2.081J/16.230J Plates and Shells

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Contents

1 Strain-Displacement Relation for Plates

1.1 1-D Strain Measure

1.1.1 Engineering Strain

Engineering strain ε is defined as the relative displacement:

$$\varepsilon = \frac{ds - ds_0}{ds_0} \tag{1}$$

where ds_0 is the increment of initial lenght and ds is the increment of current length.

1.1.2 Green-Lagrangian Strain

Instead of comparing the length, one can compare the square of lengths:

$$\begin{aligned} E &= \frac{ds^2 - ds_0^2}{2ds_0^2} \\ &= \frac{ds - ds_0}{ds_0}\frac{ds + ds_0}{2ds_0} \end{aligned} \tag{2}$$

Where $ds \to ds_0$, the second term is Eq. (2) tends to unity, and the Green strain measure and the engineering strain become identical. Equation (2) can be put into an equivalnet form:

$$ds^2 - ds_0^2 = 2E ds_0^2 \tag{3}$$

which will now be generalized to the 3-D case.

1.2 3-D Strain Measure

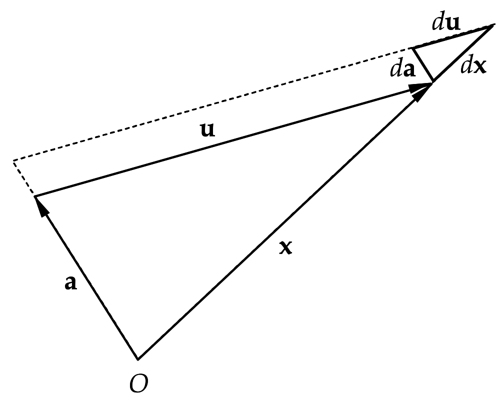
1.2.1 Derivation of Green-Lagrangian Strain Tensor for Plates

Let define the following quanties:

- $\mathbf{a} = [a_i]$: vector of the initial (material) coordinate system
- $\mathbf{x} = [x_i]$: vector of the current (spatial) coordinate system
- $\mathbf{u} = [u_i]$: displacement vector

where the index $i = 1, 2, 3$. The relation between those quantities is:

$$\begin{aligned} x_i &= a_i + u_i \\ dx_i &= da_i + du_i \end{aligned} \tag{4}$$



Now, the squares of the initial and the current length increment can be written in terms of a_i and u_i:

$$ds_0^2 = da_i da_j \delta_{ij} \tag{5}$$

$$\begin{aligned} ds^2 &= dx_i dx_j \delta_{ij} \\ &= (da_i + du_i)(da_j + du_j)\,\delta_{ij} \end{aligned} \tag{6}$$

where the Kronecker tensor δ_{ij} reads:

$$\delta_{ij} = \begin{vmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{vmatrix} \tag{7}$$

The vector $\mathbf{u}$ can be considered as a function of:

- the initial (material) coordinate system, $\mathbf{u}(\mathbf{a})$, which leads to Lagrangian descripion, *or*
- the current (spatial) coordinates, $\mathbf{u}(\mathbf{x})$, which leads to the Eulerian description

In structural mechanics, the Lagrangian description is preferable:

$$\begin{aligned} u_i &= u_i(a_i) \\ du_i &= \frac{\partial u_i}{\partial a_k} da_k = u_{i,k}\ \ da_k \\ du_j &= \frac{\partial u_j}{\partial a_l} da_l = u_{j,l}\ \ da_l \end{aligned} \tag{8}$$

Let us calculated the difference in the length square:

$$ds^2 - ds_0^2 = (da_i + du_i)(da_j + du_j)\,\delta_{ij} - da_i da_j \delta_{ij} \tag{9}$$

Using Eq. (8) and the definition of δ_{ij}, the difference in the length square can be transformed into:

$$
\begin{aligned}
ds^2 - ds_0^2 &= (du_j da_i + du_i da_j + du_i du_j)\,\delta_{ij} \\
&= (u_{j,l}\ da_l\ da_i + u_{i,k}\ da_k\ da_j + u_{i,k}\ da_k\ u_{j,l}\ da_l)\,\delta_{ij} \\
&= [u_{j,l}\ (da_j\delta_{jl})\ da_i + u_{i,k}\ (da_i\delta_{ik})\ da_j + u_{i,k}\ u_{j,l}\ (da_i\delta_{ik})\ (da_j\delta_{jl})]\,\delta_{ij} \\
&= (u_{j,i} + u_{i,j} + u_{i,k}\ u_{j,k})\,da_i da_j \\
&= 2E_{ij}\ da_i da_j
\end{aligned}
\tag{10}
$$

where, by analogy with the 1-D case, the Lagrangian or Green strain tensor E_{ij} is defined:

$$
E_{ij} = \frac{1}{2}\,(u_{i,j} + u_{j,i} + u_{k,i}\ u_{k,j}) \tag{11}
$$

In the case of small displacement gradient ($u_{k,i} \ll 1$), the second nonlinear term can be neglected leading to the defintion of the infinitesimal strain tensor:

$$
\varepsilon_{ij} = \frac{1}{2}\,(u_{i,j} + u_{j,i}) \tag{12}
$$

From the defintion, the strain tensor is symmetric $\varepsilon_{ij} = \varepsilon_{ji}$, which can be seen by intechanign the indices i for j and j for i. In the moderately large deflection theory of structures, the nonlinear terms are important. Therefore, Eq. (11) will be used as a starting point in the development of the general theory of plates.

Components of Green-Lagrangian Strain Tensor Let define the following range convention for indices:

- Greek letters: $\alpha, \beta, ... = 1, 2$
- Roman letters: $i, j, ... = 1, 2, 3$

With this range convention, the Roman letters are also written as:

$$
\begin{aligned}
i &= \alpha, 3 \\
j &= \beta, 3
\end{aligned}
\tag{13}
$$

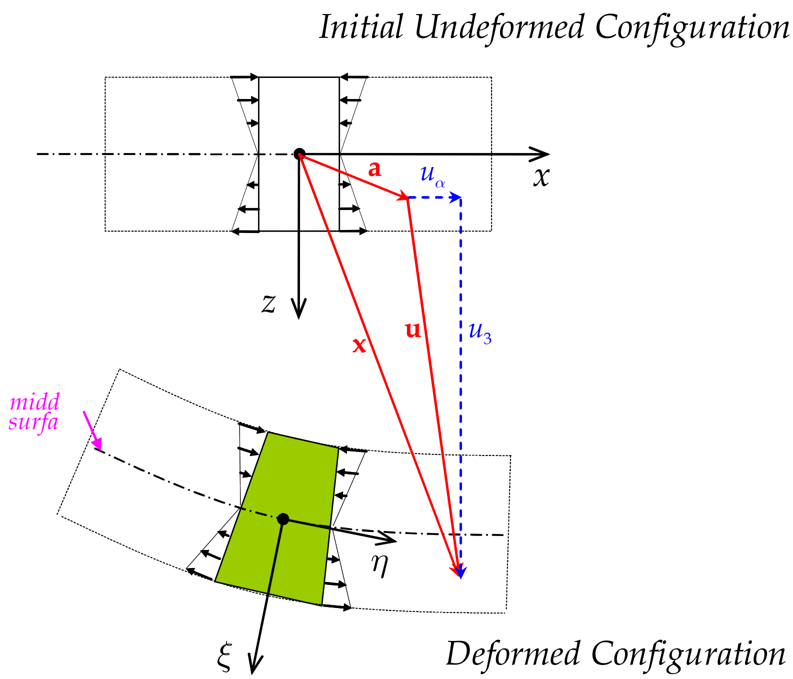
The Lagrangian or Green strain tensor can be expressed:

$$
E_{ij} = \left|\begin{array}{cc:c}
E_{11} & E_{12} & E_{13} \\
E_{21} & E_{22} & E_{23} \\
\hdashline
E_{31} & E_{32} & E_{33}
\end{array}\right| = \left|\begin{array}{c:c}
E_{\alpha\beta} & E_{3\beta} \\
\hdashline
E_{\alpha 3} & E_{33}
\end{array}\right|
$$

where $E_{\alpha\beta}$ is the in-plane component of strain tensor, $E_{\alpha 3}$ and $E_{3\beta}$ are out-of-plane shear components of strain tensor, and E_{33} is the through-thickness component of strain tensor. Similarily, displacement vector can be divided into two components:

$$u_i = \begin{vmatrix} u_1 \\ u_2 \\ \cdots \\ u_3 \end{vmatrix} = \begin{vmatrix} u \\ v \\ \cdots \\ w \end{vmatrix} = \begin{vmatrix} u_\alpha \\ \cdots \\ w \end{vmatrix}$$

where u_α is the in-plane components of the displacement vector, and $u_3 = w$ is the out-of-plane components of the displacement vector and also called as the transverse displacement.



