
You may refer to the text and other class materials as well as your own notes and scripts.

For Quiz 3 (on Unit III) *you will need a calculator* (for arithmetic operations and standard functions); note, however, that laptops, tablets, and smartphones are *not* permitted.

To simulate real quiz conditions you should complete all the questions in this sampler in 90 minutes.

NAME _____

There are a total of 100 points: five questions, each worth 20 points.

All questions are multiple choice; in all cases circle one and only one answer.

We include a blank page at the end which you may use for any derivations, but note that we do not refer to your work and in any event on the quiz *your grade will be determined solely by your multiple choice selections*.

You may assume that all arithmetic operations are performed exactly (with *no* floating point truncation or round-off errors). We may display the numbers in the multiple-choice options to just a few digits, however you should always be able to clearly discriminate the correct answer from the incorrect answers.

Question 1 (20 points)

You are given a matrix u of size 4×1 ,

$$u = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

in MATLAB `u = [2;1;1;1]`, and a second matrix v of size 4×1 ,

$$v = \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

in MATLAB `v = [0.5;1;0;0]`. Here the size of a matrix, $m \times n$, refers to the number of rows (m) and number of columns (n) in the matrix. (In order to avoid any ambiguity we shall refer solely to matrices in this question, although of course some of the entities are vectors.)

We now perform the matrix-matrix product “ u times v^T ,”

$$w = u(v^T), \tag{1}$$

where recall that T denotes transpose.

(i) (3 points) The size of the matrix w is

(a) 1×1

(b) 1×4

(c) 4×4

(d) 4×1

(ii) (3 points) The operation of equation (1) is implemented in MATLAB as

(a) `w = u.*v`

(b) `w = v.*u`

(c) `w = u'*v`

(d) `w = u*v'`

(iii) (4 points) The value of $w(1,1)$ is

(a) 0.5

(b) 1.0

(c) 2.5

(d) 2.0

We now perform the matrix-matrix product “ u^T times v ,”

$$z = (u^T)v, \tag{2}$$

where again T denotes transpose.

(iv) (3 points) The size of the matrix z is

(a) 1×1

(b) 1×4

(c) 4×4

(d) 4×1

(v) (3 points) The operation of equation (2) is implemented in MATLAB as

(a) $z = u.*v$

(b) $z = v.*u$

(c) $z = u'*v$

(d) $z = u*v'$

(vi) (4 points) The value of $z(1,1)$ is

(a) 0.5

(b) 1.0

(c) 2.5

(d) 2.0

Question 2 (20 points)

We run the script

```
% begin script

clear

n = 1000;

A = zeros(1000,1000);

A(400,404) = 1.0;
A(678,2) = 8.0;
A(678,999) = 4.0;
A(705,678) = 2.0;
A(705,999) = 10.0;

w = ones(1000,1);

v = A*w;

% end script
```

Note that A is a $1,000 \times 1,000$ matrix with *only five non-zero entries*, and that w is a $1,000 \times 1$ vector of *all ones*.

(i) (5 points) The value of $v(678)$ is

(a) 0

(b) 8

(c) 12

(d) 14

Hint: Consider the “row interpretation” of the matrix-vector product.

(ii) (5 points) The value of $v(999)$ is

(a) 0

(b) 8

(c) 12

(d) 14

Hint: Consider the “row interpretation” of the matrix-vector product.

We (now `clear` all variables and) define a $1,000 \times 5$ matrix B (in MATLAB, `B`) and a 5×1 vector w (in MATLAB, `w`) as

$$B = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}, \quad w = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{pmatrix},$$

(the j^{th} column of B consists of “all j ’s”). We then evaluate

$$v = Bw$$

and find the maximum of the vector v as

$$M = \max_{1 \leq i \leq 1000} v_i . \quad (3)$$

If you wish to see the corresponding MATLAB script, refer to Appendix A at the end of the sampler.

(iii) (10 points) The value of M from equation (3) is

(a) -2

(b) 0

(c) 2

(d) 4

(e) 5

Hint: Consider the “column interpretation” of the matrix-vector product.

Question 3 (20 points)

We consider linear regression for a particular simple case: a single independent variable ($p = 1$), which we denote x ; as always, a single dependent variable, y ; a model with only $n = 2$ terms,

$$Y_{\text{model}}(x; \beta) = \beta_0 + \beta_1 x, \quad (4)$$

(hence $h_0(x) = 1$, $h_1(x) = x$); and $m = 3$ *noisy* measurements given by $(x_1 = 0, Y_1 = 0)$, $(x_2 = 1, Y_2 = 0)$, $(x_3 = 2, Y_3 = 2)$. We denote by Y the 3×1 column vector $(0 \ 0 \ 2)^T$, and by β the 2×1 column vector $(\beta_0 \ \beta_1)^T$.

