2.094

Finite Element Analysis of Solids and Fluids

Spring 2008

Homework 1 - Solution

		Assigned:	02/07/2008
Instructor:	Prof. K. J. Bathe	Due:	02/14/2008

Problem 1 (10 points):

In this problem, the principle of virtual work reduces to

$$\int_{{}^tV} {}^t\tau_{22}\ {}_t\bar{e}_{22} d\ {}^tV = \int_{{}^tS_f} \bar{u}_2^{{}^tS_f}\ {}^tf_{x_2} d\ {}^tS_f$$

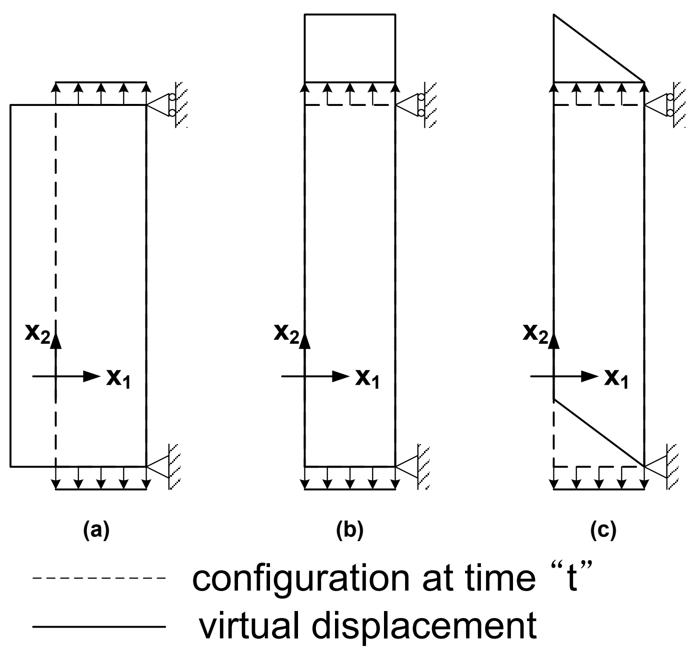

Figure 1. Three simple independent virtual displacement patterns.

(a) a simple tension in the x_1 direction; (b) a simple tension in the x_2 direction; (c) a simple shear

(a) A simple extension in x_1 direction; $\bar{u}_1 = ({}^t x_1 - 1)$ and $\bar{u}_2 = 0$

$$ {}_t\bar{e}_{22} = 0 $$
$$ \bar{u}_2^{\,{}^tS_f} = 0 $$

Therefore,

$$ \int_{{}^tV} {}^t\tau_{22}\ {}_t\bar{e}_{22}\, d\,{}^tV = 0 $$

$$ \int_{{}^tS_f} \bar{u}_2^{\,{}^tS_f}\ {}^tf_{x_2}\, d\,{}^tS_f = 0 $$

Check!

(b) A simple extension in x_2 direction; $\bar{u}_1 = 0$ and $\bar{u}_2 = ({}^t x_2 + 1)$

$$ {}_t\bar{e}_{22} = 1 $$

$\bar{u}_2^{\,{}^tS_f} = 4$ (on the top surface) and $\bar{u}_2^{\,{}^tS_f} = 0$ (on the bottom surface)

Therefore,

$$ \int_{{}^tV} {}^t\tau_{22}\ {}_t\bar{e}_{22}\, d\,{}^tV = \int_{{}^tV} 20 \times 1\, d\,{}^tV = 20 \times\ {}^tV = 80 $$
$$ \int_{{}^tS_f} \bar{u}_2^{\,{}^tS_f}\ {}^tf_{x_2}\, d\,{}^tS_f = \int_{{}^tS_f} 4 \times 20\, d\,{}^tS_f = 80 \times\ {}^tS_f = 80 $$

Check!