Assumptions of the von Karman Theory The von Karman thoery of moderately large deflection of plates assumes:

1. The plate is thin. The thickness h is much smaller than the typical plate dimension, $h \ll L$.

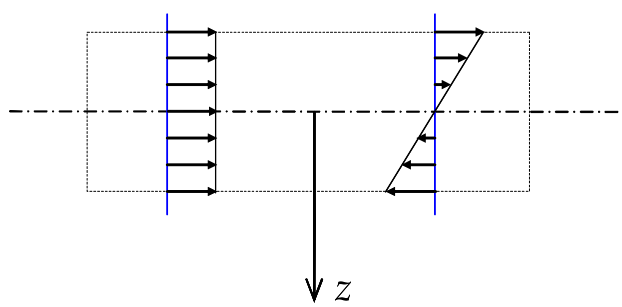
2. The magnitude of the transverse deflection is of the same order as the thickness of plate, $|w| = O(h)$. In practice, the present theory is still a good engineering approximation for deflections up to ten plate thickness.

3. Gradients of in-plane displacements $u_{\alpha},_{\beta}$ are small so that their product or square can be neglected.

4. Love-Kirchhoff hypothesis is satisfied. In-plane displacements are a linear function of the $z-$coordiate (3-coordinate).

$$u_{\alpha} = u_{\alpha}^{\circ} - z\ u_{3},_{\alpha} \tag{14}$$

where u_{α}° is the displacement of the middle surface, which is independent of $z-$coordinate, i.e. $u_{\alpha}^{\circ},_{3} = 0$; and $u_{3},_{\alpha}$ is the slope which is negative for the *"smiling"* beam.



5. The out-of-plane displacement is independent of the $z-$coordiante, i.e. $u_{3},_{3} = 0$.

1.2.2 Specification of Strain-Displacement Relation for Plates

In the theory of moderately large deflections, the strain-displacement relation can be specified for plates.

In-Plane Terms of the Strain Tensors From the general expression, Eq. (11), the 2-D in-plane componets of the strain tensor are:

$$E_{\alpha\beta} = \frac{1}{2}\left(u_{\alpha},_{\beta} + u_{\beta},_{\alpha} + u_{k},_{\alpha}\ u_{k},_{\beta}\right) \tag{15}$$

Here, consider the last, nonlinear term:

$$\begin{aligned} u_{k},_{\alpha}\ u_{k},_{\beta} &= u_{1},_{\alpha}\ u_{1},_{\beta} + u_{2},_{\alpha}\ u_{2},_{\beta} + u_{3},_{\alpha}\ u_{3},_{\beta} \\ &= u_{\gamma},_{\alpha}\ u_{\gamma},_{\beta} + u_{3},_{\alpha}\ u_{3},_{\beta} \end{aligned} \tag{16}$$

In the view of the Assumption 3, the first term in the above equation is zero, $u_{\gamma},_{\alpha}\ u_{\gamma},_{\beta} \simeq 0$. Therefore, the 2-D in-plane components of strain tensor reads:

$$E_{\alpha\beta} = \frac{1}{2}\left(u_{\alpha},_{\beta} + u_{\beta},_{\alpha} + w,_{\alpha}\ w,_{\beta}\right) \tag{17}$$

where $w = u_3$. Introducing Eq. (14) into Eq. (17), i.e. applying Love-Kirchhoff hypothesis, one gets:

$$E_{\alpha\beta} = \frac{1}{2}\left[\left(u^{\circ}_{\alpha} - z\ w_{,\alpha}\right)_{,\beta} + \left(u^{\circ}_{\beta} - z\ w_{,\beta}\right)_{,\alpha} + w_{,\alpha}\ w_{,\beta}\right] \tag{18}$$

$$= \frac{1}{2}\left(u^{\circ}_{\alpha,\beta} + u^{\circ}_{\beta,\alpha} - 2\ z\ w_{,\alpha\beta} + w_{,\alpha}\ w_{,\beta}\right)$$

$$= \frac{1}{2}\left(u^{\circ}_{\alpha,\beta} + u^{\circ}_{\beta,\alpha}\right) - z\ w_{,\alpha\beta} + \frac{1}{2} w_{,\alpha}\ w_{,\beta}$$

From the definiton of the curvature, one gets:

$$\kappa_{\alpha\beta} = -w_{,\alpha\beta} \tag{19}$$

Now, Eq. (18) can be re-casted in the form:

$$E_{\alpha\beta} = E^{\circ}_{\alpha\beta} + z\ \kappa_{\alpha\beta} \tag{20}$$

where the strain tensor of the middle surface $E^{\circ}_{\alpha\beta}$ is composed of a linear and a nonlinear term:

$$E^{\circ}_{\alpha\beta} = \frac{1}{2}\left(u^{\circ}_{\alpha,\beta} + u^{\circ}_{\beta,\alpha}\right) + \frac{1}{2} w_{,\alpha}\ w_{,\beta} \tag{21}$$

In the limiting case of small displacements, the second term can be neglected as compared to the first term. In the classical bending theory of plate, the in-plane displacements are assumed to be zero $u_\alpha = 0$ so that strains are only due to the curvatue:

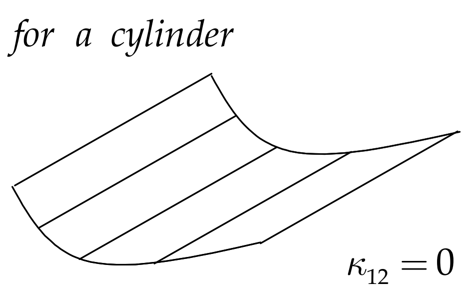
$$E_{\alpha\beta} = z\ \kappa_{\alpha\beta} \tag{22}$$

where

$$\kappa_{\alpha\beta} = \begin{vmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{vmatrix} = - \begin{vmatrix} \dfrac{\partial^2 w}{\partial x^2} & \dfrac{\partial^2 w}{\partial x \partial y} \\ \dfrac{\partial^2 w}{\partial x \partial y} & \dfrac{\partial^2 w}{\partial y^2} \end{vmatrix} = -w_{,\alpha\beta} \tag{23}$$

In the above equation, κ_{11} and κ_{22} are curvatures of the cylindrical bending, and κ_{12} is the twist which tells how the slope in the $x-$direction changes with the $y-$direction:

$$\kappa_{12} = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right)$$



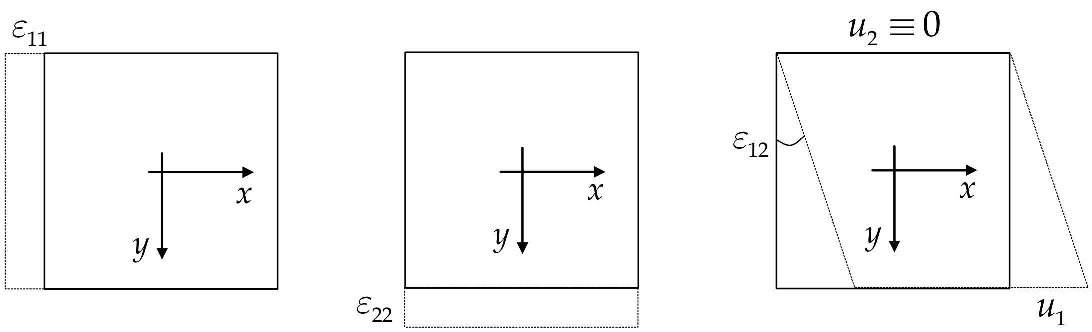
Interpretation of the linear terms: $\frac{1}{2}\left(u^{\circ}_{\alpha},_{\beta} + u^{\circ}_{\beta},_{\alpha}\right)$ Each component can be expressed in the followings:

$$\varepsilon_{11} = \frac{1}{2}\left(u_1,_1 + u_1,_1\right) = u_1,_1 = \frac{du_1}{dx} \tag{24}$$

$$\varepsilon_{22} = \frac{1}{2}\left(u_2,_2 + u_2,_2\right) = u_2,_2 = \frac{du_2}{dy} \tag{25}$$

$$\varepsilon_{12} = \frac{1}{2}\left(u_1,_2 + u_2,_1\right) = \frac{1}{2}\left(\frac{du_1}{dy} + \frac{du_2}{dx}\right) \tag{26}$$

$$\varepsilon_{12}|_{if\ \ u_2=0} = \frac{1}{2}\frac{du_1}{dy}$$



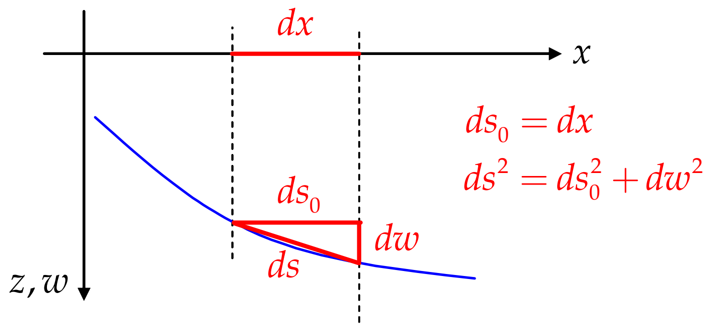
Therefore, ε_{11} and ε_{22} are the tensile strain in the two directions, and ε_{12} is the change of angles, i.e. shear strain.

