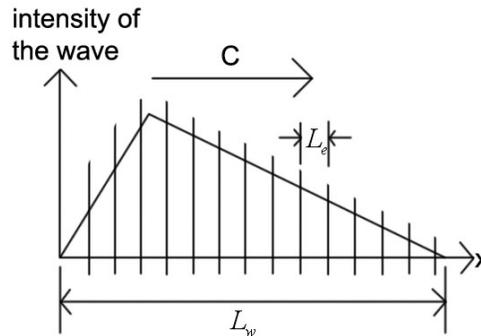


Lecture 20 - Wave Propagation Response

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Quiz #2: Closed book, 6 pages of notes, no calculators. Covers all materials including this week's lectures.

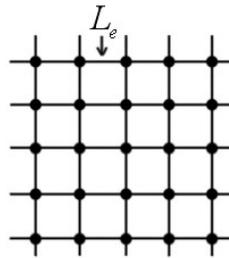


$$C = \frac{L_w}{t_w} \quad (C \text{ depends on material properties})$$

For this system, C is the wave speed (given), L_w is the critical wavelength to be represented, t_w is the total time for this wave to travel past a point, L_e is the “effective length” of a finite element, and is equivalent to $\frac{L_w}{n}$ (given). To solve, we should use $\Delta t = \frac{L_e}{C}$.

Mesh

L_e should be smaller than the shortest wave length we want to pick up. To establish a mesh, we use low-order



elements (4-node elements in 2D, 8-node elements in 3D). We use the central difference method, which requires stability. We need to ensure that $\Delta t \leq \Delta t_{cr} = \frac{2}{\omega_n}$. Recall that in nonlinear analysis, the wave speed changes.

We know that

$$\omega_n \leq \max_m \{ \omega_n^{(m)} \}$$

where ω_n is the largest frequency of an assembled finite element mesh and $\max_m \{ \omega_n^{(m)} \}$ is the largest element frequency of all elements in the mesh. Then we can use

$$\Delta t = \frac{2}{\max_m \{ \omega_n^{(m)} \}}$$

and conservatively, we use a slightly smaller value.

How to Find $\omega_n^{(m)}$

For lower-order elements, we have bounds for $\omega_n^{(m)}$. (See Sections 9.3/9.4 and Table 9.5 that gives formulae for $\omega_n^{(m)}$.) Proof: We know

$$\omega_n^2 = \frac{\phi_n^T \left(\sum_m \mathbf{K}^{(m)} \right) \phi_n}{\phi_n^T \left(\sum_m \mathbf{M}^{(m)} \right) \phi_n} \quad (\text{a})$$

$$\omega_n^2 = \frac{\sum_m \mathcal{U}^{(m)}}{\sum_m \mathcal{J}^{(m)}} \quad (\text{b})$$

$$\begin{aligned} \mathcal{U}^{(m)} &= \phi_n^T \mathbf{K}^{(m)} \phi_n \\ \mathcal{J}^{(m)} &= \phi_n^T \mathbf{M}^{(m)} \phi_n \end{aligned}$$

Note that $\mathbf{K}^{(m)}$ is of the same size as \mathbf{K} . Also, for an element (m) , since ϕ_n is not an eigenvector for $\mathbf{K}^{(m)}$:

$$\begin{aligned} \frac{\phi_n^T \mathbf{K}^{(m)} \phi_n}{\phi_n^T \mathbf{M}^{(m)} \phi_n} &\leq \left(\omega_n^{(m)} \right)^2 \\ \mathcal{U}^{(m)} &\leq \mathcal{J}^{(m)} \left(\omega_n^{(m)} \right)^2 \end{aligned} \quad (\text{c})$$

Plug (c) into (b) and we have

$$\omega_n^2 \leq \frac{\sum_m \mathcal{J}^{(m)} \left(\omega_n^{(m)} \right)^2}{\sum_m \mathcal{J}^{(m)}} \leq \max_m \left\{ \left(\omega_n^{(m)} \right)^2 \right\}$$

In practice, wave propagation problems are very difficult to solve, due to reflections, absorptions, and the many different wave types (shear waves, etc.). As mentioned, the central difference method is almost always used for wave propagation solutions, because wave propagation modeling needs fine meshes.

