# Unit III

## Linear Algebra I
## and
## Statistical Regression



# Motivation



## 2.086 Roadmap:

(Numerical) Calculus
Discretization: (finite) $h$
Unit I

(Numerical) Probability and Statistics
Estimation from "Data"
Unit II

Unit IV

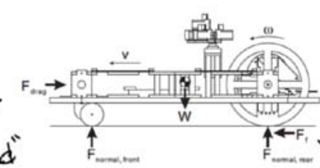
Unit III

(Numerical) Linear Algebra
Solution
Unit V, (VI)
Unit VII (nonlinear)

MATLAB: Implementation



## Friction Coefficient, $\mu_s$: Role (static)




or "load"

$$F_{f,\text{static}} \leq \mu_s \underbrace{F_{\text{normal,rear}}}_{F_{f,\text{static}}^{\max}}$$

## Measurement of $F_{f,\text{static}}^{\text{max}}$ : $F_{f,\text{static}}^{\text{max,meas}}$

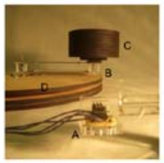



Figure 2: Experimental setup for friction measurement: Force transducer (A) is connected to contact area (B) by a thin wire. Normal force is exerted on the contact area by load stack (C). Tangential force is applied using turntable (D) via the friction between the turntable surface and the contact area. Apparatus and photograph courtesy of James Penn.

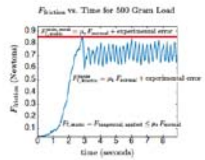



Figure 3: Sample data for one friction measurement, yielding one data point for $F_{f,\text{static}}^{\text{max,meas}}$. Data courtesy of James Penn.



## Estimation of $\mu_s$

model: hypothesis

$$F_{f,\text{static}}^{\text{max}}\big(F_{\text{normal,applied}},\ \overset{(\text{nominal})}{A_{\text{surface}}};\ \beta^{\text{true}}\big)$$

$$= \underset{0\,?}{\beta_0^{\text{true}}} + \underset{\mu_s\,?}{\beta_1^{\text{true}}}\, F_{\text{normal,applied}} + \underset{0\,?}{\beta_2^{\text{true}}}\, A_{\text{surface}}$$

measurements: $1 \le i \le m$

$$F_{f,\text{static}\,i}^{\text{max,meas}} = \beta_0^{\text{true}} + \beta_1^{\text{true}}\, F_{\text{normal,applied}\,i} + \beta_2^{\text{true}}\, A_{\text{surface}\,i} + \epsilon_i$$

noise



Questions:

What assumptions can we make on the noise?
(Can we verify these assumptions?)

How can we estimate $\beta_0^{\text{true}}, \beta_1^{\text{true}}, \beta_2^{\text{true}}$ from $F_{f,\text{static}\,i=1,\dots m}^{\text{max,meas}}$?

Can we develop confidence intervals?

How will $m$ affect our estimates?

(How can we best choose $F_{\text{normal,applied}\,i}$, $A_{\text{surface}\,i}$?)

(What if our model is biased – there is no $\beta^{\text{true}}$?)

How can we implement efficiently in MATLAB?





Linear Algebra Ia:
Basic Operations

# Matrices

$A \in \mathbb{R}^{m \times n}$ $\quad$ $m \times n$ matrix $\quad$ $m$ rows, $n$ columns

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & & & \\ \vdots & & \ddots & \\ A_{m1} & & & A_{mn} \end{pmatrix} \begin{array}{l} \leftarrow \text{row 1} \\ \leftarrow \text{row 2} \\ \\ \leftarrow \text{row } m \end{array}$$

$\uparrow$ col 1 $\quad$ $\uparrow$ col $n$

$A_{ij}$ row, col

$1 \le i \le m$

$1 \le j \le n$

SQUARE: $m = n$ $(n \times n)$

Special case: vector

row vector: $1 \times n$ matrix, $v \in \mathbb{R}^{1 \times n}$

→ col vector: $m \times 1$ matrix, $w \in \mathbb{R}^{m \times 1}$ $\quad \mathbb{R}^m$



the transpose operator: $^T$ $\qquad$ Matlab $'$

$$\underset{n \times m}{(A^T)_{ji}} = \underset{m \times n}{A_{ij}} \qquad 1 \le i \le m, \quad 1 \le j \le n$$

switch rows and columns

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

$$v = \begin{pmatrix} 0 & 1 & \tfrac{1}{2} \end{pmatrix}, \quad v^T = \begin{pmatrix} 0 \\ 1 \\ \tfrac{1}{2} \end{pmatrix}$$

row vector$^T$ = col vector

Note: $(A^T)^T = A$.