Interpretation of the nonlinear term: $\frac{1}{2}w,_{\alpha}\ \ w,_{\beta}$ Let $\alpha = 1$ and $\beta = 1$. Then, the nonlienar term reads:

$$\frac{1}{2}w,_{\alpha}\ \ w,_{\beta}\Big|_{\alpha=1,\beta=1} = \frac{1}{2}\frac{dw}{dx}\frac{dw}{dx} = \frac{1}{2}\left(\frac{dw}{dx}\right)^2 \tag{27}$$

One can also obtain the same quantity by the defintion of 1-D Green-Lagrangian strain:

$$E = \frac{ds^2 - ds_0^2}{2ds_0^2} \simeq \frac{\left(ds_0^2 + dw^2\right) - ds_0^2}{2ds_0^2} = \frac{1}{2}\left(\frac{dw}{ds_0}\right)^2 = \frac{1}{2}\left(\frac{dw}{dx}\right)^2 \tag{28}$$



Thus, the conclusion is that the nonlinear term $\frac{1}{2}w_{,\alpha}\ w_{,\beta}$ represents the change of length of the plate element due to finite rotations.

Out-Of-Plane Terms of the Strain Tensors Refering to the definition introduced in Section 1.2.1, there are three other componets of the strain tensor: $E_{3\beta}$, $E_{\alpha 3}$ and E_{33}. Using the general expression for the components of the strain tensor, Eq. (11), it can be shown that the application of Assumption 4 and 5 lead to the following expressions:

$$\begin{aligned} E_{3\beta} &= \frac{1}{2}\left(u_{3,\beta} + u_{\beta,3} + u_{k,3}\ u_{k,\beta}\right) \\ &= \frac{1}{2}\left[u_{3,\beta} + u_{\beta,3} + \left(u_{1,3}\ u_{1,\beta} + u_{2,3}\ u_{2,\beta} + u_{3,3}\ u_{3,\beta}\right)\right] \\ &= \frac{1}{2}\left[u_{3,\beta} - u_{3,\beta} + \left(-u_{3,1}\ u_{1,\beta} - u_{3,2}\ u_{2,\beta}\right)\right] \\ &= \frac{1}{2}\left(-u_{3,1}\ u_{1,\beta} - u_{3,2}\ u_{2,\beta}\right) \\ &= -\frac{1}{2}w_{,\gamma}\ u_{\gamma,\beta} \end{aligned} \tag{29}$$

$$\begin{aligned} E_{\alpha 3} &= \frac{1}{2}\left(u_{\alpha,3} + u_{3,\alpha} + u_{k,\alpha}\ u_{k,3}\right) \\ &= \frac{1}{2}\left[u_{\alpha,3} + u_{3,\alpha} + \left(u_{1,\alpha}\ u_{1,3} + u_{2,\alpha}\ u_{2,3} + u_{3,\alpha}\ u_{3,3}\right)\right] \\ &= \frac{1}{2}\left[-u_{3,\alpha} + u_{3,\alpha} + \left(-u_{1,\alpha}\ u_{3,1} - u_{2,\alpha}\ u_{3,2}\right)\right] \\ &= \frac{1}{2}\left(-u_{1,\alpha}\ u_{3,1} - u_{2,\alpha}\ u_{3,2}\right) \\ &= -\frac{1}{2}w_{,\gamma}\ u_{\gamma,\alpha} \end{aligned} \tag{30}$$

$$\begin{aligned}E_{33} &= \frac{1}{2}\left(u_{3,3} + u_{3,3} + u_{k,3}\ u_{k,3}\right) \\ &= u_{3,3} + \frac{1}{2}\left[(u_{1,3})^2 + (u_{2,3})^2 + (u_{3,3})^2\right] \\ &= \frac{1}{2}\left[(u_{1,3})^2 + (u_{2,3})^2\right] \\ &= \frac{1}{2}\left[(-u_{3,1})^2 + (-u_{3,2})^2\right] \\ &= \frac{1}{2} w_{,\gamma}\ w_{,\gamma}\end{aligned} \tag{31}$$

The above are all second order terms which vanish for small deflection theory of plates. In the theory of moderately larege deflection of plates, the out-of-plate shear strains as well as the through-thickness strain is not zero. Therefore, an assumption *"plane remains plane,"* expressed by Eq. (14), does not mean that *"normal remains normal."* The existance of the out-of-plane shear strain means that lines originally normal to the middle surface do not remain normal to the deformed plate. However, the incremental work of these strains with the corresponding stresses is negligible:

$$E_{3\beta}\sigma_{3\beta},\ E_{\alpha 3}\sigma_{\alpha 3} \text{ and } E_{33}\sigma_{33}, \text{ are small} \tag{32}$$

because the corresponding stress $\sigma_{3\beta}$, $\sigma_{\alpha 3}$ and σ_{33} are small as compared to the in-plane stress $\sigma_{\alpha\beta}$. One can conclude that the elastic strain energy (and even plastic dissipation) is well approximated using the plane strain assumption:

$$\int_h \frac{1}{2}\sigma_{ij}\varepsilon_{ij} dz \simeq \int_h \frac{1}{2}\sigma_{\alpha\beta}\varepsilon_{\alpha\beta} dz \tag{33}$$

2 Derivation of Constitutive Equations for Plates

2.1 Definitions of Bending Moment and Axial Force

Hook's law in plane stress reads:

$$\sigma_{\alpha\beta} = \frac{E}{1-\nu^2}\left[(1-\nu)\ \varepsilon_{\alpha\beta} + \nu\ \varepsilon_{\gamma\gamma}\ \delta_{\alpha\beta}\right] \tag{34}$$

In terms of components:

$$\begin{aligned} \sigma_{xx} &= \frac{E}{1-\nu^2}\left(\varepsilon_{xx} + \nu\ \varepsilon_{yy}\right) \\ \sigma_{yy} &= \frac{E}{1-\nu^2}\left(\varepsilon_{yy} + \nu\ \varepsilon_{xx}\right) \\ \sigma_{xy} &= \frac{E}{1+\nu}\varepsilon_{xy} \end{aligned} \tag{35}$$

Here, strain tensor can be obtained from the strain-displacement relations:

$$\varepsilon_{\alpha\beta} = \varepsilon^{\circ}_{\alpha\beta} + z\ \kappa_{\alpha\beta} \tag{36}$$

Now, define the tensor of bending moment:

$$M_{\alpha\beta} \equiv \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha\beta}\ z\ dz \tag{37}$$

and the tensor of axial force (membrane force):

$$N_{\alpha\beta} \equiv \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha\beta}\ dz \tag{38}$$

2.2 Bending Energy

2.2.1 Bending Moment

Let us assume that $\varepsilon^{\circ}_{\alpha\beta} = 0$. The bending moment $M_{\alpha\beta}$ can be calculated:

$$\begin{aligned} M_{\alpha\beta} &= \frac{E}{1-\nu^2}\int_{-\frac{h}{2}}^{\frac{h}{2}} \left[(1-\nu)\ \varepsilon_{\alpha\beta} + \nu\ \varepsilon_{\gamma\gamma}\ \delta_{\alpha\beta}\right]\ z\ dz \\ &= \frac{E}{1-\nu^2}\ \left[(1-\nu)\ \varepsilon^{\circ}_{\alpha\beta} + \nu\ \varepsilon^{\circ}_{\gamma\gamma}\ \delta_{\alpha\beta}\right]\int_{-\frac{h}{2}}^{\frac{h}{2}} z\ dz \\ &\quad + \frac{E}{1-\nu^2}\ \left[(1-\nu)\ \kappa_{\alpha\beta} + \nu\ \kappa_{\gamma\gamma}\ \delta_{\alpha\beta}\right]\int_{-\frac{h}{2}}^{\frac{h}{2}} z^2\ dz \\ &= \frac{Eh^3}{12\left(1-\nu^2\right)}\ \left[(1-\nu)\ \kappa_{\alpha\beta} + \nu\ \kappa_{\gamma\gamma}\ \delta_{\alpha\beta}\right] \end{aligned} \tag{39}$$