(i) (3 points) The “regression matrix” X is given by

(a) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 2 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$

Recall that $(X\beta)_i$ (the i^{th} component of the vector $X\beta$) is equal to $Y_{\text{model}}(x_i; \beta)$ for the measurement points x_i , $i = 1, 2, 3$.

(ii) (3 points) The equation

$$X\beta = Y \quad (5)$$

has, for our *particular* (noisy) measurements and hence Y vector $(0 \ 0 \ 2)^T$ indicated above,

(a) no solution for β

(b) a unique solution for β

(c) many solutions for β

Hint: Consider the relationship of equation (5) to the problem of putting a straight line through three points; plot the experimental data (x_i, Y_i) , $i = 1, 2, 3$.

(iii) (5 points) The matrix $X^T X$ is given by

(a) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix}$

(b) $\begin{pmatrix} 3 & 3 \\ 3 & 5 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 4 \end{pmatrix}$

(d) $\begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix}$

(iv) (4 points) The matrix $X^T Y$ is given by

(a) $\begin{pmatrix} 3 \\ 9 \end{pmatrix}$

(b) $\begin{pmatrix} 4 \\ 7 \end{pmatrix}$

(c) $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$

(d) $\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$

(v) (5 points) The least squares solution $\hat{\beta}$ to our problem is

(a) $\hat{\beta}_0 = -0.3333$, $\hat{\beta}_1 = 1.0$

(b) $\hat{\beta}_0 = -0.1667$, $\hat{\beta}_1 = 1.5$

(c) $\hat{\beta}_0 = 0$, $\hat{\beta}_1 = 1.0$

(d) $\hat{\beta}_0 = -0.5$, $\hat{\beta}_1 = 1.2$

(Recall that the least-squares solution $\hat{\beta}$ is the particular β which minimizes the norm of the residual squared (i.e., $\|Y - X\beta\|^2$).

Hint: Recall the simple formula for the inverse of a 2×2 matrix.

Question 4 (20 points)

A naturalist studying a pair of geese devises a camouflaged scale to provide weight data for each of the geese (henceforth denoted Goose 1 and Goose 2). Sometimes a single goose will walk onto the scale but on other occasions both geese walk onto the scale at the same time. On a given day the naturalist records in the logbook

Observation 1: Goose 1 on scale; scale transducer indicates 12.3 kilos.

Observation 2: Goose 1 and Goose 2 on scale; scale transducer indicates 26.3 kilos.

Observation 3: Goose 2 on scale; scale transducer indicates 13.1 kilos.

We wish to infer from this data a good estimate for the weights of each of the geese.

In what follows we shall denote by W_1 the weight of Goose 1 (in kilos) and by W_2 the weight of Goose 2 (in kilos); we further denote by W the 2×1 vector

$$W \equiv \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}. \quad (6)$$

We further define the 3×1 vector of scale readings

$$S \equiv \begin{pmatrix} 12.3 \\ 26.3 \\ 13.1 \end{pmatrix} \quad (7)$$

(where again the units are kilos). The naturalist's three observations above can be summarized as

$$BW = S + \mathcal{E}, \quad (8)$$

where B is a matrix and \mathcal{E} is a 3×1 noise vector. We shall assume that the noise vector \mathcal{E} (in kilos) relates to transducer error and does not depend on the weight on the scale.

(i) (8 points) The matrix B is given by

(a) $\begin{pmatrix} 12.3 & 0 \\ 26.3 & 26.3 \\ 0 & 13.1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 12.3 & 0 & -1 \\ 26.3 & 26.3 & -2 \\ 0 & 13.1 & -1 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 1 & 0 \end{pmatrix}$

(ii) (4 points) The matrix equation (system of linear equations) $BW = S$ has

- (a) a unique solution for W
- (b) no solution for W
- (c) many solutions for W
- (iii) (8 points) The weight vector \hat{W} which

minimizes the norm of the residual squared, $\|S - BW\|^2$, over all vectors W

satisfies the linear system of equations

$$(a) \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} W = \begin{pmatrix} 12.3 \\ 26.3 \end{pmatrix}$$

$$(b) \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} W = \begin{pmatrix} 38.6 \\ 39.4 \end{pmatrix}$$

$$(c) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} W = \begin{pmatrix} 12.3 \\ 26.3 \end{pmatrix}$$

$$(d) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} W = \begin{pmatrix} 38.6 \\ 39.4 \end{pmatrix}$$

Question 5 (20 points)

The average skin friction coefficient (nondimensional and positive), \overline{C}_f , for a turbulent boundary layer depends on the Reynolds number (nondimensional and positive), Re , as

$$\overline{C}_f = \alpha(\text{Re}^\rho), \quad (9)$$

where α and ρ are unknown constants (i.e., which do not depend on Re). Note that α is a positive multiplicative factor and ρ is the exponent of Re . The above form would lead to a nonlinear regression problem. We thus define $y = \log(\overline{C}_f)$ and $x = \log(\text{Re})$ such that now y may be expressed as $y = Y_{\text{model}}(x; \beta)$ with

$$Y_{\text{model}}(x; \beta) = \beta_0 + \beta_1 x \quad (10)$$

for β_0 and β_1 independent of x . Here \log refers to the natural logarithm (with corresponding MATLAB built-in function `log`). This form, you will agree, is amenable to linear regression analysis.

