2.092/2.093 — Finite Element Analysis of Solids & Fluids I	Fall '09
Lecture 18 - Modeling for Dynamic Analysis & Solution	
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From last lecture,

$$\boldsymbol{M}\ddot{\boldsymbol{U}} + \boldsymbol{C}\dot{\boldsymbol{U}} + \boldsymbol{K}\boldsymbol{U} = \boldsymbol{R}(t) \quad ; \quad {}^{0}\boldsymbol{U} \ , \ {}^{0}\dot{\boldsymbol{U}} \tag{1}$$

$\boldsymbol{K}\boldsymbol{U} = \boldsymbol{F}_I$, the internal force calculated from the element stresses.

Mode Superposition

The mode superposition method transforms the displacements so as to decouple the governing equation (1). Thus, consider:

$$\boldsymbol{U} = \sum_{i=1}^{n} \boldsymbol{\phi}_i x_i \tag{2}$$

We start with the general solution, where $\boldsymbol{\phi}_i$ is an eigenvector. Then Eq. (1) becomes

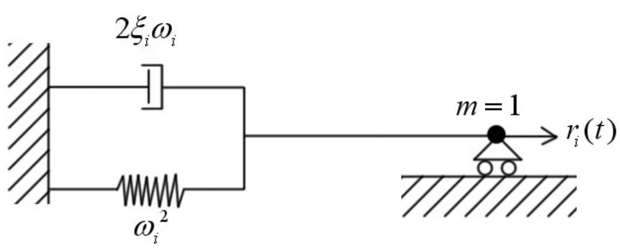
$$\ddot{x}_i + 2\xi_i\omega_i\dot{x}_i + \omega_i^2 x_i = \boldsymbol{\phi}_i^T\boldsymbol{R} = r_i \quad (i = 1, \ldots, n) \tag{3}$$

For damping, assume a diagonal $\boldsymbol{C}$ matrix:

$$\boldsymbol{\Phi}^T\boldsymbol{C}\boldsymbol{\Phi} = \begin{bmatrix} \ddots & & \text{zeros} \\ & 2\xi_i\omega_i & \\ \text{zeros} & & \ddots \end{bmatrix}$$

$$\boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{\phi}_1 & \ldots & \boldsymbol{\phi}_n \end{bmatrix}$$

The initial conditions are ${}^{0}x_i = \boldsymbol{\phi}_i^T\boldsymbol{M}\,{}^{0}\boldsymbol{U}$, ${}^{0}\dot{x}_i = \boldsymbol{\phi}_i^T\boldsymbol{M}\,{}^{0}\dot{\boldsymbol{U}}$. We consider and solve n such single-DOF systems:



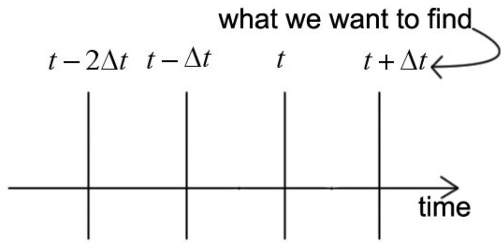
The mass m is 1, and the stiffness is ω_i^2. In Eq. (1) we have fully coupled equations. By performing the transformation, we obtain n decoupled equations. r_i can be a complicated function of time.

Direct Integration

In direct integration, no transformation is performed.

I. Explicit Method: Central Difference Method

The explicit method evaluates Eq. (1) at time t to obtain the solution at time $t + \Delta t$. Assume we



already have the values for ${}^{t}\boldsymbol{U}$, ${}^{t-\Delta t}\boldsymbol{U}$, ${}^{t-2\Delta t}\boldsymbol{U}$, ... Consider the following three linearly independent equations:

$$\boldsymbol{M}\,{}^{t}\ddot{\boldsymbol{U}} + \boldsymbol{C}\,{}^{t}\dot{\boldsymbol{U}} + \boldsymbol{K}\,{}^{t}\boldsymbol{U} = {}^{t}\boldsymbol{R} \tag{4}$$

$${}^{t}\ddot{\boldsymbol{U}} = \tfrac{1}{\Delta t^2}\left({}^{t+\Delta t}\boldsymbol{U} - 2\,{}^{t}\boldsymbol{U} + {}^{t-\Delta t}\boldsymbol{U}\right) \tag{5}$$

$${}^{t}\dot{\boldsymbol{U}} = \tfrac{1}{(2\Delta t)}\left({}^{t+\Delta t}\boldsymbol{U} - {}^{t-\Delta t}\boldsymbol{U}\right) \tag{6}$$

These equations can be solved for ${}^{t+\Delta t}\boldsymbol{U}$. Assume $\boldsymbol{C} = \boldsymbol{0}$ and $\boldsymbol{M}$ = diagonal mass matrix $\boldsymbol{M}_l$,

$$\frac{1}{(\Delta t)^2}\boldsymbol{M}_l\,{}^{t+\Delta t}\boldsymbol{U} = {}^{t}\hat{\boldsymbol{R}} \tag{7}$$

All known quantities go to the right-hand side into ${}^{t}\hat{\boldsymbol{R}}$. $\boldsymbol{M}_l$ is a diagonal matrix, hence we have

$${}^{t+\Delta t}\boldsymbol{U}_i = \frac{(\Delta t)^2}{(\boldsymbol{M}_l)_{ii}}\,{}^{t}\hat{\boldsymbol{R}}_i$$

for every ith component. If $(\boldsymbol{M}_l)_{ii}$ is zero, the equation can not be solved. This corresponds to an infinite frequency. For the method to be stable, we must have

$$\text{The Condition of Stability:}\quad \Delta t \le \Delta t_{cr} = \frac{T_n}{\pi} = \frac{2}{\omega_n} \tag{8}$$

If $\boldsymbol{C}$ is diagonal as well, the method still works in the same way! Note that $\boldsymbol{K}$ only appears in the right-hand side of the equation.