Here, we define the bending rigidity of a plate D as follows:

$$D = \frac{Eh^3}{12\,(1-\nu^2)} \tag{40}$$

Now, one gets the moment-curvature relations:

$$\boxed{M_{\alpha\beta} = D\ \left[(1-\nu)\ \kappa_{\alpha\beta} + \nu\ \kappa_{\gamma\gamma}\ \delta_{\alpha\beta}\right]} \tag{41}$$

$$M_{\alpha\beta} = \begin{vmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{vmatrix} \tag{42}$$

where $M_{12} = M_{21}$ due to symmetry.

$$\begin{aligned} M_{11} &= D\ \left(\kappa_{11} + \nu\ \kappa_{22}\right) \\ M_{22} &= D\ \left(\kappa_{22} + \nu\ \kappa_{11}\right) \\ M_{12} &= D\ \left(1-\nu\right)\ \kappa_{12} \end{aligned} \tag{43}$$

2.2.2 Bending Energy Density

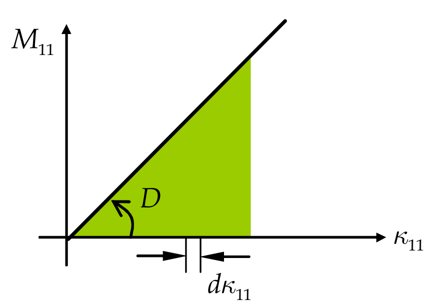
One -Dimensional Case Here, we use the hat notation for a function of certain argument such as:

$$\begin{aligned} M_{11} &= \hat{M}_{11}\left(\kappa_{11}\right) \\ &= D\ \kappa_{11} \end{aligned} \tag{44}$$

Then, the bending energy density $\bar{U}_b$ reads :

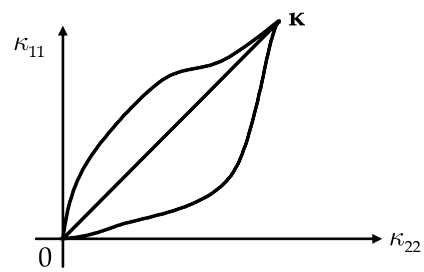
$$\begin{aligned} \bar{U}_b &= \int_0^{\bar{\kappa}_{11}} \hat{M}_{11}\left(\kappa_{11}\right)\ d\kappa_{11} \\ &= D\ \int_0^{\bar{\kappa}_{11}} \kappa_{11}\ d\kappa_{11} \\ &= \frac{1}{2}\ D\ \left(\bar{\kappa}_{11}\right)^2 \end{aligned} \tag{45}$$

$$\bar{U}_b = \frac{1}{2} M_{11}\ \bar{\kappa}_{11} \tag{46}$$



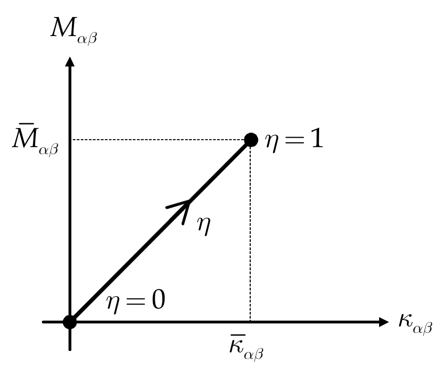
General Case General definition of the bending energy density reads:

$$\bar{U}_b = \oint M_{\alpha\beta} \; d\kappa_{\alpha\beta} \tag{47}$$



Calculate the energy density stored when the curvature reaches a given value $\bar{\kappa}_{\alpha\beta}$. Consider a straight loading path:

$$\begin{aligned} \kappa_{\alpha\beta} &= \eta \; \bar{\kappa}_{\alpha\beta} \\ d\kappa_{\alpha\beta} &= \bar{\kappa}_{\alpha\beta} \; d\eta \end{aligned} \tag{48}$$



$$\begin{aligned} M_{\alpha\beta} &= \hat{M}_{\alpha\beta} \left(\kappa_{\alpha\beta}\right) \\ &= \hat{M}_{\alpha\beta} \left(\eta \; \bar{\kappa}_{\alpha\beta}\right) \\ &= \eta \; \hat{M}_{\alpha\beta} \left(\bar{\kappa}_{\alpha\beta}\right) \end{aligned} \tag{49}$$

where $\hat{M}_{\alpha\beta}(\kappa_{\alpha\beta})$ is a homogeneous function of degree one.

$$\begin{aligned}\bar{U}_b &= \oint \hat{M}_{\alpha\beta}(\kappa_{\alpha\beta})\ d\kappa_{\alpha\beta} \\ &= \int_0^1 \eta\ \hat{M}_{\alpha\beta}(\bar{\kappa}_{\alpha\beta})\ \bar{\kappa}_{\alpha\beta}\ d\eta \\ &= \hat{M}_{\alpha\beta}(\bar{\kappa}_{\alpha\beta})\ \bar{\kappa}_{\alpha\beta} \int_0^1 \eta\ d\eta \\ &= \frac{1}{2}\ \hat{M}_{\alpha\beta}(\bar{\kappa}_{\alpha\beta})\ \bar{\kappa}_{\alpha\beta} \\ &= \frac{1}{2}\ M_{\alpha\beta}\ \bar{\kappa}_{\alpha\beta}\end{aligned} \tag{50}$$

Now, the bending energy density reads:

$$\begin{aligned}\bar{U}_b &= \frac{D}{2}\ \left[(1-\nu)\ \bar{\kappa}_{\alpha\beta} + \nu\ \bar{\kappa}_{\gamma\gamma}\ \delta_{\alpha\beta}\right]\ \bar{\kappa}_{\alpha\beta} \\ &= \frac{D}{2}\ \left[(1-\nu)\ \bar{\kappa}_{\alpha\beta}\ \bar{\kappa}_{\alpha\beta} + \nu\ \bar{\kappa}_{\gamma\gamma}\ \bar{\kappa}_{\alpha\beta}\ \delta_{\alpha\beta}\right] \\ &= \frac{D}{2}\ \left[(1-\nu)\ \bar{\kappa}_{\alpha\beta}\ \bar{\kappa}_{\alpha\beta} + \nu\ (\bar{\kappa}_{\gamma\gamma})^2\right]\end{aligned} \tag{51}$$

The bending energy density expressed in terms of components:

$$\begin{aligned}\bar{U}_b &= \frac{D}{2}\ \left\{(1-\nu)\left[(\bar{\kappa}_{11})^2 + 2\ (\bar{\kappa}_{12})^2 + (\bar{\kappa}_{22})^2\right] + \nu\ (\bar{\kappa}_{11} + \bar{\kappa}_{22})^2\right\} \\ &= \frac{D}{2}\ \left\{(1-\nu)\left[(\bar{\kappa}_{11} + \bar{\kappa}_{22})^2 - 2\ \bar{\kappa}_{11}\ \bar{\kappa}_{22} + 2\ (\bar{\kappa}_{12})^2\right] + \nu\ (\bar{\kappa}_{11} + \bar{\kappa}_{22})^2\right\} \\ &= \frac{D}{2}\ \left\{\left[(\bar{\kappa}_{11} + \bar{\kappa}_{22})^2 - 2\ \bar{\kappa}_{11}\ \bar{\kappa}_{22} + 2\ (\bar{\kappa}_{12})^2\right] - \nu\ \left[-2\ \bar{\kappa}_{11}\ \bar{\kappa}_{22} + 2\ (\bar{\kappa}_{12})^2\right]\right\} \\ &= \frac{D}{2}\ \left\{(\bar{\kappa}_{11} + \bar{\kappa}_{22})^2 - 2\ \bar{\kappa}_{11}\ \bar{\kappa}_{22} + 2\ (\bar{\kappa}_{12})^2 - \nu\ \left[-2\ \bar{\kappa}_{11}\ \bar{\kappa}_{22} + 2\ (\bar{\kappa}_{12})^2\right]\right\} \\ &= \frac{D}{2}\ \left\{(\bar{\kappa}_{11} + \bar{\kappa}_{22})^2 + 2\ (1-\nu)\left[-\bar{\kappa}_{11}\ \bar{\kappa}_{22} + (\bar{\kappa}_{12})^2\right]\right\}\end{aligned} \tag{52}$$

$$\boxed{\bar{U}_b = \frac{D}{2}\left\{(\bar{\kappa}_{11} + \bar{\kappa}_{22})^2 - 2\ (1-\nu)\left[\bar{\kappa}_{11}\ \bar{\kappa}_{22} - (\bar{\kappa}_{12})^2\right]\right\}} \tag{53}$$