(c) A simple shear; $\bar{u}_1 = 0$ and $\bar{u}_2 = (1 - {}^t x_1)$

$$ {}_t\bar{e}_{22} = 0 $$
$$ \bar{u}_2^{\,{}^tS_f} = 1 - {}^t x_1 $$

Therefore,

$$ \int_{{}^tV} {}^t\tau_{22}\ {}_t\bar{e}_{22}\, d\,{}^tV = 0 $$

$$ \int_{{}^tS_f} \bar{u}_2^{\,{}^tS_f}\ {}^tf_{x_2}\, d\,{}^tS_f = \int_{{}^tS_{f_1}} 20 \times (1 - {}^t x_1)\, d\,{}^tS_{f_1} + \int_{{}^tS_{f_2}} (-20) \times (1 - {}^t x_1)\, d\,{}^tS_{f_2} $$
$$ = 20 \int_0^1 (1 - {}^t x_1)\, d\,{}^t x_1 - 20 \int_0^1 (1 - {}^t x_1)\, d\,{}^t x_1 = 0 $$

(on the top surface) (on the bottom surface)

Check!

Problem 2 (20 points):

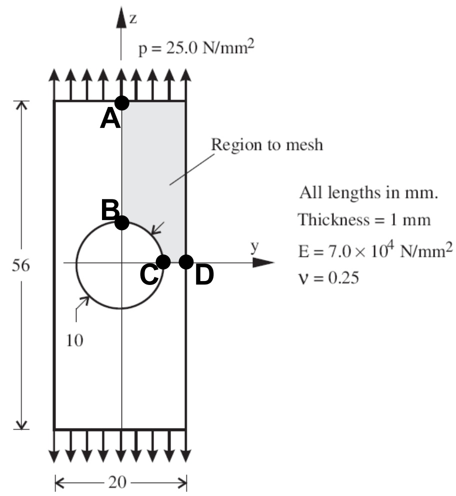

Figure 2. Plate with a hole in tension

In this solution, the horizontal symmetry line corresponds to the line CD and the vertical symmetry line corresponds to the line AB. The distance of the horizontal line is measured from the point C and the distance of the vertical line is measured from the point A. (See Figure 2.)

The stresses are shown through Figures 3 and 6. We can see that the solutions obtained with 9-node elements are better than those obtained with 4-node elements. One possible way, in general, to check calculated solutions is to see whether the stress boundary conditions are satisfied. Here, we know that we should have $\boldsymbol{\tau_{zz}|_A = 25}$ due to the applied traction and $\boldsymbol{\tau_{zz}|_B = \tau_{yy}\big|_C = \tau_{yy}\big|_D = 0}$ because no tractions are applied at the points B,C, and D. The solutions obtained with 9-node elements are closer to these exact values than those obtained with 4-node elements. In this problem we also have the analytical solutions for the $\boldsymbol{\tau_{zz}}$ at point C and D, see reference 1. The solutions are compared in Figure 7.

[1] Timoshenko, S.P. and Goodier, J.N., Theory of Elasticity, Third Edition, McGraw-Hill, 1970, pp. 94-95.

