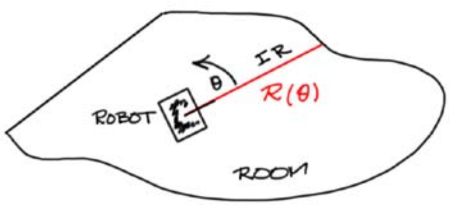
Differentiation

Motivation

Robotic Perimeter Estimation



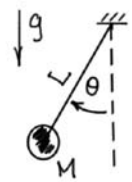
Perimeter = $\int_0^{2\pi} \left(\left(\frac{dR}{d\theta} \right)^2 + R^2 \right)^{1/2} d\theta$

~~direct differentiation of "noisy" data~~

$\Downarrow$

"average": filtering; regression; ...

Nonlinear Dynamical Systems: Pendulum



$$\frac{d\theta}{dt} = \omega, \qquad \theta(0) = \theta_0$$

fit to data

$$\underbrace{\frac{d\omega}{dt} = -\frac{g}{L}\sin\theta - b\omega - c|\omega|\omega, \qquad \omega(0) = 0}_{(\theta(t),\,\omega(t))}$$

$\theta_0 \ll 1$: $(\theta_{LIN}(t), \omega_{LIN}(t))$

$\theta_0 \not\ll 1$: $(\tilde{\theta}_{\Delta t}(t), \tilde{\omega}_{\Delta t}(t))$:

discretization

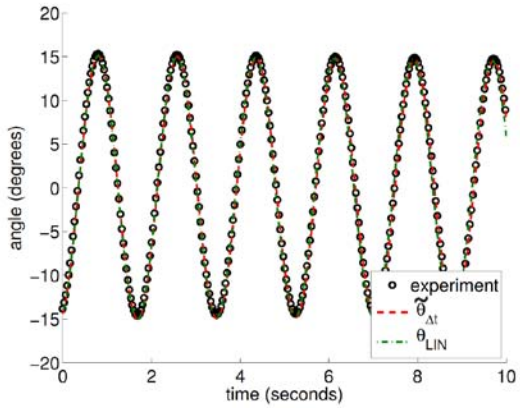
$$\frac{d\theta}{dt} \approx \frac{\tilde{\theta}(t+\Delta t) - \tilde{\theta}(t)}{\Delta t}$$

$$\frac{d\omega}{dt} \approx \frac{\tilde{\omega}(t+\Delta t) - \tilde{\omega}(t)}{\Delta t}$$

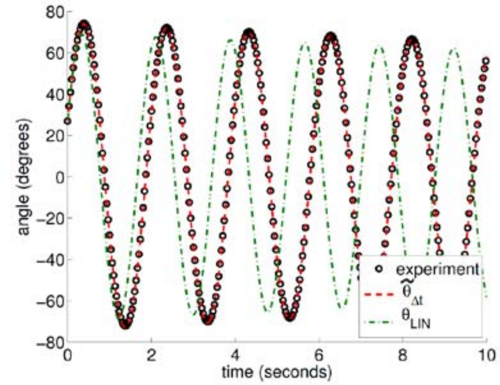
differential → algebraic

small θ_0 :

Yano, Penn

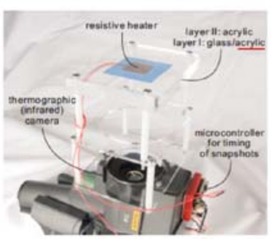


θ_0 large :



"Material Vision"

Penn, Tonn



$$\rho c \frac{\partial T}{\partial t} = k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right)$$

$$q_{in} = k \frac{\partial T}{\partial z} \quad \text{on patch}$$

$$T = T_{amb} \quad \text{at } t = 0$$

$$T(t, x, y, z; k, \rho c)$$

$$\Downarrow$$

$$\tilde{T}_{\Delta t, \Delta x, \Delta y, \Delta z}(t, x, y, z; k, \rho c)$$

$$\frac{\partial}{\partial t} \approx \left(\frac{\partial}{\partial t} \right)_{\Delta t}, \quad \frac{\partial^2}{\partial x^2} \approx \left(\frac{\partial^2}{\partial x^2} \right)_{\Delta x}, \ldots$$

experiment

model

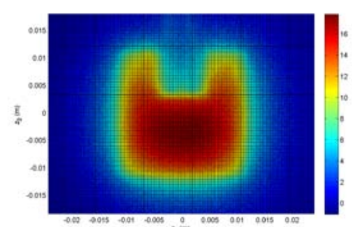
$$T_{exp}(t, x, y, z; k^*, \rho c^*) \rightarrow \text{COMPARE} \leftarrow \tilde{T}_{\Delta t, \Delta x, \Delta y, \Delta z}(t, x, y, z; k, \rho c)$$