2.2.3 Total Bending Energy

The total bending energy is the integral of the bending energy density over the area of plate:

$$U_b = \int_S \bar{U}_b\ dA \tag{54}$$

2.3 Membrane Energy

2.3.1 Axial Force

Assume that $\kappa_{\alpha\beta} = 0$. The axial force can be calculated:

$$\begin{aligned} N_{\alpha\beta} &= \frac{E}{1-\nu^2} \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[(1-\nu)\ \varepsilon_{\alpha\beta} + \nu\ \varepsilon_{\gamma\gamma}\ \delta_{\alpha\beta}\right]\ dz \\ &= \frac{E}{1-\nu^2} \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[(1-\nu)\ \varepsilon^\circ_{\alpha\beta} + \nu\ \varepsilon^\circ_{\gamma\gamma}\ \delta_{\alpha\beta}\right]\ dz \\ &\quad + \frac{E}{1-\nu^2} \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[(1-\nu)\ \kappa_{\alpha\beta} + \nu\ \kappa_{\gamma\gamma}\ \delta_{\alpha\beta}\right]\ z\ dz \\ &= \frac{E}{1-\nu^2} \left[(1-\nu)\ \varepsilon^\circ_{\alpha\beta} + \nu\ \varepsilon^\circ_{\gamma\gamma}\ \delta_{\alpha\beta}\right] \int_{-\frac{h}{2}}^{\frac{h}{2}} dz \\ &\quad + \frac{E}{1-\nu^2} \left[(1-\nu)\ \kappa_{\alpha\beta} + \nu\ \kappa_{\gamma\gamma}\ \delta_{\alpha\beta}\right] \int_{-\frac{h}{2}}^{\frac{h}{2}} z\ dz \\ &= \frac{Eh}{1-\nu^2} \left[(1-\nu)\ \varepsilon^\circ_{\alpha\beta} + \nu\ \varepsilon^\circ_{\gamma\gamma}\ \delta_{\alpha\beta}\right] \end{aligned} \tag{55}$$

Here, we define the axial rigidity of a plate C as follows:

$$C = \frac{Eh}{1-\nu^2} \tag{56}$$

Now, one gets the membrane force-extension relation:

$$\boxed{N_{\alpha\beta} = C\ \left[(1-\nu)\ \varepsilon^\circ_{\alpha\beta} + \nu\ \varepsilon^\circ_{\gamma\gamma}\ \delta_{\alpha\beta}\right]} \tag{57}$$

$$N_{\alpha\beta} = \begin{vmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{vmatrix} \tag{58}$$

where $N_{12} = N_{21}$ due to symmetry.

$$\begin{aligned} N_{11} &= C\ \ (\varepsilon^\circ_{11} + \nu\ \varepsilon^\circ_{22}) \\ N_{22} &= C\ \ (\varepsilon^\circ_{22} + \nu\ \varepsilon^\circ_{11}) \\ N_{12} &= C\ \ (1-\nu)\ \ \varepsilon^\circ_{11} \end{aligned} \tag{59}$$

2.3.2 Membrane Energy Density

Using the similar definition used in the calculation of the bending energy density, the extension energy (membrane energy) reads:

$$\bar{U}_m = \oint N_{\alpha\beta}\ d\varepsilon^\circ_{\alpha\beta} \tag{60}$$

Calculate the energy stored when the extension reaches a given value $\bar{\varepsilon}^{\circ}_{\alpha\beta}$. Consider a straight loading path:

$$\begin{aligned} \varepsilon^{\circ}_{\alpha\beta} &= \eta \ \bar{\varepsilon}^{\circ}_{\alpha\beta} \\ d\varepsilon^{\circ}_{\alpha\beta} &= \bar{\varepsilon}^{\circ}_{\alpha\beta} \ d\eta \end{aligned} \tag{61}$$

$$\begin{aligned} N_{\alpha\beta} &= \hat{N}_{\alpha\beta}\left(\varepsilon^{\circ}_{\alpha\beta}\right) \\ &= \hat{N}_{\alpha\beta}\left(\eta \ \bar{\varepsilon}^{\circ}_{\alpha\beta}\right) \\ &= \eta \ \hat{N}_{\alpha\beta}\left(\bar{\varepsilon}^{\circ}_{\alpha\beta}\right) \end{aligned} \tag{62}$$

where $\hat{N}_{\alpha\beta}\left(\varepsilon^{\circ}_{\alpha\beta}\right)$ is a homogeneous function of degree one.

$$\begin{aligned} \bar{U}_m &= \int_0^{\bar{\varepsilon}^{\circ}_{\alpha\beta}} \hat{N}_{\alpha\beta}\left(\varepsilon^{\circ}_{\alpha\beta}\right) \ d\varepsilon^{\circ}_{\alpha\beta} \\ &= \int_0^1 \eta \ \hat{N}_{\alpha\beta}\left(\bar{\varepsilon}^{\circ}_{\alpha\beta}\right) \ \bar{\varepsilon}^{\circ}_{\alpha\beta} \ d\eta \\ &= \frac{1}{2} \ \hat{N}_{\alpha\beta}\left(\bar{\varepsilon}^{\circ}_{\alpha\beta}\right) \ \bar{\varepsilon}^{\circ}_{\alpha\beta} \\ &= \frac{1}{2} \ N_{\alpha\beta} \ \bar{\varepsilon}^{\circ}_{\alpha\beta} \end{aligned} \tag{63}$$

Now, the extension energy reads:

$$\begin{aligned} \bar{U}_m &= \frac{C}{2} \ \left[(1-\nu) \ \bar{\varepsilon}^{\circ}_{\alpha\beta} + \nu \ \bar{\varepsilon}^{\circ}_{\gamma\gamma} \ \delta_{\alpha\beta}\right] \ \bar{\varepsilon}^{\circ}_{\alpha\beta} \\ &= \frac{C}{2} \ \left[(1-\nu) \ \bar{\varepsilon}^{\circ}_{\alpha\beta} \ \bar{\varepsilon}^{\circ}_{\alpha\beta} + \nu \ \left(\bar{\varepsilon}^{\circ}_{\gamma\gamma}\right)^2\right] \end{aligned} \tag{64}$$

The extension energy expressed in terms of components:

$$\begin{aligned} \bar{U}_m &= \frac{C}{2} \ \left\{(1-\nu) \ \left[(\bar{\varepsilon}^{\circ}_{11})^2 + 2 \ (\bar{\varepsilon}^{\circ}_{12})^2 + (\bar{\varepsilon}^{\circ}_{22})^2\right] + \nu \ (\bar{\varepsilon}^{\circ}_{11} + \bar{\varepsilon}^{\circ}_{22})^2\right\} \\ &= \frac{C}{2} \ \left\{(1-\nu) \ \left[(\bar{\varepsilon}^{\circ}_{11} + \bar{\varepsilon}^{\circ}_{22})^2 - 2 \ \bar{\varepsilon}^{\circ}_{11}\bar{\varepsilon}^{\circ}_{22} + 2 \ (\bar{\varepsilon}^{\circ}_{12})^2\right] + \nu \ (\bar{\varepsilon}^{\circ}_{11} + \bar{\varepsilon}^{\circ}_{22})^2\right\} \\ &= \frac{C}{2} \ \left\{(\bar{\varepsilon}^{\circ}_{11} + \bar{\varepsilon}^{\circ}_{22})^2 - 2 \ \bar{\varepsilon}^{\circ}_{11}\bar{\varepsilon}^{\circ}_{22} + 2 \ (\bar{\varepsilon}^{\circ}_{12})^2 - \nu\left[-2 \ \bar{\varepsilon}^{\circ}_{11}\bar{\varepsilon}^{\circ}_{22} + 2 \ (\bar{\varepsilon}^{\circ}_{12})^2\right]\right\} \\ &= \frac{C}{2} \ \left\{(\bar{\varepsilon}^{\circ}_{11} + \bar{\varepsilon}^{\circ}_{22})^2 + 2 \ (1-\nu)\left[-\bar{\varepsilon}^{\circ}_{11}\bar{\varepsilon}^{\circ}_{22} + (\bar{\varepsilon}^{\circ}_{12})^2\right]\right\} \end{aligned} \tag{65}$$

$$\boxed{\bar{U}_m = \frac{C}{2}\left\{(\bar{\varepsilon}^{\circ}_{11} + \bar{\varepsilon}^{\circ}_{22})^2 - 2 \ (1-\nu)\left[\bar{\varepsilon}^{\circ}_{11}\bar{\varepsilon}^{\circ}_{22} - (\bar{\varepsilon}^{\circ}_{12})^2\right]\right\}} \tag{66}$$

2.3.3 Total Membrane Energy

The total membrane is the integral of the membrane energy density over the area of plate::

$$U_m = \int_S \bar{U}_m \; dS \tag{67}$$

3 Development of Equation of Equilibrium and Boundary Conditions Using Variational Approach

3.1 Bending Theory of Plates

3.1.1 Total Potential Energy

The total potential energy of the system Π reads:

$$\Pi = U_b - V_b \tag{68}$$

where U_b is the bending energy stored in the plate, and V_b is the work of external forces.

