Unit V Quiz Question Sampler **2.086/2.090 Fall 2012**

You may refer to the text and other class materials as well as your own notes and scripts.

For Quiz 4 (on Unit V) you will not *need* a calculator; however, if you like, you may use a calculator to confirm arithmetic results. In any event, laptops, tablets, and smartphones are *not* permitted.

To simulate real quiz conditions you should complete all the questions in this sampler in 90 minutes.

NAME ______________________________________

There are a total of 100 points: four questions, each worth 25 points.

All questions are multiple choice; in all cases circle one and only one answer.

We include a blank page at the end which you may use for any derivations, but note that we do not refer to your work and in any event on the quiz *your grade will be determined solely by your multiple choice selections*.

You may assume that all arithmetic operations are performed exactly (with *no* floating point truncation or round-off errors). We may display the numbers in the multiple-choice options to just a few digits, however you should always be able to clearly discriminate the correct answer from the incorrect answers.

Question 1 (25 points).

We consider in this problem the system of linear equations

$$Au = f \tag{1}$$

where A is a given 2×2 matrix, f is a given 2×1 vector, and u is the 2×1 vector we wish to find.

We introduce two matrices

$$A^{\mathrm{I}} = \begin{pmatrix} 1 & -2 \\ 3 & -6 \end{pmatrix}, \tag{2}$$

and

$$A^{\mathrm{II}} = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}, \tag{3}$$

which will be relevant in Parts (i),(ii) and Parts (iii),(iv) respectively.

In Parts (i),(ii), A of equation (1) is given by A^{I} of equation (2). (In other words, we consider the system $A^{\mathrm{I}}u = f$.)

(i) (6.25 points) For $f = (1/2 \quad 3/2)^{\mathrm{T}}$ (recall $^{\mathrm{T}}$ denotes transpose),

(a) $A^{\mathrm{I}}u = f$ has *a unique solution*

(b) $A^{\mathrm{I}}u = f$ has *no solution*

(c) $A^{\mathrm{I}}u = f$ has an infinity of solutions of the form

$$u = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

for any (real number) α

(d) $A^{\mathrm{I}}u = f$ has an infinity of solutions of the form

$$u = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

for any (real number) α.

(*ii*) (6.25 points) For $f = (1/2 \quad 1)^{\mathrm{T}}$,

(*a*) $A^{\mathrm{I}} u = f$ has *a unique solution*

(*b*) $A^{\mathrm{I}} u = f$ has *no solution*

(*c*) $A^{\mathrm{I}} u = f$ has an infinity of solutions of the form

$$u = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

for any (real number) α

(*d*) $A^{\mathrm{I}} u = f$ has an infinity of solutions of the form

$$u = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

for any (real number) α.

Now, in Parts (*iii*), (*iv*), A of equation (1) is given by A^{II} of equation (3). (In other words, we consider the system $A^{\mathrm{II}} u = f$.)

(*iii*) (6.25 points) For $f = (1/2 \quad 3/2)^{\mathrm{T}}$ (recall $^{\mathrm{T}}$ denotes transpose),

(*a*) $A^{\mathrm{II}} u = f$ has *a unique solution*

(*b*) $A^{\mathrm{II}} u = f$ has *no solution*

(*c*) $A^{\mathrm{II}} u = f$ has an infinity of solutions of the form

$$u = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

for any (real number) α

(*d*) $A^{\mathrm{II}} u = f$ has an infinity of solutions of the form

$$u = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

for any (real number) α.

(iv) (6.25 points) For $f = (1/2 \quad 1)^{\mathrm{T}}$,

(a) $A^{\mathrm{II}} u = f$ has *a unique solution*

(b) $A^{\mathrm{II}} u = f$ has *no solution*

(c) $A^{\mathrm{II}} u = f$ has an infinity of solutions of the form

$$u = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

for any (real number) α

(d) $A^{\mathrm{II}} u = f$ has an infinity of solutions of the form

$$u = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

for any (real number) α.

Question 2 (25 points).