$$\Downarrow$$

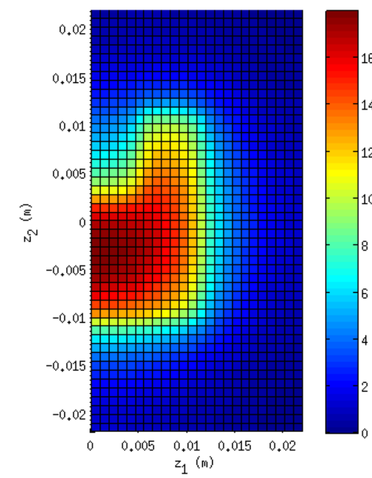
$$(k_{est}, \rho c_{est})$$

$$\Downarrow$$

material type



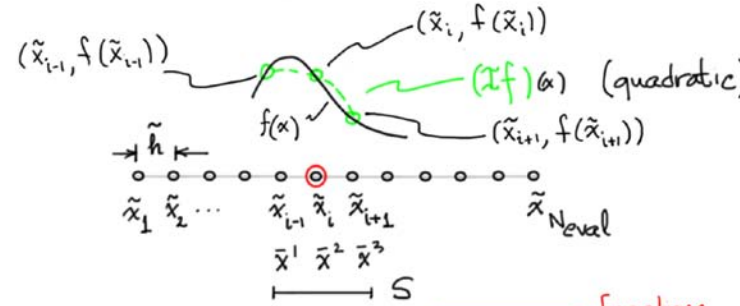
experiment
$(k^*, \rho c^*)$



model
$(k_{est}, \rho c_{est})$

from Interpolation to Differentiation

Given $f(\tilde{x}_i)$, $1 \le i \le N_{eval}$ — values at points:



function

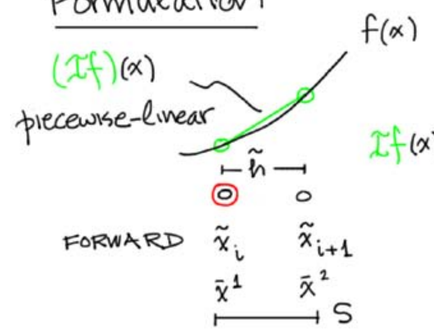
$$f'(\tilde{x}_i) \approx f'_{\tilde{h}}(\tilde{x}_i) = (\mathcal{I}f)'(\tilde{x}_i)$$

$$f''(\tilde{x}_i) \approx f''_{\tilde{h}}(\tilde{x}_i) = (\mathcal{I}f)''(\tilde{x}_i)$$

$$\vdots$$

First Derivative: Forward Difference

Formulation



$$\mathcal{I}f(x) = f(\tilde{x}_i) + \frac{f(\tilde{x}_{i+1}) - f(\tilde{x}_i)}{\tilde{h}}(x - \tilde{x}_i)$$

$$\Downarrow$$

$$(\mathcal{I}f)'(x) = \frac{f(\tilde{x}_{i+1}) - f(\tilde{x}_i)}{\tilde{h}}$$

$$f'_{\tilde{h}}(\tilde{x}_i) = (\mathcal{I}f)'(\tilde{x}_i) = \frac{f(\tilde{x}_{i+1}) - f(\tilde{x}_i)}{\tilde{h}}$$

FINITE DIFFERENCE

Error Analysis

$$\tilde{x}_{i+1} = \tilde{x}_i + \tilde{h}$$

$$f(\tilde{x}_{i+1}) = f(\tilde{x}_i + \tilde{h})$$

$$= f(\tilde{x}_i) + f'(\tilde{x}_i)\tilde{h} + \frac{1}{2} f''(\xi)\tilde{h}^2 \qquad \text{Taylor series}$$

$$\tilde{x}_i \le \xi \le \tilde{x}_{i+1}$$

$$\Downarrow$$

$$\Big| f'(\tilde{x}_i) - \Big(\overbrace{\frac{f(\tilde{x}_{i+1}) - f(\tilde{x}_i)}{\tilde{h}}}^{f'_{\tilde{h}}(\tilde{x}_i)} \Big) \Big|$$

$$= \Big| f'(\tilde{x}_i) - \Big(\frac{\big[f(\tilde{x}_i) + f'(\tilde{x}_i)\tilde{h} + \frac{1}{2} f''(\xi)\tilde{h}^2 \big] - f(\tilde{x}_i)}{\tilde{h}} \Big) \Big|$$

$$= \Big| \frac{1}{2} f''(\xi)\tilde{h} \Big|$$

$$\le \underbrace{\frac{1}{2} \max_{\tilde{x}_i \le x \le \tilde{x}_{i+1}} | f''(x) | \tilde{h}}$$

