

Lecture 16 - Solution of Dynamic Equilibrium Equations, cont'd

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Reading assignment: Sections 9.1-9.3

Recall from our last lecture the general dynamic equilibrium equation and initial conditions:

$$M\ddot{U} + C\dot{U} + KU = R(t) \quad ; \quad {}^0U, {}^0\dot{U} \quad (1)$$

This equation can be solved by:

- Mode superposition
- Direct integration

Mode Superposition

$$K\phi_i = \omega_i^2 M\phi_i \quad (2)$$

The ω_i^2 are the eigenvalues and ϕ_i are the eigenvectors for this system. Solve for ω_i^2, ϕ_i :

$$0 \leq \underbrace{\omega_1^2}_{\text{for } \phi_1} \leq \underbrace{\omega_2^2}_{\text{for } \phi_2} \leq \dots \leq \underbrace{\omega_n^2}_{\text{for } \phi_n}$$

where each ϕ_i refers to a mode shape.

Aside: Consider, picking “a” ϕ ,

$$K\phi = \alpha M\tilde{\phi} \quad (3)$$

where α is a nonzero scalar. Obviously, $K\left(\frac{1}{\alpha}\phi\right) = M\tilde{\phi} = R$. If $\tilde{\phi}$ is an eigenvector, then the load R obtained using $\tilde{\phi}$ gives us back the vector $\tilde{\phi}$ (now scaled by $\frac{1}{\alpha}$).

We also used orthonormality to establish that:

$$\phi_i^T M\phi_j = \delta_{ij}$$

$$\phi_i^T K\phi_j = \omega_i^2 \delta_{ij}$$

The definition of the Rayleigh quotient is

$$\rho(\phi) = \frac{\phi^T K\phi}{\phi^T M\phi}$$

where ϕ can be any vector. So, we have

$$\rho(\phi_i) = \omega_i^2$$

$$\omega_1^2 \leq \rho(\phi_i) \leq \omega_n^2$$

Recall that the strain energy for any vector ϕ is $\frac{1}{2}\phi^T K\phi$. Thus, the strain energy corresponding to a displacement vector (for which $\phi^T M\phi = 1$) is bounded by $\frac{1}{2}$ times the eigenvalues ω_1^2 and ω_n^2 (frequencies squared, $\lambda = \omega^2$). In mode superposition, we use

$$U = \Phi X = \sum_{i=1}^n \phi_i x_i \quad ; \quad \Omega^2 = \begin{bmatrix} \omega_1^2 & & \text{zeros} \\ & \ddots & \\ \text{zeros} & & \omega_n^2 \end{bmatrix}$$

Then, from (1), we now have

$$\ddot{\mathbf{X}} + \Phi^T \mathbf{C} \Phi \dot{\mathbf{X}} + \Omega^2 \mathbf{X} = \Phi^T \mathbf{R} \quad (4)$$

$$\ddot{x} + 2\xi_i \omega_i \dot{x}_i + \omega_i^2 x_i = \phi_i^T \mathbf{R} = r_i \quad (5)$$

Here, $2\xi_i \omega_i \dot{x}_i$ is the term for damping, and ξ_i is the damping ratio. Therefore, from Eq. (5), we now have n fully decoupled equations (each one of a single-DOF model). Assume $\xi_i = 0$. Then, Eq. (5) becomes

$$\ddot{x}_i + \omega_i^2 x_i = r_i \quad ; \quad i = 1, 2, \dots, n \quad (6)$$

We see that $r_i = \phi_i^T \mathbf{R}$, and the initial conditions 0x_i , ${}^0\dot{x}_i$ are next established.

Initial Conditions

$${}^0\mathbf{U} = \Phi {}^0\mathbf{X} \quad \rightarrow \quad (\Phi^T \mathbf{M}) {}^0\mathbf{U} = \underbrace{\Phi^T \mathbf{M} \Phi}_I {}^0\mathbf{X}$$

Thus,

$${}^0\mathbf{X} = \Phi^T \mathbf{M} {}^0\mathbf{U} \quad ; \quad {}^0\dot{\mathbf{X}} = \Phi^T \mathbf{M} {}^0\dot{\mathbf{U}} \quad (7)$$

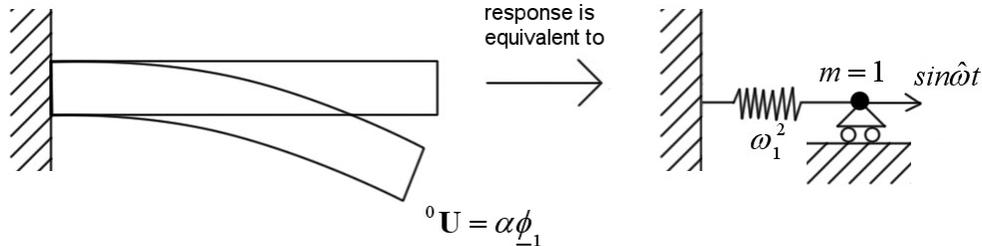
Consider the following simple case

$${}^0\dot{\mathbf{U}} = \mathbf{0} \quad ; \quad \mathbf{R} = \mathbf{0} \quad ; \quad {}^0\mathbf{U} = \alpha \phi_1$$

Then, Eqs. (6) and (7) reduce to $\ddot{x}_1 + \omega_1^2 x_1 = 0$ with ${}^0x_1 = \alpha$, ${}^0\dot{x}_1 = 0$. Also $x_i = 0$, $i \geq 2$. The overall system response is

$$\mathbf{U}(t) = \phi_1 x_1(t)$$

The response is in only one mode! If the beam is initially displaced in only one mode and let go, it will

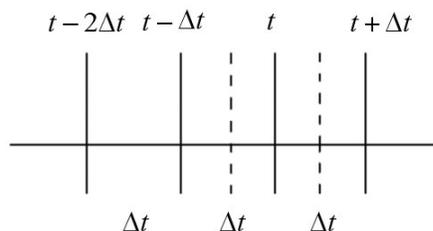


vibrate **only** in that mode. If damping exists, the structure will still vibrate in that single mode, but the amplitude will decay over time.