In beams, rotational DOFs result in high frequencies, so rotational mass is frequently set to zero. If we have zero masses in the model, we need to use static condensation.

$$\begin{bmatrix} \mathbf{K}_{aa} & \mathbf{K}_{ab} \\ \mathbf{K}_{ba} & \mathbf{K}_{bb} \end{bmatrix} \begin{bmatrix} \phi_a \\ \phi_b \end{bmatrix} = \omega^2 \begin{bmatrix} \mathbf{M}_a & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \phi_a \\ \phi_b \end{bmatrix}$$

No mass on b-DOFs

Use the 2nd equation to eliminate ϕ_b from the 1st equation.

$$\mathbf{K}_{ba} \phi_a + \mathbf{K}_{bb} \phi_b = \mathbf{0}$$

$$\phi_b = -\mathbf{K}_{bb}^{-1} \mathbf{K}_{ba} \phi_a$$

We obtain

$$\left(\mathbf{K}_{aa} - \mathbf{K}_{ab} \mathbf{K}_{bb}^{-1} \mathbf{K}_{ba} \right) \phi_a = \omega^2 \mathbf{M}_a \phi_a$$

It is the same as Gauss elimination on b-DOFs. Hence, if there are zero masses, we use static condensation prior to the use of the central difference method.

All we have discussed regarding transient/dynamic analysis is also applicable (with modifications!) in heat transfer & fluid flow analysis.

Transient Heat Transfer Analysis

The governing finite element equation is

$$\mathbf{C}\dot{\boldsymbol{\theta}} + \mathbf{K}\boldsymbol{\theta} = \mathbf{Q} \quad (\text{given } {}^0\boldsymbol{\theta}) \quad (\text{A})$$

where \mathbf{C} is the heat capacity matrix, \mathbf{K} is the conductivity matrix, and \mathbf{Q} is the heat flow input vector.

I. Mode Superposition

Frequently, heat transfer solutions can be obtained with coarser meshes. Let's examine the system

$$\mathbf{C}\dot{\boldsymbol{\theta}} + \mathbf{K}\boldsymbol{\theta} = \mathbf{0} \quad (\text{no heat flow input})$$

Assume

$$\begin{aligned} \boldsymbol{\theta} = e^{-\lambda t} \boldsymbol{\phi} &\rightarrow -\lambda e^{-\lambda t} \mathbf{C} \boldsymbol{\phi} + \mathbf{K} e^{-\lambda t} \boldsymbol{\phi} = \mathbf{0} \\ \mathbf{K} \boldsymbol{\phi} &= \lambda \mathbf{C} \boldsymbol{\phi} \end{aligned} \quad (\text{B})$$

The eigenvalues are $0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$, and the eigenvectors are $\phi_1, \phi_2, \dots, \phi_n$. For constant temperature over the mesh, we have $\lambda_1 = 0$. Also:

$$\boldsymbol{\phi}_i^T \mathbf{C} \boldsymbol{\phi}_j = \delta_{ij}$$

$$\boldsymbol{\phi}_i^T \mathbf{K} \boldsymbol{\phi}_j = \lambda_i \delta_{ij}$$

We obtain the n decoupled equations.

