2.092/2.093 — Finite Element Analysis of Solids & Fluids I Fall '09

Lecture 15 - Solution of Dynamic Equilibrium Equations

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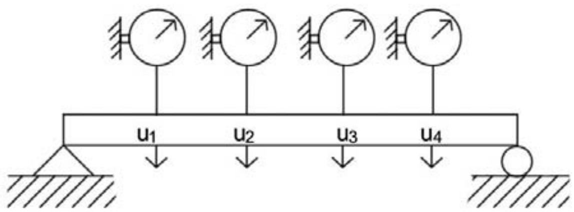
In the last lecture, we described a physical setup that demonstrates the technique of Gauss elimination. We used clamps on each DOF and removed one clamp for one step of Gauss elimination.

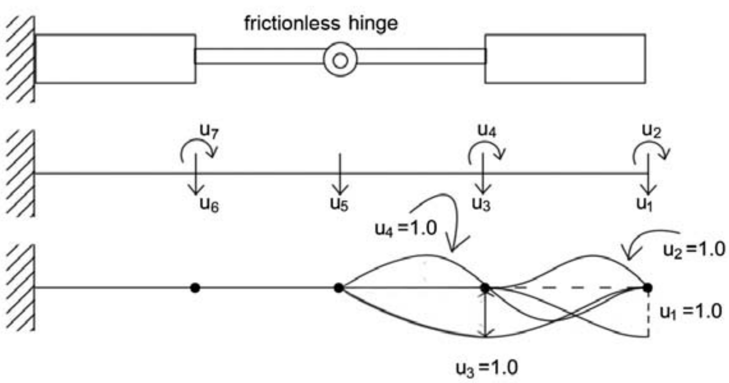


$$\begin{bmatrix} \otimes & \times & \times & \times \\ \times & \otimes & \times & \times \\ \times & \times & \otimes & \times \\ \times & \times & \times & \otimes \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

$\otimes$ should be positive, and should remain positive.

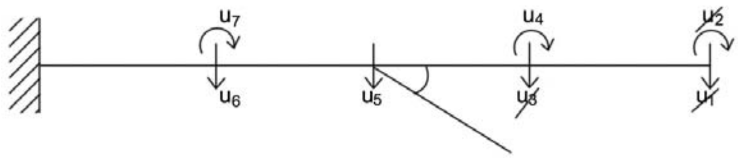
Our rule: Remove clamps one at a time, in the order we would perform Gauss elimination. If there is "a" clamp "seeing" no more stiffness after having removed some clamp(s), the structure is unstable.

Example



$$K = \begin{bmatrix} \times & & & & & \\ & \times & & & & \\ & & \times & & & \\ & & & \times & & \\ & & & & \times & \\ & & & & & \times \end{bmatrix}$$

All diagonal terms are positive. However, there will be a zero diagonal entry after Gauss elimination has been performed for the 3rd DOF.



after 3 Gauss elimination of u_1, u_2 and u_3, u_4 sees no stiffness

Solution of dynamic equilibrium equations

Consider a system with n DOFs:

$$\boldsymbol{M}\ddot{\boldsymbol{U}} + \boldsymbol{C}\dot{\boldsymbol{U}} + \underbrace{\boldsymbol{K}\boldsymbol{U}}_{\boldsymbol{F}_I} = \boldsymbol{R}(t) \tag{1}$$

with initial conditions

$$\boldsymbol{U}\big|_{t=0} = {}^0\boldsymbol{U} \quad ; \quad \dot{\boldsymbol{U}}\big|_{t=0} = {}^0\dot{\boldsymbol{U}}$$

The term $\boldsymbol{C}\dot{\boldsymbol{U}}$ will be discussed later. Our methods for solving (1) are:

- Mode superposition: We first transform the equation and then integrate.
- Direct integration: We integrate the equation directly!

First, let's transform Eq. (1). Assume we use

$$\boldsymbol{U}(t) = \underset{n\times n}{\boldsymbol{P}} \; \underset{n\times 1}{\boldsymbol{X}}(t) \tag{2}$$

The function $\boldsymbol{P}$ is independent of time. Substitute this into Eq.(1) to obtain

$$\boldsymbol{P}^T\boldsymbol{M}\boldsymbol{P}\ddot{\boldsymbol{X}} + \boldsymbol{P}^T\boldsymbol{C}\boldsymbol{P}\dot{\boldsymbol{X}} + \boldsymbol{P}^T\boldsymbol{K}\boldsymbol{P}\boldsymbol{X} = \boldsymbol{P}^T\boldsymbol{R} \tag{A}$$

The best $\boldsymbol{P}$ matrix would diagonalize the matrix, thereby decoupling the equations. To obtain a "wonderful" $\boldsymbol{P}$, consider

$$\boldsymbol{M}\ddot{\boldsymbol{U}} + \boldsymbol{K}\boldsymbol{U} = \boldsymbol{0} \quad \text{(free vibration)}$$

$$\boldsymbol{U} = \boldsymbol{\phi}\sin\omega\,(t - t_0)$$

Then,

$$-\omega^2\boldsymbol{M}\boldsymbol{\phi}\sin\omega\,(t-t_0) + \boldsymbol{K}\boldsymbol{\phi}\sin\omega\,(t-t_0) = \boldsymbol{0} \tag{a}$$

For (a) to hold,

$$\boldsymbol{K}\boldsymbol{\phi} = \omega^2\boldsymbol{M}\boldsymbol{\phi}$$

$$\left(\boldsymbol{K} - \omega^2\boldsymbol{M}\right)\boldsymbol{\phi} = \boldsymbol{0}$$