# Vector Operations

$v, w$ same size: col $m$-vector or row $n$-vector

1) Addition (say col $m$-vector)

$$\underset{\substack{\uparrow \\ \text{col } m\text{-vector}}}{u} = v + \underset{\substack{\uparrow \\ \text{real number}}}{\alpha} w$$

$$u_i = v_i + \alpha w_i, \quad 1 \le i \le m$$

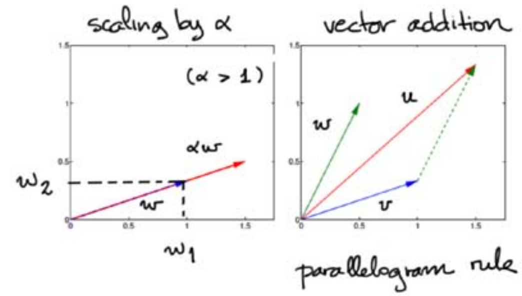
$\uparrow$ $\underbrace{\text{scale all elements of } w \text{ by } \alpha \Rightarrow \alpha w}$

$u \Leftarrow \underbrace{\text{add corresponding elements of } v \text{ and } \alpha w}$

Matlab: `u = v + α*w` "O-O"



"picture" in $\mathbb{R}^2$




Cartesian representation

## 2) Inner Product (dot product)

$$\beta \; (= v^T w) = \sum_{i=1}^{m} v_i w_i \qquad \text{Matlab: } v' * w$$

$\Rightarrow$ Norm (2-norm, Euclidean norm, ...)

$$\|v\| = \sqrt{v^T v} = \left( \sum_{i=1}^{m} v_i^2 \right)^{1/2} \qquad \text{Matlab: norm}$$

note in $\mathbb{R}^2$, $\|v\| = \underbrace{(v_1^2 + v_2^2)^{1/2}}_{\text{usual notion of length}}$

(NOT Matlab "length")



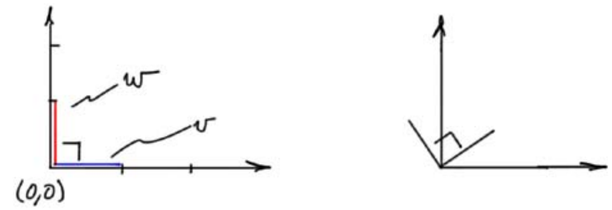
orthogonal vectors: $v^T w = 0$

orthonormal vectors: $v^T w = 0$ AND $\|v\| = \|w\| = 1$

in $\mathbb{R}^2$: $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $w = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$v^T w = \sum_{i=1}^{2} v_i w_i = 0 \; ; \quad \|v\| = (1^2 + 0^2)^{1/2} = 1 \; , \quad \|w\| = 1$$

$\Rightarrow$ $v$ and $w$ are orthonormal vectors






## Matrix Operations

### 1) Addition (and scaling)

$A$ $m \times n$, $B$ $m \times n$ (same size) $\qquad \alpha$ scalar

$$C = A + \alpha B$$

$$C_{ij} = A_{ij} + \alpha B_{ij} \; , \quad 1 \le i \le m, \; 1 \le j \le n$$

element by element addition

Matlab: $C = A + \alpha * B$



### 2) Multiplication

$A$ $m_1 \times n_1$ , $B$ $m_2 \times n_2$

$$\boxed{\begin{array}{c}\text{REQUIREMENT} \\ n_1 = m_2\end{array}}$$

$$\overset{m_1 \times n_2}{C} = \overset{m_1 \times n_1}{A} \; \overset{m_2 \times n_2}{B}$$

match

$$C_{ij} = \sum_{k=1}^{n_1} A_{ik} B_{kj} \; , \quad 1 \le i \le m_1 \, , \; 1 \le j \le n_2$$