We shall assume that the model of equation (10) is *bias-free*: there exists a unique $\beta^{\text{true}} \equiv (\beta_0^{\text{true}} \ \beta_1^{\text{true}})^T$ such that equation (10) exactly predicts $\log(\overline{C}_f)$.

We further provide $m \times 1$ arrays `xpts` (values of $\log(\text{Re})$) and `Y` (values of $\log(\overline{C}_f)$) corresponding to (noisy) measurements (`xpts(i)`, `Y(i)`), $1 \leq i \leq m$, for $m = 52$. We assume that the measurements are given by

$$Y(i) = Y_{\text{model}}(\text{xpts}(i); \beta^{\text{true}}) + \epsilon(i), \quad 1 \leq i \leq m,$$

where the noise $\epsilon(i)$ satisfies our assumptions N1, N2, and N3 (normal zero-mean, homoscedastic with standard deviation σ , uncorrelated in x).

We now run the script*

```
% begin script

m = 52;

X = [ones(m,1), xpts];
betahat = X \ Y;

% end script
```

(i) (5 points) Based on the regression analysis above, our estimate for α is

- (a) `betahat(1)`
- (b) `exp(betahat(1))`
- (c) `log(betahat(1))`
- (d) `-betahat(1)`

where `exp` is the MATLAB built-in exponential function.

Hint: Consider the log transformation of equation (9) to equation (10).

*Note that prior to running the script the MATLAB workspace contains only `xpts` and `Y`.

(ii) (5 points) Based on the regression analysis above, our estimate for ρ is

(a) `betahat(2)`

(b) `exp(betahat(2))`

(c) `log(betahat(2))`

(d) `-betahat(2)`

Hint: Consider the log transformation of equation (9) to equation (10).

(iii) (5 points) We now calculate, based on the data provided, joint 0.95-confidence-level confidence intervals for β_0^{true} and β_1^{true} , I_0^{joint} and I_1^{joint} , respectively. We obtain for I_1^{joint} (the confidence interval for β_1^{true}) the interval $[-0.1560, -0.1262]$. With confidence level 0.95 we may then conclude that (over the range of Re covered by the experimental data)

(a) \bar{C}_f increases with increasing Re

(b) \bar{C}_f decreases with increasing Re

(c) \bar{C}_f is independent of Re

(d) \bar{C}_f is always negative

(iv) (5 points) The length of the confidence interval of Question 5(iii) is deemed too large and hence the accuracy of our β_1^{true} estimate potentially too low. To remedy the situation you should

(a) increase the number of measurements from $m = 52$ to $m^{\text{new}} (> m)$, and then re-perform the regression analysis

(b) retain the $m = 52$ measurements but replace the model of equation (10) (which has $n = 2$ terms) by a new model (which has $n = 52$ terms)

$$Y_{\text{model}}^{\text{new}}(x; \beta) = \sum_{j=0}^{51} \beta_j x^j \quad (= \beta_0 + \beta_1 x + \cdots + \beta_{51} x^{51}),$$

and then re-perform the regression analysis

Appendix A
Matlab Script for Question 2(*iii*)

```
% begin script

clear

m = 1000;

B = zeros(1000,5);
for i = 1:m
    for j = 1:5
        B(i,j) = j;
    end
end

w = zeros(5,1);
w(3) = 1.0;
w(5) = -1.0;

v = B*w;
M = max(v);

% end script
```

where `max` is the MATLAB built-in function which returns the maximum of a vector.

Answer Key

Q1 (i) (c) 4×4

(ii) (d) $\mathbf{w} = \mathbf{u} * \mathbf{v}'$

(iii) (b) 1.0

(iv) (a) 1×1

(v) (c) $\mathbf{z} = \mathbf{u}' * \mathbf{v}$

(vi) (d) 2.0

Q2 (i) (c) 12

(ii) (a) 0

(iii) (a) -2

Q3 (i) (b) $\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}$

(ii) (a) no solution for β

(iii) (b) $\begin{pmatrix} 3 & 3 \\ 3 & 5 \end{pmatrix}$

(iv) (c) $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$

(v) (a) $\hat{\beta}_0 = -0.3333, \hat{\beta}_1 = 1.0$

Q4 (i) (b) $\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$

(ii) (b) no solution for W

(iii) (b) $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} W = \begin{pmatrix} 38.6 \\ 39.4 \end{pmatrix}$,

Q5 (i) (b) `exp(betahat(1))`

(ii) (a) `betahat(2)`

(iii) (b) \bar{C}_f decreases with increasing Re

(iv) (a) increase the number of measurements

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