II. Implicit Method: Trapezoidal Rule

An implicit method evaluates Eq. (1) at time $t+\Delta t$ to obtain the solution at time $t+\Delta t$.

$$\boldsymbol{M}\,{}^{t+\Delta t}\ddot{\boldsymbol{U}} + \boldsymbol{C}\,{}^{t+\Delta t}\dot{\boldsymbol{U}} + \boldsymbol{K}\,{}^{t+\Delta t}\boldsymbol{U} = {}^{t+\Delta t}\boldsymbol{R} \tag{9}$$

$${}^{t+\Delta t}\dot{\boldsymbol{U}} = {}^{t}\dot{\boldsymbol{U}} + \tfrac{1}{2}\left({}^{t+\Delta t}\ddot{\boldsymbol{U}} + {}^{t}\ddot{\boldsymbol{U}}\right)\Delta t \tag{10}$$

The last term in Eq. (10) tells why the trapezoidal rule is also called the constant average acceleration method.

We need one more linearly independent equation to solve the system.

$${}^{t+\Delta t}\boldsymbol{U} = {}^{t}\boldsymbol{U} + \Delta t\,{}^{t}\dot{\boldsymbol{U}} + \frac{1}{4}\left({}^{t+\Delta t}\ddot{\boldsymbol{U}} + {}^{t}\ddot{\boldsymbol{U}}\right)(\Delta t)^2 \tag{11}$$

Here, the last two terms are incremental displacements. Substituting Eqs. (10) and (11) into (9):

$$(c_1\boldsymbol{M} + c_2\boldsymbol{C} + \boldsymbol{K})\,{}^{t+\Delta t}\boldsymbol{U} = {}^{t+\Delta t}\hat{\boldsymbol{R}} \tag{12}$$

where c_1 and c_2 are constants, and are given in the textbook (see Sections 9.1-9.3 for more information). ${}^{t+\Delta t}\hat{\boldsymbol{R}}$ is obtained from known quantities. The larger Δt is, the smaller $c_1\boldsymbol{M} + c_2\boldsymbol{C}$ becomes.

This method is unconditionally stable. In other words, there is no condition on the time step size to have stability. (Not all implicit methods are unconditionally stable.) In numerical analysis, stability and accuracy are distinct requirements. Stability is the first fundamental requirement. But even if the scheme is stable, the result will not be accurate unless a sufficiently small time step has been used.

- For conditionally stable explicit methods → The time step Δt is chosen for stability and accuracy.
- For unconditionally stable implicit methods → The time step Δt is chosen for accuracy.

How to Construct $\boldsymbol{C}$

Rayleigh damping is widely used. For the $\boldsymbol{C}$ matrix, assume

$$\boldsymbol{C} = \alpha \boldsymbol{M} + \beta \boldsymbol{K}$$

where α and β are constants to be selected.

$$\boldsymbol{\phi}_i^T \boldsymbol{C} \boldsymbol{\phi}_j = 2\xi_i \omega_i \delta_{ij} \quad \begin{cases} \text{if } i = j,\ \delta_{ij} = 1 \\ \text{if } i \neq j,\ \delta_{ij} = 0 \end{cases} \tag{13}$$

For two values of Eq. (13) we obtain

$$\begin{aligned} \boldsymbol{\phi}_i^T \left(\alpha \boldsymbol{M} + \beta \boldsymbol{K}\right) \boldsymbol{\phi}_i &= 2\xi_i \omega_i \\ \alpha + \beta \omega_i^2 &= 2\xi_i \omega_i \end{aligned} \tag{14}$$

Let's use $i = 1$ and $i = 2$. We get two independent equations:

$$\begin{aligned} \alpha + \beta \omega_1^2 &= 2\xi_1 \omega_1 \\ \alpha + \beta \omega_2^2 &= 2\xi_2 \omega_2 \end{aligned} \tag{15}$$

which we can solve for α and β. (Obviously, we must have $\omega_1 \neq \omega_2$.) Then we use Eq. (14) to estimate what damping ratios are implicitly assumed in the remaining frequencies.

$$\begin{aligned} \xi_i &= \frac{1}{2\omega_i}\left(\alpha + \beta \omega_i^2\right) \\ &= \frac{\alpha}{2\omega_i} + \frac{\beta}{2}\omega_i \end{aligned}$$

Hence,

$$\xi_i = \frac{\alpha}{2\omega_i} + \frac{\beta}{2}\omega_i, \quad i = 3, 4, \ldots, n \tag{16}$$

where $\frac{\alpha}{2\omega_i}$ is the (low-frequency) mass-proportional damping, and $\frac{\beta}{2}\omega_i$ is the (high-frequency) stiffness-proportional damping. See the textbook for examples on how this method may be applied when more than two damping ratios need to be matched approximately.