We consider the system of three springs and masses shown in Figure 1.

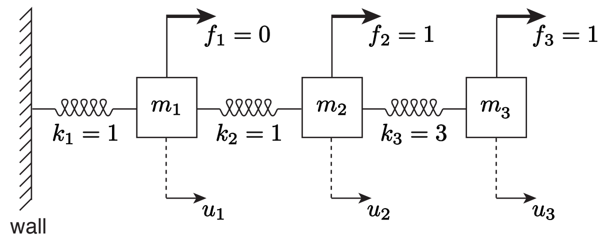

Figure 1: The spring-mass system for Question 2.

The equilibrium displacements satisfy the linear system of three equations in three unknowns, $Au = f$, given by

$$\underbrace{\begin{pmatrix} 2 & -1 & 0 \\ -1 & 4 & -3 \\ 0 & -3 & 3 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}}_{u} = \underbrace{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}_{f}. \tag{4}$$

The matrix A is SPD (*Symmetric Positive Definite*). Note you should only consider the particular right-hand side f (forces) indicated.

We now reduce the system by Gaussian Elimination to form $Uu = \hat{f}$, where U is an upper triangular matrix. We may then find u by Back Substitution. Note that we do *not* perform any partial pivoting — reordering of the rows of A — for stability (since the matrix is SPD), and furthermore we do *not* perform any reordering of the columns of the matrix A for optimization: we work directly on the matrix A as given by equation (4).

(i) (5 points) The element $U_{2\,2}$ (i.e., the entry in the i = second row, j = second column) of U is given by

(a) 4

(b) 5/2

(c) 9/2

(d) 7/2

Note: If at any point you are confused about which entry (row, column) we are referring to in a question, please *ask*.

(*ii*) (5 points) The element U_{33} (i.e., the entry in the i = third row, j = third column) of U is given by

(*a*) 2/5

(*b*) 3/7

(*c*) 4/9

(*d*) 3

(*iii*) (5 points) The element $\hat{f}_2$ (i.e., the second entry in the $\hat{f}$ vector) is given by

(*a*) 1

(*b*) 3/2

(*c*) 1/2

(*d*) 0

(*iv*) (5 points) The element $\hat{f}_3$ (i.e., the third entry in the $\hat{f}$ vector) is given by

(*a*) 1

(*b*) 5/3

(*c*) 13/7

(*d*) 9/5

(*v*) (5 points) The displacement of the third mass, u_3, is given by

(*a*) 13/3

(*b*) 9/2

(*c*) 1/3

(*d*) 1

Question 3 (*25 points*).

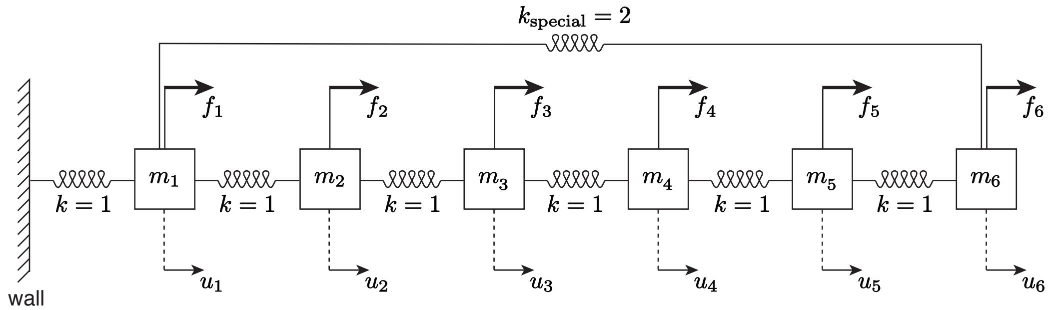

Figure 2: The spring-mass system for Question 3. Note that all the spring constants are unity, $k = 1$, except for the "special" spring which links mass 1 and mass 6, $k_{\text{special}} = 2$. Note that all the springs are described by the linear Hooke relation.