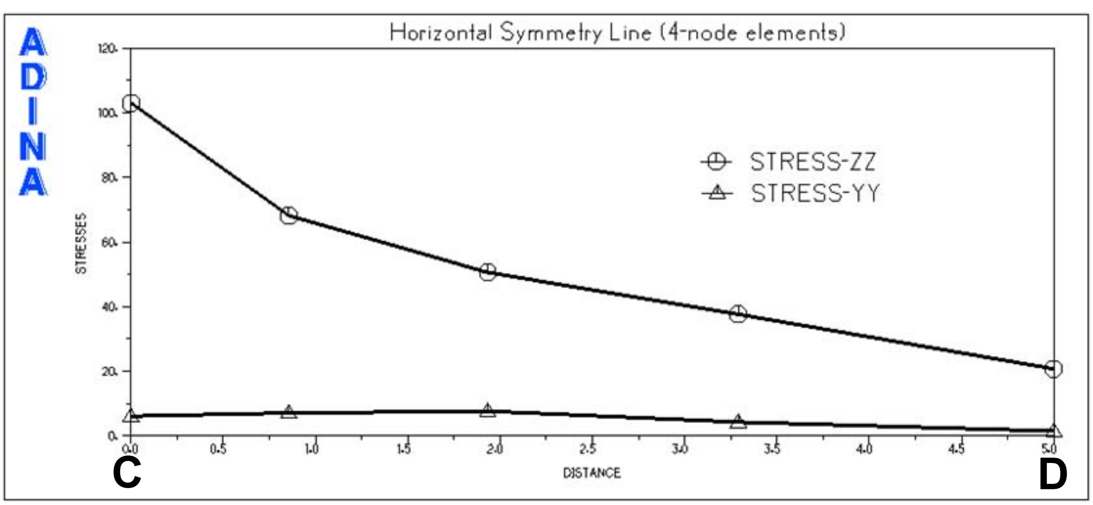

Figure 3. Stresses on the horizontal symmetry line solved using 4-node elements

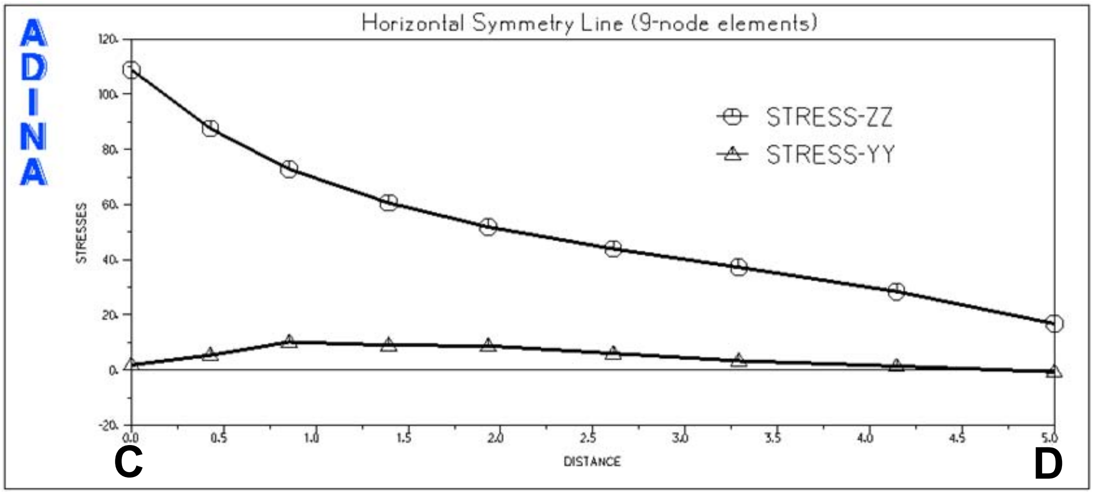

Figure 4. Stresses on the horizontal symmetry line solved using 9-node elements

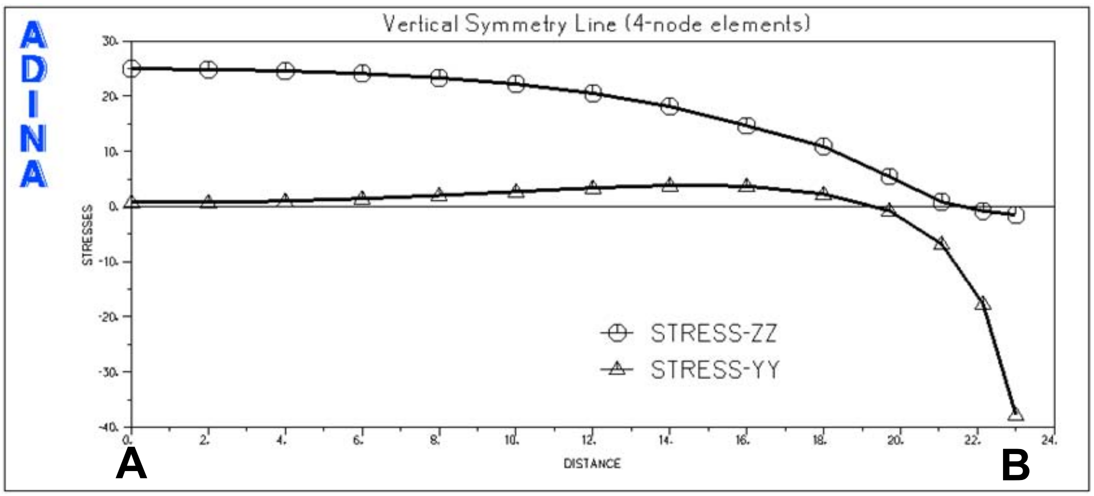