Direct Integration

I. Explicit integration

Consider the central difference method:



We assume the solution is known up to time t and use the central difference method to determine the solution at time $t + \Delta t$. At time t , Eq. (1) gives us

$$\mathbf{M} {}^t\ddot{\mathbf{U}} + \mathbf{C} {}^t\dot{\mathbf{U}} + \underbrace{\mathbf{K} {}^t\mathbf{U}}_{{}^t\mathbf{F}_I} = {}^t\mathbf{R}$$

The internal forces, ${}^t\mathbf{F}_I$, are calculated from the element stresses. The unknown in the equation is ${}^{t+\Delta t}\mathbf{U}$:

$${}^t\dot{\mathbf{U}} = \frac{1}{2\Delta t} ({}^{t+\Delta t}\mathbf{U} - {}^{t-\Delta t}\mathbf{U}) \quad (\text{A})$$

$${}^t\ddot{\mathbf{U}} = \left(\frac{1}{\Delta t}\right) \left({}^{t+\frac{\Delta t}{2}}\dot{\mathbf{U}} - {}^{t-\frac{\Delta t}{2}}\dot{\mathbf{U}}\right) \quad (\text{a})$$

$${}^{t+\frac{\Delta t}{2}}\dot{\mathbf{U}} = \left(\frac{1}{\Delta t}\right) ({}^{t+\Delta t}\mathbf{U} - {}^t\mathbf{U}) \quad , \quad {}^{t-\frac{\Delta t}{2}}\dot{\mathbf{U}} = \left(\frac{1}{\Delta t}\right) ({}^t\mathbf{U} - {}^{t-\Delta t}\mathbf{U}) \quad (\text{b})$$

Now substitute (b) into (a):

$${}^t\ddot{\mathbf{U}} = \frac{1}{(\Delta t)^2} ({}^{t+\Delta t}\mathbf{U} - 2{}^t\mathbf{U} + {}^{t-\Delta t}\mathbf{U}) \quad (\text{B})$$

Use Eqs. (A) and (B) in Eq. (1) applied at time t to obtain

$$\boxed{(c_1\mathbf{M} + c_2\mathbf{C}) {}^{t+\Delta t}\mathbf{U} = {}^t\hat{\mathbf{R}}} \quad (\text{8})$$

where c_1 and c_2 are constants, and ${}^t\hat{\mathbf{R}}$ is constructed from known values. Next, assume $\mathbf{C} = \mathbf{0}$, and \mathbf{M} is diagonal.

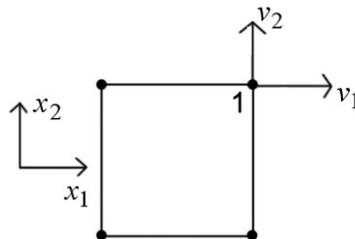
$$\mathbf{M} = \begin{bmatrix} \times & & & & \\ & \times & & \text{zeros} & \\ & & \ddots & & \\ & \text{zeros} & & \times & \\ & & & & \times \end{bmatrix}$$

Then (8) involves no factorization and is very efficiently solved. If one diagonal element in \mathbf{M} is zero, however, the method would not work! Actually, for stability, we must have

$$\Delta t \leq \frac{T_n}{\pi} = \Delta t_{critical}$$

where $T_n = \frac{2\pi}{\omega_n}$ and ω_n is the highest natural frequency. If we use a time step greater than $\Delta t_{critical}$ the solution will diverge quickly.

Example: The Mass Matrix \mathbf{M} for a 4-node 2D Element



$$\mathbf{M}_{\text{consistent}} = \int_V \rho \mathbf{H}^T \mathbf{H} dV$$

To obtain $\mathbf{M}_{\text{lumped}}$, place $\frac{1}{4}$ of the total mass at each node.

$$\mathbf{M}_{8 \times 8} = \begin{bmatrix} \times & & & \text{zeros} \\ & \times & & \\ & & \ddots & \\ & & & \times \\ \text{zeros} & & & & \times \end{bmatrix} = \text{diagonal matrix}$$

In the central difference method, ${}^{t+\Delta t}\mathbf{U}$ is calculated using Eq. (1) at time t . Therefore, the method is called an explicit integration method.

II. Implicit integration, e.g. trapezoidal rule (special case of Newmark method)

These methods use Eq. (1) at time $t + \Delta t$ to obtain ${}^{t+\Delta t}\mathbf{U}$. We will explore them further in the next lecture.

Explicit method: Equilibrium at time (t) \rightarrow results into displacements at time ($t + \Delta t$)

Implicit method: Equilibrium at time ($t + \Delta t$) \rightarrow results into displacements at time ($t + \Delta t$)

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2.092 / 2.093 Finite Element Analysis of Solids and Fluids I
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