Bending Energy

$$\begin{aligned} U_b &= \frac{1}{2}\int_S M_{\alpha\beta}\ \kappa_{\alpha\beta}\ dS \\ &= -\frac{1}{2}\int_S M_{\alpha\beta}\ w,_{\alpha\beta}\ dS \end{aligned} \tag{69}$$

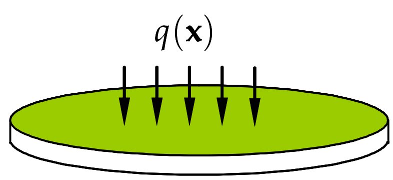
where the geometrical relation $\kappa_{\alpha\beta} = -w,_{\alpha\beta}$ has been used.

Work of External Forces

Plate Loading Lateral load:

$$q(\mathbf{x}) = q(x_\alpha) \tag{70}$$

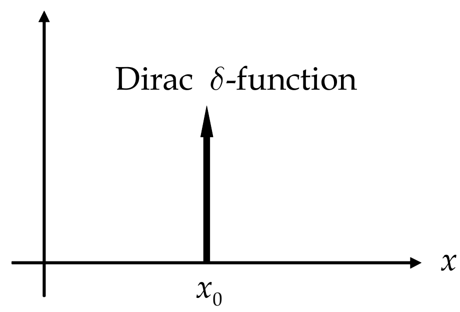
This is distributed load measured in $[N/m^2]$ or $[lb/in^2]$ force per unit area of the middle surface of the plate.



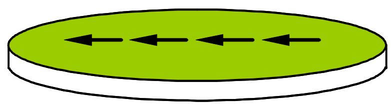
The distributed load contains concentrated load P as a special case:

$$P(x_0, y_0) = P_0 \delta(x - x_0)\, \delta(y - y_0) \tag{71}$$

where δ is the Dirac delta function, $[x_{0,}\ y_0]$ is the coordinate of the application of the concentrated force, and P_0 is the load intensity.

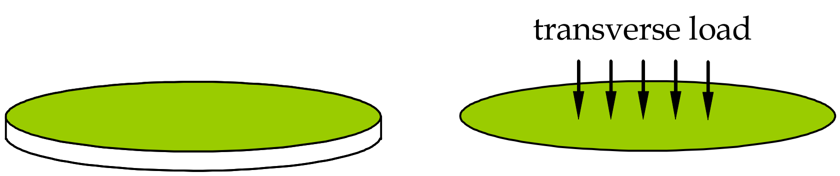


𝕹𝕺𝕿𝔼 *The shearing loads on the lateral surface of ice are normally not considered in the theory of thin plates.*

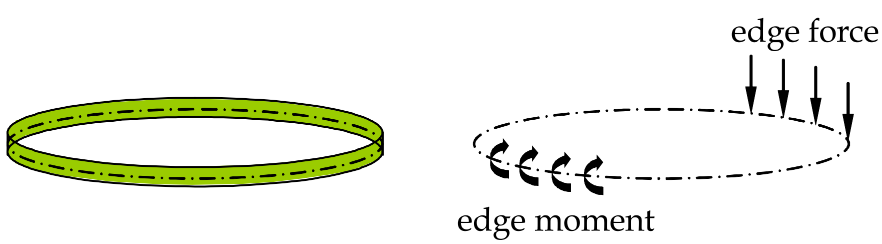


Load Classification

- Load applied at the horizontal surfaces.

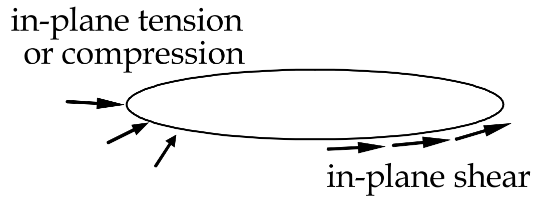


- Load applied at the lateral surfaces.



Loads are assumed to be applied to the middle plane of the plate

𝕹𝕺𝕿𝔼 *Other type of loading such as shear or in-plane tension or compression do not deflect laterally the plate and therefore are not considered in the bending theory.*

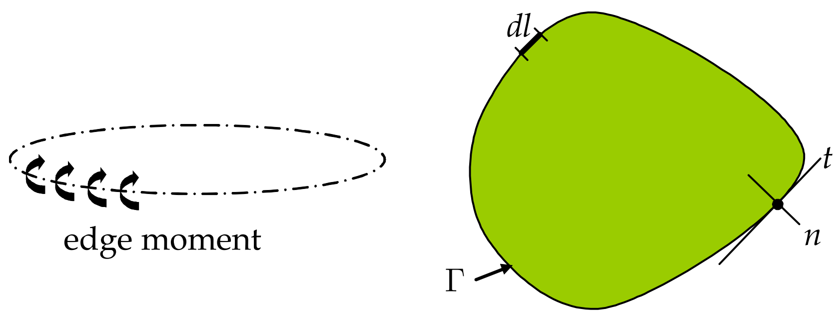


Potential Energy due to Lateral Load q Lateral (transverse) load does work on transverse deflection:

$$\int_S q\ w\ dS \tag{72}$$

This is also called a work of external forces.

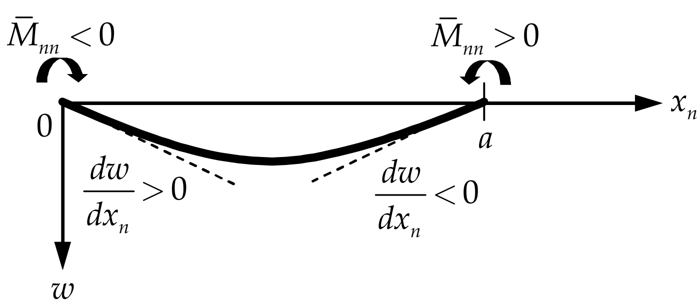
Potential Energy due to Edge Moment The conjugate kinematic variable associated with the edge moment is the edge rotation dw/dx_n.



We apply only the normal bending moment M_{nn}:

$$-\int_\Gamma \bar{M}_{nn}\ \frac{dw}{dx_n}\ dl \tag{73}$$

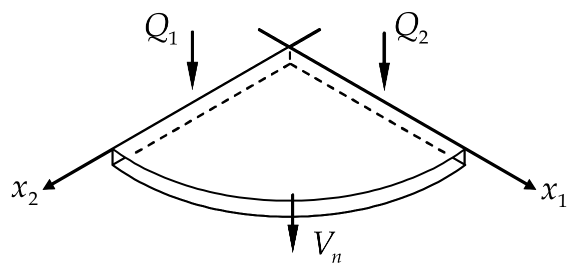
where the minus sign is included because positive bending moment results in a negative rotation and negative moment produces positive rotation.



At the edge, $M_{tt} = 0$ and $M_{tn} = 0$.

Potential Energy due to Edge Forces

$$\int_{\Gamma} \bar{V}_n \; w \; dl \tag{74}$$



Potential Energy due to All External Forces Now, the work of external forces reads:

$$V_b = \int_S q \; w \; dS - \int_{\Gamma} \bar{M}_{nn} \; \frac{dw}{dx_n} \; dl + \int_{\Gamma} \bar{V}_n \; w \; dl \tag{75}$$

3.1.2 First Variation of the Total Potential Energy

The total potential energy reads:

$$\begin{aligned} \Pi = & -\frac{1}{2} \int_S M_{\alpha\beta} \; w_{,\alpha\beta} \; dS \\ & - \int_S q \; w \; dS + \int_{\Gamma} \bar{M}_{nn} \; \frac{dw}{dx_n} \; dl - \int_{\Gamma} \bar{V}_n \; w \; dl \end{aligned} \tag{76}$$

First variation of the total potential energy $\delta\Pi$ is expressed:

$$\begin{aligned} \delta\Pi = & -\int_S M_{\alpha\beta} \; \delta w_{,\alpha\beta} \; dS \\ & - \int_S q \; \delta w \; dS + \int_{\Gamma} \bar{M}_{nn} \; \delta \left(\frac{dw}{dx_n} \right) \; dl - \int_{\Gamma} \bar{V}_n \; \delta w \; dl \end{aligned} \tag{77}$$