Matlab: $C = A * B$ "O-O" $\qquad$ NOT $A .* B$

requires same size

a) inner product (revisited)

$v$ $m \times 1$ , $w$ $m \times 1$

$$\underset{1\times 1}{\beta} = \underset{1\times m}{v^T} \underset{m\times 1}{w}$$

"A" "B"

Matlab: $\beta = v' * w$ NOT $v * w$

$= \text{sum}(v .* w)$

$$\beta = (v_1 \; v_2 \; \ldots \; v_m) \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{pmatrix}$$

$2m$ FLOPs

"2-handed"

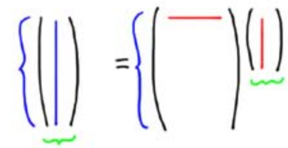
$$\beta = \sum_{i=1}^{m} v_i w_i$$



b) matrix-vector product

$A$ $m \times n$, $w$ $n \times 1$

$$\underset{m\times 1}{v} = \underset{m\times n}{A} \; \underset{n\times 1}{w}$$



$$v_{i1} = \sum_{k=1}^{n} A_{ik} w_{k1} \quad 1 \le i \le m$$

Matlab: $v = A * w$



"done out"

$$v_1 = A_{11} w_1 + A_{12} w_2 + \cdots A_{1n} w_n \quad 2n \text{ FLOPs}$$

$$v_2 = A_{21} w_1 + A_{22} w_2 + \cdots A_{2n} w_n \quad 2n \text{ FLOPs}$$

$$\vdots \qquad\qquad \vdots$$

$$v_m = A_{m1} w_1 + A_{m2} w_2 + \cdots A_{mn} w_n \quad 2n \text{ FLOPs}$$

$2mn$ FLOPs



(i) row perspective : "2-handed"

$$v_1 = (\text{row 1 of } A) \begin{pmatrix} \\ w \\ \\ \end{pmatrix}$$ inner product (of $n$ vectors)

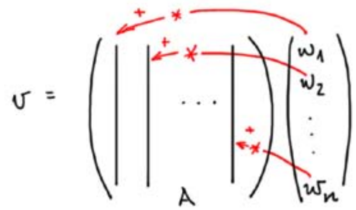
$$= (A_{11} \; A_{12} \cdots A_{1n}) \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = A_{11} w_1 + A_{12} w_2 + \cdots A_{1n} w_n$$

$$\boxed{\begin{array}{c} v_i = (\text{row } i \text{ of } A) \begin{pmatrix} | \\ w \\ | \end{pmatrix}, \; 1 \le i \le m \\ \Rightarrow Aw \text{ calculated as } m \text{ inner products of } n \text{ vectors} \end{array}}$$

(ii) column perspective: "1-handed"

$$v = \begin{pmatrix} A_{11} \\ A_{21} \\ \vdots \\ A_{m1} \end{pmatrix} \underbrace{w_1}_{\text{scaling}} + \begin{pmatrix} A_{12} \\ A_{22} \\ \vdots \\ A_{m2} \end{pmatrix} w_2 + \cdots \begin{pmatrix} A_{1n} \\ A_{2n} \\ \vdots \\ A_{mn} \end{pmatrix} w_n$$

col 1 of A  col 2 of A  . . .  col $n$ of A




$\boxed{\text{Aw calculated as linear combination (coefficients } w_i, 1 \le i \le n) \text{ of } n \text{ column } m\text{-vectors (columns of A)}}$



example:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \qquad w = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

2-handed: $v_1 = (1\ 2\ 3)\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 5$   $v = \begin{pmatrix} 5 \\ 11 \end{pmatrix}$

