2.087 Differential Equations and Linear Algebra, Fall 2014

Homework #5

Date Issued: Wednesday 8 October, 2014

Date Due: Wednesday 15 October, 2014, 9:30AM (bring hard copy to lecture)

As described in the course policies document, this is one of 5 homeworks you will complete in this course. Each of these count as 6% of your total grade. Full credit can generally only be earned by showing your work. This often includes making clear and well-labeled plots.

1) (20 points)

Two salt water storage tanks are connected to each other so that water is pumped from Tank 1 to Tank 2 with flow rate r along one pipe, and water is pumped back from Tank 2 to Tank 1 also with flow rate r along another pipe. The amounts of salt $x_1(t)$ and $x_2(t)$ in the two tanks therefore satisfy the differential equations:

$$\frac{dx_1}{dt} = -k_1x_1 + k_2x_2,$$

$$\frac{dx_2}{dt} = k_1x_1 - k_2x_2,$$

where $k_i = r/V_i$. If the flow rate r=10 liters/sec and the volumes of the two tanks are V_1=50 liters and V_2=25 liters, then:

a. Solve for the volume of salt in each tank as a function of time for the initial conditions: $x_1(0)$=15 liters and $x_2(0)$=0 liters.

b. Determine the final amounts of salt in each of the two tanks.

2) (20 points) (20 points) Confirm that the equation $\begin{bmatrix} 1 & -1 & 2 \\ -1 & 3 & -4 \\ -1 & 5 & -6 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 2 \\ -3 \\ -4 \end{bmatrix}$

is satisfied by the whole family of solutions $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 1.5 \end{bmatrix} + \alpha \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$. Further, confirm that $\mathbf{y} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

is the null space of the matrix, that is, confirm that $\begin{bmatrix} 1 & -1 & 2 \\ -1 & 3 & -4 \\ -1 & 5 & -6 \end{bmatrix}\mathbf{y} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

3) (20 points) Find the complete solution to these ODEs assuming intial conditions $y(0)=y'(0)=0$

a. $y'' + 16y = e^{3x}$

b. $y'' - y' - 6y = 2\sin(3x)$

c. $y'' - y' - 2y = 3x + 4$

4) (20 points)

Consider a forced mechanical system governed by the following equation (this is typical of unbalanced rotating devices such as an unbalanced flywheel):

$$mx'' + \gamma x' + kx = mA\omega^2 \cos \omega t$$

where m is the mass, $x(t)$ is the displacement, k is the spring constant, A is a measure of the imbalance of the system, and ω is the frequency of forcing.

a. Find the general solution to this equation.

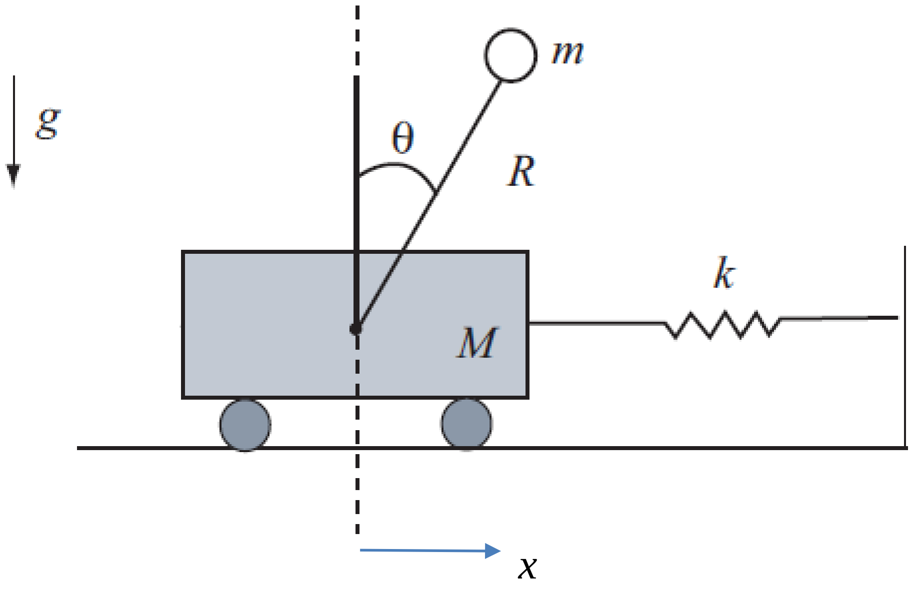
b. Show that the amplitude of oscillation is $\rho mA/k$, where $\rho = k\omega^2 \left[(k - m\omega^2)^2 + (\gamma\omega)^2\right]^{-1/2}$

c. If there is no damping in the system, what is the resonant frequency?

d. Show that with damping, the maximum amplitude occurs at the frequency $\omega_m^2 = \frac{k}{m}\left(\frac{2mk}{2mk-\gamma^2}\right)$ (hint: to save excessive algebra, define $\alpha = \omega^2$ and seek to maximize ρ^2 with respect to α; and you can assume $\gamma^2 < 2mk$)

e. How does the damped resonant frequency compare to the undamped resonant frequency?

5) (20 points) For the system depicted below



a) (5 points) Write the equations of motion including state variables x, Θ, and any derivatives of those as appropriate.

b) (5 points) Consider the case k=100N/m, m=1g, M=10kg, g=9.8m/s^2, and R=1m. Given initial conditions x=1cm, Θ=15deg, carry out five steps of forward Euler solution of the equations of motion with step size Δt=0.1 sec.

c) (5 points) Write the linearized equations of motion about the equilibrium configuration x=0, Θ=0 in the form $\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \mathbf{A} \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix}$

d) (5 pts) Determine the eigenvalues of **A** and comment on the implications for stability of the system.

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