We shall transform now the first integral with the help of the Gauss theorem. First note that from the rule of the product differentiation:

$$M_{\alpha\beta} \; \delta w_{,\alpha\beta} = \left(M_{\alpha\beta} \; \delta w_{,\alpha} \right)_{,\beta} - M_{\alpha\beta,\beta} \; \delta w_{,\alpha} \tag{78}$$

then

$$\int_S M_{\alpha\beta} \; \delta w_{,\alpha\beta} \; dS = \int_S \left(M_{\alpha\beta} \; \delta w_{,\alpha} \right)_{,\beta} \; dS - \int_S M_{\alpha\beta,\beta} \; \delta w_{,\alpha} \; dS \tag{79}$$

Now, the first integral on the right hand side of the above equation transforms to the line integral:

$$\int_S M_{\alpha\beta} \; \delta w_{,\alpha\beta} \; dS = \int_{\Gamma} M_{\alpha\beta} \; \delta w_{,\alpha} \; n_{\beta} \; dl - \int_S M_{\alpha\beta,\beta} \; \delta w_{,\alpha} \; dS \tag{80}$$

The integrand of the second integral on the right hand side of the above equation transform to:

$$M_{\alpha\beta},_\beta\ \ \delta w,_\alpha = (M_{\alpha\beta},_\beta\ \ \delta w)\,,_\alpha - M_{\alpha\beta},_{\alpha\beta}\ \ \delta w \tag{81}$$

which results in:

$$\begin{aligned}\int_S M_{\alpha\beta}\ \delta w,_{\alpha\beta}\ \ dS = &\int_\Gamma M_{\alpha\beta}\ \delta w,_\alpha\ \ n_\beta\ dl \\ &- \int_S (M_{\alpha\beta},_\beta\ \ \delta w)\,,_\alpha\ \ dS + \int_S M_{\alpha\beta},_{\alpha\beta}\ \ \delta w\ dS\end{aligned} \tag{82}$$

upon which the application of the Gauss rule gives:

$$\begin{aligned}\int_S M_{\alpha\beta}\ \delta w,_{\alpha\beta}\ \ dS = &\int_\Gamma M_{\alpha\beta}\ \delta w,_\alpha\ \ n_\beta\ dl \\ &- \int_\Gamma M_{\alpha\beta},_\beta\ \ \delta w\ n_\alpha\ dl + \int_S M_{\alpha\beta},_{\alpha\beta}\ \ \delta w\ dS\end{aligned} \tag{83}$$

We can return now to the expression for $\delta\Pi$ and substitute there the transformed first integral:

$$\begin{aligned}\delta\Pi = &\int_S (-M_{\alpha\beta},_{\alpha\beta} - q)\ \ \delta w\ dS \\ &+ \int_\Gamma M_{\alpha\beta},_\beta\ \ \delta w\ n_\alpha\ dl - \int_\Gamma \bar{V}_n\ \delta w\ dl \\ &- \int_\Gamma M_{\alpha\beta}\ \delta w,_\alpha\ \ n_\beta\ dl + \int_\Gamma \bar{M}_{nn}\ \delta w,_n\ \ dl\end{aligned} \tag{84}$$

where $\delta w,_n = \delta\left(\frac{dw}{dx_n}\right)$. It is seen that integrals involving the prescribed forces $\bar{M}_{nn}$ and $\bar{V}_n$ are written in a local coordinate system $x_\gamma\ \ \{x_n,\ x_t\}$ while the remaining two integrals over the contour Γ are written in the global coordinate system x_α. In order to make comparison, we have to decide on one coordinate system. We choose the local system.

Consider the *first* integral:

$$\int_\Gamma (M_{\alpha\beta},_\beta\ \ n_\alpha)\ \ \delta w\ dl \tag{85}$$

The term in the parenthesis is a scalar quantity and thus remain unchanged with respect to the rotation of coordinate system. In the local system x_γ, the line integral becomes:

$$\int_\Gamma (M_{\gamma\delta},_\delta\ \ n_\gamma)\ \ \delta w\ dl \tag{86}$$

where $\gamma = 1$ is the normal direction n, and $\gamma = 2$ is the tangential direction t. The coordinates of the unit normal vector is the local system are $n_\gamma\ \ \{1,\ 0\}$. Hence,

$$\begin{aligned}\int_\Gamma (M_{\gamma\delta},_\delta\ \ n_\gamma)\ \ \delta w\ dl &= \int_\Gamma (M_{1\delta},_\delta\ \ n_1 + M_{2\delta},_\delta\ \ n_2)\ \ \delta w\ dl \\ &= \int_\Gamma M_{1\delta},_\delta\ \ \delta w\ dl\end{aligned} \tag{87}$$

Furthermore, the integrand reads:

$$M_{1\delta},_{\delta} = M_{11},_1 + M_{12},_2 \\ = \frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{12}}{\partial x_2} = \frac{\partial M_{nn}}{\partial x_n} + \frac{\partial M_{nt}}{\partial x_t} \tag{88}$$

and we call it the shear force in the normal direction n and denote:

$$Q_n \equiv M_{n\delta},_{\delta} \qquad \delta = \{1,\ 0\} \text{ or } \{n,\ t\} \tag{89}$$

Now, we can combine two line integrals in the equation of first variation of the total potential energy:

$$\int_{\Gamma} \left(Q_n - \bar{V}_n\right)\ \delta w\ dl \tag{90}$$

How the *remaining* integral is transformed?

$$\int_{\Gamma} (M_{\alpha\beta}\ n_{\beta})\ \delta w,_{\alpha}\ dl = \int_{\Gamma} (M_{\gamma\delta}\ n_{\delta})\ \delta w,_{\gamma}\ dl \tag{91}$$

Because it is a scalar quantity, we simply switch indices from global system (α and β) to local (γ and δ). As before n_{δ} $\{1, 0\}$ so after summing with respect to δ, we have:

$$\int_{\Gamma} (M_{\gamma 1}\ n_1 + M_{\gamma 2}\ n_2)\ \delta w,_{\gamma}\ dl = \int_{\Gamma} M_{\gamma 1}\ \delta w,_{\gamma}\ dl \\ = \int_{\Gamma} M_{\gamma n}\ \delta w,_{\gamma}\ dl \\ = \int_{\Gamma} (M_{nn}\ \delta w,_n + M_{tn}\ \delta w,_t)\ dl \tag{92}$$

The first term can be absorbed with the line integral representing potential energy of bending moment:

$$-\int_{\Gamma} \left(M_{nn} - \bar{M}_{nn}\right)\ \delta w,_n\ dl \tag{93}$$

There remains though one integral which does not fit to anything. Since the boundary term must be equilibrated, it is suspected that this term might belong to the shearing force term, at least partially:

$$\int_{\Gamma} M_{tn}\ \delta w,_t\ dl \qquad \text{transverse term} \tag{94}$$

In order to compare this term with the shearing force term, we have to make this term comparable as far as the kinematic quantity describing variation is concerned. One integral involves δw and the other one $\delta w,_t$. Note that $\partial w,_t = \partial\left(\delta w\right)/\partial x_t$ is the derivative of the function δw in the tangential direction, i.e. direction along the curve Γ. This means that we can integrate by parts along Γ. Thus,

$$M_{tn}\ \delta w,_t = \left(M_{tn}\ \delta w\right),_t - M_{tn},_t\ \delta w \tag{95}$$

$$\int_\Gamma M_{tn}\ \delta w,_t\ \ dl = \int_\Gamma (M_{tn}\ \delta w)\ ,_t\ \ dl - \int_\Gamma M_{tn},_t\ \ \delta w\ dl \tag{96}$$

The first term in the right hand is equal to the value of the integrand calculated at the beginning and end of the integration path:

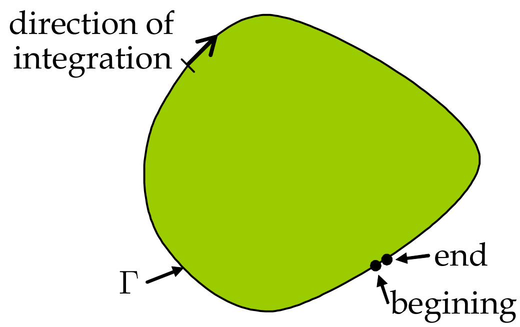
$$\int_\Gamma (M_{tn}\ \delta w)\ ,_t\ \ dl = M_{tn}\ \delta w|_{beginning}^{end} \tag{97}$$

Consider now two cases.