$v_2 = (4\ 5\ 6)\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 11$

1-handed:

$$v = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \cdot 0 + \begin{pmatrix} 2 \\ 5 \end{pmatrix} \cdot 1 + \begin{pmatrix} 3 \\ 6 \end{pmatrix} \cdot 1$$

$$= \begin{pmatrix} 2+3 \\ 5+6 \end{pmatrix} = \begin{pmatrix} 5 \\ 11 \end{pmatrix}$$



(c) matrix-matrix product   $\underset{m_1 \times n_2}{C} = \underset{m_1 \times n_1}{A}\ \underset{m_2 \times n_2}{B}$

(i) 2-handed

$$C_{ij} = \underset{m_1 \text{ rows}}{(\text{row } i \text{ of } A)} \underset{\substack{\text{col } j \text{ of } B \\ n_2 \text{ columns}}}{\begin{pmatrix} \\ \\ \end{pmatrix}}$$

$\boxed{m_1 n_2 \text{ inner products of } n_1\text{-vectors}}$

$2 m_1 n_2 n_1$ FLOPs

(ii) 1-handed

$$\text{col } j \text{ of } C = \overset{m_1 \times n_1}{A} \underset{\substack{\text{col } j \text{ of } B \\ n_2 \text{ columns}}}{\begin{pmatrix} \\ \\ \end{pmatrix}}$$

$\boxed{n_2 \quad \underset{m_1 \times n_1}{A}\ \underset{n_1 \times 1}{w} \text{ matrix-vector products}}$

$2 n_2 m_1 n_1$ FLOPs



## Matrix Rules

(for appropriate "sizes")

$$A + B = B + A$$
$$(A + B) C = AC + BC$$
$$ABC = A(BC) = (AB)C$$
$$(ABC)^T = C^T B^T A^T$$

all very useful.

But note (even if both defined, even if square)

$$AB \neq BA \text{ in general.}$$

example: AB vs BA (when both defined)

$$A = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$AB = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 1$$

$$BA = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$



## Some Special (Types of) Matrices

square $n \times n$

• the identity matrix $I$ matrix "1"

$$I = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \qquad \begin{array}{l} I_{ij} = 0 \text{ unless } i = j \\ I_{ii} = 1, \quad 1 \le i \le n \end{array}$$

$$I w = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots \\ & & \ddots & \\ 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = w$$

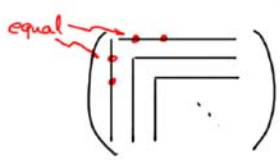


• a matrix is diagonal if

$$A = \begin{pmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{pmatrix} \qquad \begin{array}{l} A_{ij} = 0 \text{ unless } i = j \\ A_{ii} = d_i, \quad 1 \le i \le n \end{array}$$

• a matrix is symmetric if square

$$A = A^T \qquad A_{ij} = A_{ji}, \quad \begin{array}{l} 1 \le i \le n \\ 1 \le j \le n \end{array}$$






## Inverse of a Matrix (preliminary encounter)

$A$ $n \times n$

Scalar case: $a \neq 0$

the inverse of $a = \frac{1}{a} = a^{-1}$ such that $a^{-1} a = \frac{a}{a} = 1$

matrix case: $A$ invertible (non-singular)

the inverse of $\underset{n \times n}{A}$ is $\underset{n \times n}{A^{-1}}$ such that $\underset{n \times n}{A^{-1}} \underset{n \times n}{A} = \underset{n \times n}{I}$

note $A^{-1} A = A A^{-1} = I$

$Aw = v$ given $w \to v$ (forward): matrix-vector product

$Aw = v$ given $v \to w$ (inverse)

$$\underbrace{A^{-1}(Aw)}_{Iw} = A^{-1} v \Rightarrow w = A^{-1} v$$

BUT NOT the GOOD COMPUTATIONAL APPROACH (usually)

7

example: $n=2$ ($2\times 2$ matrix $A$)

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$A$ non-singular if
$ad - bc \neq 0$

since

$$\underbrace{\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}_{A^{-1}} \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{A} = \frac{1}{ad-bc}\begin{pmatrix} da-bc & db-bd \\ -ca+ac & -cb+ad \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{I}$$

2.086 Numerical Computation for Mechanical Engineers
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