$$\dot{\eta}_i + \lambda_i \eta_i = q_i \quad (i = 1, \dots, n) \quad ; \quad {}^0\eta_i \quad (\text{C})$$

$$\boldsymbol{\theta} = \sum_{i=1}^n \boldsymbol{\phi}_i \eta_i$$

$$\boldsymbol{\Phi}^T \mathbf{C} \boldsymbol{\Phi} = \mathbf{I} \quad ; \quad \boldsymbol{\Phi} = [\boldsymbol{\phi}_1 \quad \dots \quad \boldsymbol{\phi}_n]$$

$$\boldsymbol{\theta} = \boldsymbol{\Phi} \boldsymbol{\eta} \quad \rightarrow \quad {}^0\boldsymbol{\eta} = \boldsymbol{\Phi}^T \mathbf{C} {}^0\boldsymbol{\theta}$$

$$\boldsymbol{\Phi}^T \mathbf{K} \boldsymbol{\Phi} = \begin{bmatrix} \ddots & & & \\ & \lambda_i & & \\ & & \ddots & \\ & & & \ddots \end{bmatrix}$$

$$\mathbf{q} = \boldsymbol{\Phi}^T \mathbf{Q}$$

$$q_i = \boldsymbol{\phi}_i^T \mathbf{Q}$$

To solve Eq. (C) or perform direct integration on Eq. (A), we can use the Euler backward method or the Euler forward method.

II. The Euler Backward Method

Reading assignment: Section 9.6

$${}^{t+\Delta t}\dot{\boldsymbol{\theta}} = \frac{({}^{t+\Delta t}\boldsymbol{\theta} - {}^t\boldsymbol{\theta})}{\Delta t}$$

Use Eq. (A) at time $t + \Delta t$ to solve for ${}^{t+\Delta t}\boldsymbol{\theta}$. This is an implicit method, and is unconditionally stable. Δt only needs to be selected for accuracy.

III. The Euler Forward Method

$${}^t\dot{\boldsymbol{\theta}} = \frac{{}^{t+\Delta t}\boldsymbol{\theta} - {}^t\boldsymbol{\theta}}{\Delta t}$$

Then, use Eq. (A) at time t .

$$\mathbf{C} {}^t\dot{\boldsymbol{\theta}} + \mathbf{K} {}^t\boldsymbol{\theta} = {}^t\mathbf{Q}$$

$$\frac{1}{\Delta t} (\mathbf{C} {}^{t+\Delta t}\boldsymbol{\theta}) = \frac{1}{\Delta t} \mathbf{C} {}^t\boldsymbol{\theta} - \mathbf{K} {}^t\boldsymbol{\theta} + {}^t\mathbf{Q}$$

If \mathbf{C} is a diagonal matrix, then no factorization is involved. The Euler forward method results in:

$$\frac{1}{\Delta t} \mathbf{C} {}^{t+\Delta t}\boldsymbol{\theta} = {}^t\hat{\mathbf{Q}}$$

The method is conditionally stable, and Δt must satisfy

$$\Delta t \leq \Delta t_{cr} = \frac{2}{\lambda_n}$$

Recall:

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{0}$$

$$\mathbf{U} = \boldsymbol{\phi} \sin \omega (t - \tau)$$

$$\ddot{\mathbf{U}} = -\omega^2 \boldsymbol{\phi} \sin \omega (t - \tau)$$

$$-\mathbf{M}\omega^2 \boldsymbol{\phi} \sin \omega (t - \tau) + \mathbf{K} \boldsymbol{\phi} \sin \omega (t - \tau) = 0$$

Hence, $\mathbf{K}\boldsymbol{\phi} = \omega^2 \mathbf{M}\boldsymbol{\phi}$ is the eigenvalue problem.

$$\dot{\eta}_i + \lambda_i \eta_i = q_i \quad \rightarrow \quad \Delta t_{cr} = \frac{2}{\lambda_n}$$

$$\ddot{x}_i + \omega_i^2 x_i = r_i \quad \rightarrow \quad \Delta t_{cr} = \frac{2}{\omega_n}$$

The explicit method uses the governing equations at time t to obtain the solution at $t + \Delta t$. The implicit method uses the governing equations at time $t + \Delta t$ to obtain the solution at $t + \Delta t$.

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2.092 / 2.093 Finite Element Analysis of Solids and Fluids I
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