We consider the system of springs and masses shown in Figure 2. Equilibrium — force balance on each mass and Hooke's law for the spring constitutive relation — leads to the system of six equations in six unknowns, $Au = f$,

$$\underbrace{\begin{pmatrix} a & -1 & 0 & 0 & 0 & c \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ c & 0 & 0 & 0 & -1 & b \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix}}_{u} = \underbrace{\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{pmatrix}}_{f} . \tag{5}$$

where we will ask you to specify a, b, and c in the questions below.* Note for the correct choices of a, b, and c the matrix A is SPD.

We now reduce the system $Au = f$ by Gaussian Elimination to form $Uu = \hat{f}$, where U is an upper triangular matrix. Note that we do *not* perform any partial pivoting — reordering of the rows of A — for stability (since the matrix is SPD), and furthermore we do *not* perform any reordering of the columns of the matrix A for optimization: we work directly on the matrix A as given by equation (5).

(i) (5 points) The value of a is

(a) 2

(b) -1

(c) -2

*We presume that all quantities are provided in consistent units.

(*d*) 3

(*e*) 4

(*ii*) (5 points) The value of b is

(*a*) 2

(*b*) -1

(*c*) -2

(*d*) 3

(*e*) 4

(*iii*) (5 points) The value of c is

(*a*) 0

(*b*) -1

(*c*) -2

(*d*) 3

(*e*) 4

(*iv*) (5 points) The number of nonzero elements in the (upper triangular) matrix U is

(*a*) 24

(*b*) 15

(*c*) 12

(*d*) 11

(*e*) 36

Hint: Consider the initial stage of Gaussian Elimination (and the fill-in process) to deduce the only possibly correct option from the available choices.

(*v*) (5 points) The number of nonzero elements in A^{-1} (the inverse of A) is

(*a*) 24

(*b*) 15

(*c*) 12

(*d*) 11

(*e*) 36

Hint: Make an educated guess based on the physical interpretation of the columns of A^{-1}.

Question 4 (25 points)

We consider the spring-mass system shown in Figure 3: there are n masses, each of which (except those near the two ends) is connected to its four nearest neighbors.

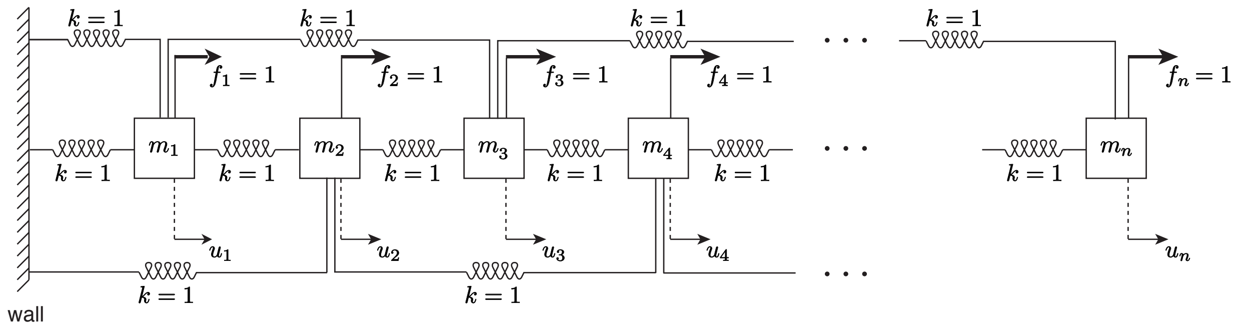

Figure 3: The spring-mass system for Question 4.

