin{matrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{matrix}$$

For the element B,

$$\{\boldsymbol{u_1}, \boldsymbol{v_1}, \boldsymbol{\theta_1}\}^T = \{\boldsymbol{U_3}, \boldsymbol{U_4}, \boldsymbol{U_5}\}^T$$

The corresponding components of the stiffness matrix are

$$\widehat{\boldsymbol{K}}_B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{matrix} U_3 \\ U_4 \\ U_5 \end{matrix}$$

Then the global stiffness matrix is

$$\boldsymbol{K} = \begin{bmatrix} a_{11} & a_{12} & a_{17} & a_{18} & 0 \\ a_{21} & a_{22} & a_{27} & a_{28} & 0 \\ a_{71} & a_{72} & a_{77}+b_{11} & a_{78}+b_{12} & b_{13} \\ a_{81} & a_{82} & a_{87}+b_{21} & a_{88}+b_{22} & b_{23} \\ 0 & 0 & b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{matrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{matrix}$$