Figure 5. Stresses on the vertical symmetry line solved using 4-node elements

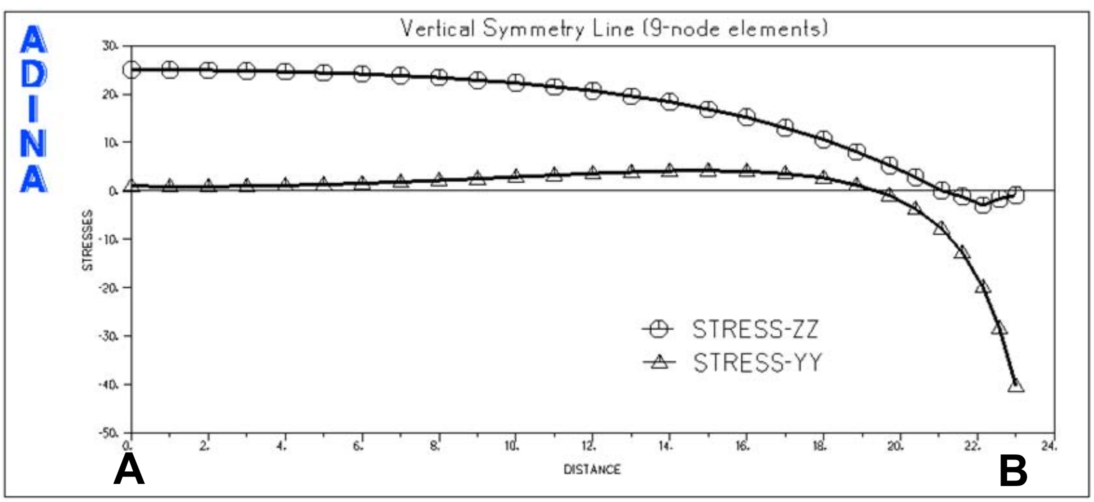

Figure 6. Stresses on the vertical symmetry line solved using 9-node elements

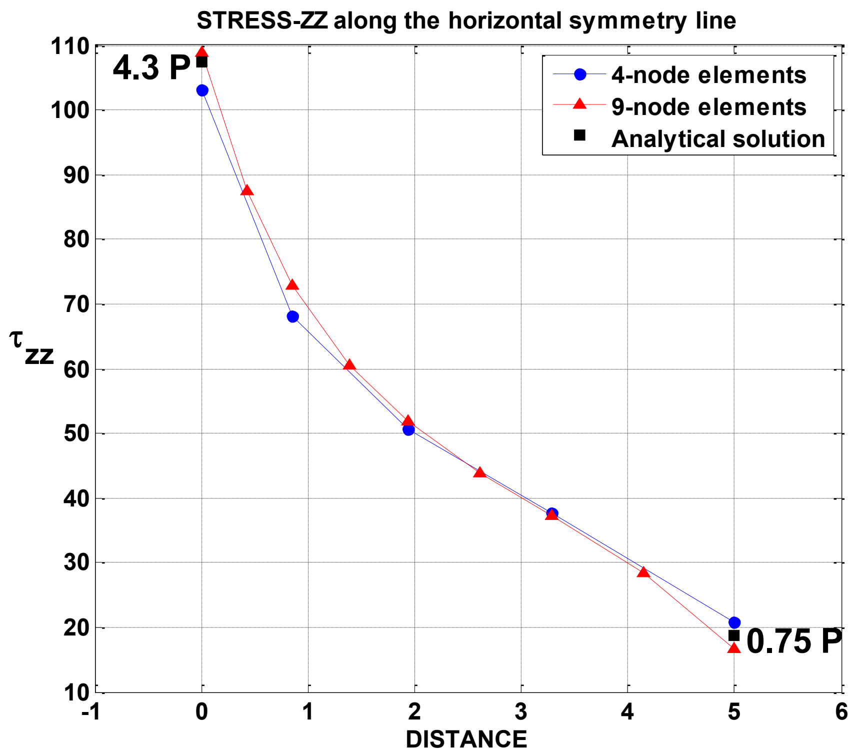

Figure 7. τ_{zz} on the horizontal symmetry line