The displacements of the masses, u, satisfies a linear system of n equations in n unknowns, $Au = f$. The MATLAB script

```
clear

n = 1000; % number of masses, assumed greater than 4

A = spalloc(n,n,5*n);

A(1,1) = 4.;
A(1,2) = -1.;
A(1,3) = -1.;
A(2,2) = 4.;
A(2,1) = -1.;
A(2,3) = -1.;
A(2,4) = -1.;
for i = 3:n-2
   A(i,i) = 4.;
   A(i,i-1) = -1.;
   A(i,i-2) = -1.;
   A(i,i+1) = -1.;
   A(i,i+2) = -1.;
end
A(n-1,n-1) = 3.;
A(n-1,n-2) = -1.;
A(n-1,n-3) = -1.;
A(n-1,n) = -1.;
A(n,n) = 2.;
A(n,n-1) = -1.;
A(n,n-2) = -1.;
```

```
f = ones(n,1);

numnonzero_of_A = nnz(A);

numtimes_compute = 20;
tic;
for itimes = 1:numtimes_compute
   u = A \ f;
end
avg_time_to_find_u = toc/numtimes_compute;
% repeat calculation numtimes_compute times to get reliable timing

PE = 0.5*u'*A*u;
```

forms the stiffness matrix A (= `A` in MATLAB) and force vector f (= `f` in MATLAB) and then solves for the displacements u (= `u` in MATLAB). Note that the matrix A is SPD.

You may assume that the MATLAB backslash operator need not perform any partial pivoting — reordering of the rows of A — for stability (since the matrix is SPD), and furthermore will not perform any reordering of the columns of the matrix A for optimization (the ordering is already the best possible): MATLAB works directly on the matrix A as given — Gaussian Elimination to obtain U (= `U` in MATLAB) and $\hat{f}$ followed by Back Substitution to obtain `u`.

We run the script for `n = 1000` as indicated above.

(*i*) (5 points) The script will set the value of `A(3,2)` to

(*a*) `4.`

(*b*) `-1.`

(*c*) `3.`

(*d*) `2.`

(*e*) `0.`

(*ii*) (5 points) The script will set the value of `numnonzero_of_A` to

(*a*) `2998`

(*b*) `3996`

(*c*) `4994`

(*d*) `6001`

Hint: How many diagonals of A are populated?

(*iii*) (5 points) The number of nonzero elements of `U` will be

(*a*) `1999`

(*b*) `2997`

(*c*) `3000`

(*d*) `500500`

(Note you do not see `U` explicitly in the script of page 10 but it is formed internally as part of the backslash operation `u = A \ f`.)

Hint: Recall Gaussian Elimination for banded matrices.

(*iv*) (5 points) The script will set the value of `PE` to

(*a*) `0`

(*b*) `-3.3231e+06`

(*c*) `[1,1]`

(*d*) `3.3411e+07`

Hint: You need not calculate the exact answer to identify the correct answer amongst the available choices.

(*v*) (5 points) For `n = 1000` we find `avg_time_to_find_u` to be `4.6e-04` seconds. We now rerun the script but we change the line `n = 1000` to `n = 10000` such that `n` is 10 times as large. (We make no other changes.)

Now, running the script for `n = 10000`, we obtain for `avg_time_to_find_u`

(*a*) `4.2e-04`

(*b*) `6.0e-02`

(*c*) `4.1e-03`

(*d*) `5.1e-01`

You should choose the most plausible answer assuming that computational time is roughly proportional to the number of FLOPs.

Answer Key

Q1 (*i*) (*c*) $A^{\mathrm{I}}u = f$ has an infinity of solutions of the form

$$u = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

for any (real number) α

(*ii*) (*b*) $A^{\mathrm{I}}u = f$ has *no solution*

(*iii*) (*a*) $A^{\mathrm{II}}u = f$ has *a unique solution*

(*iv*) (*a*) $A^{\mathrm{II}}u = f$ has *a unique solution*

Q2 (*i*) (*d*) 7/2

(*ii*) (*b*) 3/7

(*iii*) (*a*) 1

(*iv*) (*c*) 13/7

(*v*) (*a*) 13/3

Q3 (*i*) (*e*) 4

(*ii*) (*d*) 3

(*iii*) (*c*) -2

(*iv*) (*b*) 15

(*v*) (*e*) 36

Q4 (*i*) (*b*) `-1.`

(*ii*) (*c*) `4994`

(*iii*) (*b*) `2997`

(*iv*) (*d*) `3.3411e+07`

(*v*) (*c*) `4.1e-03`