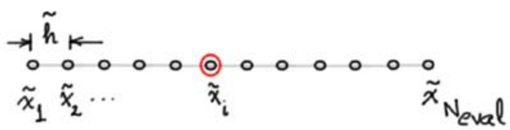
$$\Updownarrow$$

exact for $f(x)$ linear

$$\le C\tilde{h}^1 \qquad p = 1: \text{ first order} \quad \text{(vs. interpolant error)}$$

DEMO

Operation Count



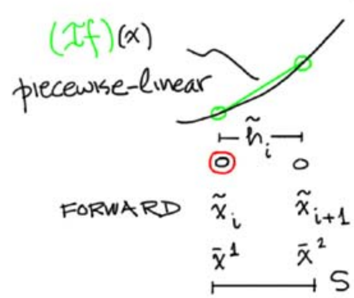
$$f'_{\tilde{h}}(\tilde{x}_i) = (\mathcal{I}f)'(\tilde{x}_i) = \frac{f(\tilde{x}_{i+1}) - f(\tilde{x}_i)}{\tilde{h}}, \quad 1 \le i \le N_{eval} - 1$$

(i) evaluate $f(\tilde{x}_i)$, $1 \le i \le N_{eval}$

(ii) calculate finite difference: $2\,N_{eval}$ FLOPs

Non-Uniform Grid:

mesh



$$\mathcal{I}f(x) = f(\tilde{x}_i) + \frac{f(\tilde{x}_{i+1}) - f(\tilde{x}_i)}{\tilde{h}_i}(x - \tilde{x}_i)$$

$$\Downarrow$$

$$(\mathcal{I}f)'(x) = \frac{f(\tilde{x}_{i+1}) - f(\tilde{x}_i)}{\tilde{h}_i}$$

$$f'_{\tilde{h}}(\tilde{x}_i) = (\mathcal{I}f)'(\tilde{x}_i) = \frac{f(\tilde{x}_{i+1}) - f(\tilde{x}_i)}{\tilde{h}_i}$$

and

$$| f'(\tilde{x}_i) - f'_{\tilde{h}}(\tilde{x}_i) | \le \frac{1}{2} \max | f'' | \cdot \tilde{h}_i^1$$

Some First-Derivative Finite Difference Formulas

$\mathcal{I}$	"stencil"	formula	order	
linear	$\tilde{x}_i \overset{\tilde{h}}{\quad} \tilde{x}_{i+1}$	$\dfrac{(f(\tilde{x}_{i+1}) - f(\tilde{x}_i))}{\tilde{h}}$	$p = 1$	FORWARD DIFFERENCE
linear	$\tilde{x}_{i-1} \overset{\tilde{h}}{\quad} \tilde{x}_i$	$\dfrac{(f(\tilde{x}_i) - f(\tilde{x}_{i-1}))}{\tilde{h}}$	$p = 1$	BACKWARD DIFFERENCE
quadratic	$\tilde{x}_{i-1} \overset{\tilde{h}}{\quad} \tilde{x}_i \overset{\tilde{h}}{\quad} \tilde{x}_{i+1}$	$\dfrac{(f(\tilde{x}_{i+1}) - f(\tilde{x}_{i-1}))}{2\tilde{h}}$ uniform mesh	$p = 2$	CENTERED DIFFERENCE

A Second Derivative Finite Difference Formula

(CENTERED)

$\mathcal{I}$	stencil	formula: $f''_{\tilde{h}}$	order
quadratic	$\tilde{x}_{i-1} \overset{\tilde{h}}{\quad} \tilde{x}_i \overset{\tilde{h}}{\quad} \tilde{x}_{i+1}$	$\dfrac{f(\tilde{x}_{i-1}) - 2f(\tilde{x}_i) + f(\tilde{x}_{i+1})}{\tilde{h}^2}$	$p = 2$

---- uniform mesh ----

Noisy Functions

Say we wish to approximate

$$f'(x)$$

but we only have access to

$$g(x) = f(x) + \underbrace{\epsilon}_{\text{amplitude}} \sin \overbrace{k}^{\text{wavelength } 2\pi/k} x$$

Then

$$g'_{\tilde{h}}(\tilde{x}_i) \to g'(\tilde{x}_i) = f'(\tilde{x}_i) + \underbrace{\epsilon k}\cos kx.$$

Say $\epsilon = .01$, $k = 100$

$$\neq f'(\tilde{x}_i)$$

large error