Let $\omega^2 = \lambda$. We have a generalized eigenvalue problem. We must have $\det\left(\boldsymbol{K} - \lambda\boldsymbol{M}\right) = 0$, and we find the solution for λ from the roots of the characteristic polynomial

$$p(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \ldots + a_n\lambda^n$$

Find the eigenvalues λ_1, λ_2, …, λ_n from $p(\lambda) = 0$ and then the eigenvectors $\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_n$ from

$$\left(\boldsymbol{K} - \lambda_i\boldsymbol{M}\right)\boldsymbol{\phi}_i = \boldsymbol{0}$$

Then, normalize $\boldsymbol{\phi}_i$ so that it satisfies $\boldsymbol{\phi}_i^T\boldsymbol{M}\boldsymbol{\phi}_i = 1$. We now have (see Chapters 2, 10)

$$0 \le \underbrace{\omega_1^2}_{\text{for } \boldsymbol{\phi}_1} \le \underbrace{\omega_2^2}_{\text{for } \boldsymbol{\phi}_2} \le \ldots \le \underbrace{\omega_n^2}_{\text{for } \boldsymbol{\phi}_n}$$

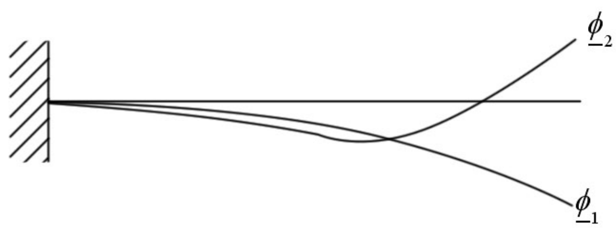
Each $\boldsymbol{\phi}_i$ represents a mode shape, and we have

$$\boldsymbol{\phi}_i^T \boldsymbol{M} \boldsymbol{\phi}_j = \delta_{ij}$$

where δ_{ij} is the Kronecker delta, so we call $\boldsymbol{\phi}_i$ $\boldsymbol{M}$-orthogonal (or $\boldsymbol{M}$-orthonormal, because $\boldsymbol{\phi}_i^T \boldsymbol{M} \boldsymbol{\phi}_i = 1$). In turn, this yields

$$\boldsymbol{\phi}_i^T \boldsymbol{K} \boldsymbol{\phi}_j = \omega_i^2 \delta_{ij}$$

Physically,



Consider $\boldsymbol{\phi}_1$:

$$\boldsymbol{\phi}_1^T \boldsymbol{M} \boldsymbol{\phi}_1 = 1$$

$$\boldsymbol{\phi}_1^T \boldsymbol{K} \boldsymbol{\phi}_1 = \omega_1^2$$

The strain energy in the beam is $\frac{1}{2}\boldsymbol{\phi}_1^T \boldsymbol{K} \boldsymbol{\phi}_1 = \frac{1}{2}\omega_1^2$. By orthonormality, also,

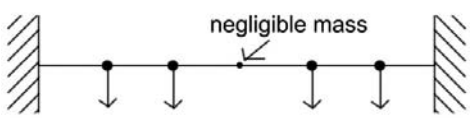
$$\boldsymbol{\phi}_2^T \boldsymbol{M} \boldsymbol{\phi}_1 = 0$$

$$\boldsymbol{\phi}_2^T \boldsymbol{M} \boldsymbol{\phi}_2 = 1$$

and

$$\boldsymbol{\phi}_2^T \boldsymbol{K} \boldsymbol{\phi}_2 = \omega_2^2$$

Consider this simple case, for which we must solve $\boldsymbol{K\phi} = \omega^2 \boldsymbol{M\phi}$:



$$\boldsymbol{M} = \begin{bmatrix} \times & & & & \\ & \times & & & \\ & & 0 & & \\ & & & \times & \\ & & & & \times \end{bmatrix}$$

Then

$$\boldsymbol{M\phi} = \frac{1}{\omega^2}\boldsymbol{K\phi} = \kappa \boldsymbol{K\phi}$$

A non-trivial solution is $\kappa = 0$, $\boldsymbol{\phi} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow \omega^2 = \infty$.

Note: $\omega_1^2 = 0$ for rigid body motion. (No strain energy!)

Now let's use $\boldsymbol{P} = [\,\boldsymbol{\phi}_1 \quad \ldots \quad \boldsymbol{\phi}_n\,]$. Then, (A) becomes

$$\ddot{\boldsymbol{X}} + \boldsymbol{P}^T\boldsymbol{C}\boldsymbol{P}\dot{\boldsymbol{X}} + \begin{bmatrix} \omega_1^2 & & \text{zeros} \\ & \ddots & \\ \text{zeros} & & \omega_n^2 \end{bmatrix} \boldsymbol{X} = \boldsymbol{P}^T\boldsymbol{R}$$

For now, let's assume no damping. (If $\boldsymbol{C} = 0$, there is no damping and the equations are decoupled.) Then, we have

$$\ddot{\boldsymbol{X}} + \boldsymbol{\Omega}^2\boldsymbol{X} = \underset{n\times n}{\boldsymbol{\Phi}}{}^T\boldsymbol{R}$$

$$\boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{\phi}_1 & \boldsymbol{\phi}_2 & \ldots & \boldsymbol{\phi}_n \end{bmatrix} \quad ; \quad \boldsymbol{\Omega}^2 = \begin{bmatrix} \omega_1^2 & & \text{zeros} \\ & \ddots & \\ \text{zeros} & & \omega_n^2 \end{bmatrix} \quad ; \quad \boldsymbol{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

So, we have

$$\ddot{x}_i + \omega_i^2 x_i = \boldsymbol{\phi}_i^T\boldsymbol{R} \quad (i = 1, \ldots, n)$$

As always, we need the initial conditions ${}^0x_i$, ${}^0\dot{x}_i$ to solve.

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