- The contour Γ is a smooth closed curve, so the value at the beginning is equal to the value at the end:

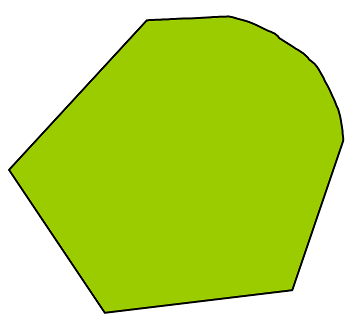
$$M_{tn}\ \delta w|_{end} - M_{tn}\ \delta w|_{beginning} = 0 \tag{98}$$

The term does not give any contribution.



- The contour Γ is piece-wise linear or composed of a finite number, k, of smooth curves with discontinuity. Therefore, the integration should be made in a piece-wise manner. Thus, the continuation of the beginning and end of each should be added:

$$\sum_k M_{tn}\ \delta w|_{beginning}^{end} \tag{99}$$



3.1.3 Equilibrium Equation and Boundary Conditions

Now, we can write the final expression for the first variation of $\delta\Pi$:

$$
\begin{aligned}
\delta\Pi = & \int_S (-M_{\alpha\beta},_{\alpha\beta} - q) \ \delta w \ dS \\
& + \int_\Gamma \left(Q_n - \bar{V}_n\right) \ \delta w \ dl - \int_\Gamma \left(M_{nn} - \bar{M}_{nn}\right) \ \delta w,_n \ dl \\
& - \left(\sum_k M_{tn} \ \delta w|_{beginning}^{end} - \int_\Gamma M_{tn},_t \ \delta w \ dl \right) \\
= & \int_S (-M_{\alpha\beta},_{\alpha\beta} - q) \ \delta w \ dS \\
& + \int_\Gamma \left(V_n - \bar{V}_n\right) \ \delta w \ dl - \int_\Gamma \left(M_{nn} - \bar{M}_{nn}\right) \ \delta w,_n \ dl \\
& - \sum_k M_{tn} \ \delta w|_{beginning}^{end}
\end{aligned}
\tag{100}
$$

where $V_n = Q_n + M_{tn},_t$ is the effective shear force.

In order to make the functional Π stationary under arbitrary variation of the displacement field δw, there must hold:

Equation of Equilibrium

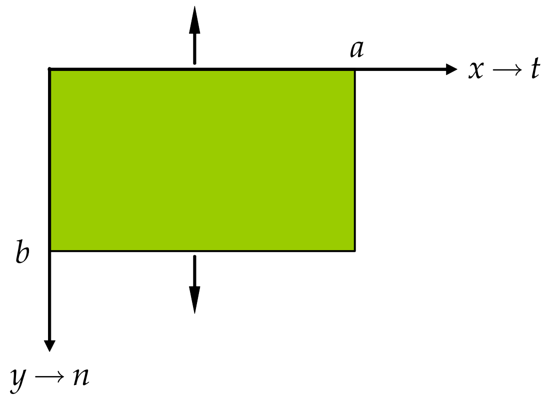
$$
M_{\alpha\beta},_{\alpha\beta} + q = 0 \qquad \text{on } S \tag{101}
$$

Boundary Conditions

$$
\begin{array}{llll}
M_{nn} - \bar{M}_{nn} = 0 & \text{or} & \delta w,_n = 0 & \text{on } \Gamma \\
V_n - \bar{V}_n = 0 & \text{or} & \delta w = 0 & \text{on } \Gamma \\
M_{nt} = 0 & \text{or} & \delta w = 0 & \text{at corner points of the contour } \Gamma
\end{array}
\tag{102}
$$

3.1.4 Specification of Equation for Rectangular Plate

Consider a rectangular plate.



Boundary Conditions For edges parallel to x-axis, the normal direction is the $y$direction.

$$\begin{aligned} M_{yy} - \bar{M}_{yy} = 0 \quad &\text{or} \quad \frac{\partial w}{\partial y} = 0 \\ V_y - \bar{V}_y = 0 \quad &\text{or} \quad w = 0 \end{aligned} \tag{103}$$

where

$$\begin{aligned} V_x &= Q_x + \frac{\partial M_{yx}}{\partial y} \\ V_y &= Q_y + \frac{\partial M_{xy}}{\partial x} \end{aligned} \tag{104}$$

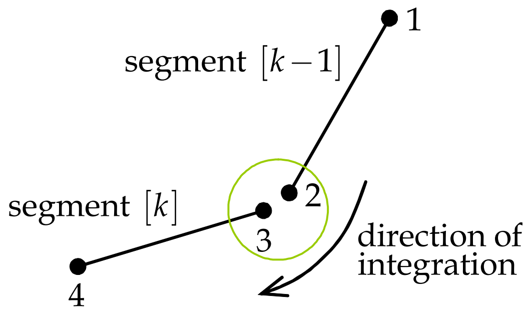
For edges parallel to y-axis, the normal direction is the x-direction.

$$\begin{aligned} M_{xx} - \bar{M}_{xx} = 0 \quad &\text{or} \quad \frac{\partial w}{\partial x} = 0 \\ V_x - \bar{V}_x = 0 \quad &\text{or} \quad w = 0 \end{aligned} \tag{105}$$

where

$$\begin{aligned} V_y &= Q_y + \frac{\partial M_{xy}}{\partial x} \\ V_x &= Q_x + \frac{\partial M_{yx}}{\partial y} \end{aligned} \tag{106}$$

Interpretation of Corner Points



Boundary condition reads:

$$\sum M_{tn}\ \delta w|_{beginning}^{end} = M_{tn}^{[2]}\ \delta w^{[2]} - M_{tn}^{[1]}\ \delta w^{[1]} + M_{tn}^{[4]}\ \delta w^{[4]} - M_{tn}^{[3]}\ \delta w^{[3]} \tag{107}$$

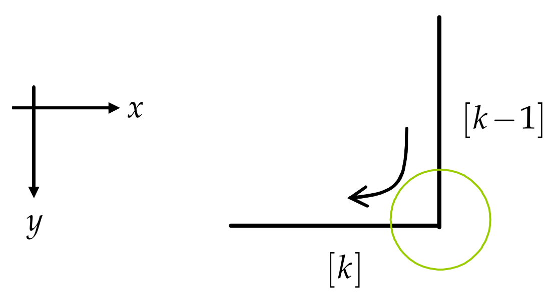
where

$$\delta w^{[3]} = \delta w^{[2]} \tag{108}$$

thus

$$\sum M_{tn}\ \delta w|_{beginning}^{end} = -M_{tn}^{[1]}\ \delta w^{[1]} + \left(M_{tn}^{[2]} - M_{tn}^{[3]}\right)\ \delta w^{[2]} + M_{tn}^{[4]}\ \delta w^{[4]} \tag{109}$$

Consider the right angle.

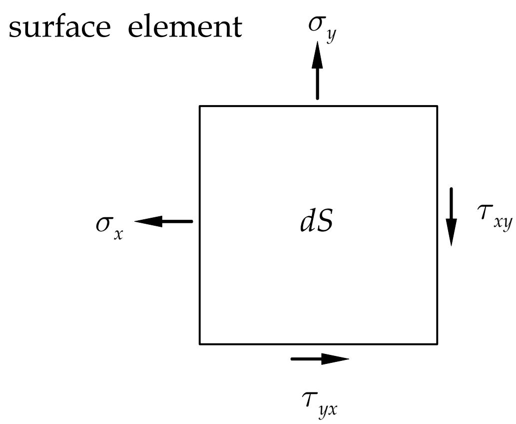


$$\begin{aligned} \text{for the } k-1 \text{ side} \qquad & n = x,\ t = y \\ \text{for the } k \text{ side} \qquad & n = y,\ t = x \end{aligned} \tag{110}$$

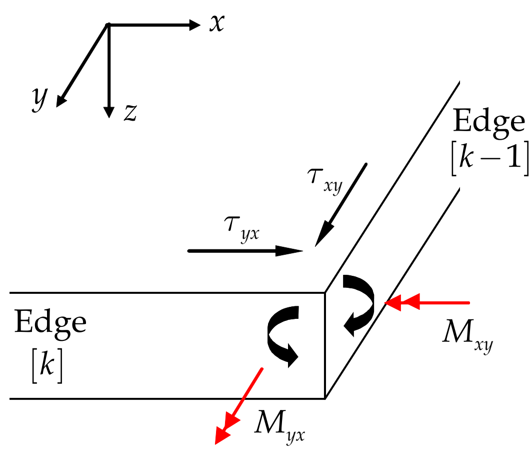
$$\left(M_{tn}\ \delta w|_{beginning}^{end}\right)_{at\ the\ right\ angle} = (M_{xy} - M_{yx})\ \delta w \tag{111}$$

Interpretation of Corner Forces Plane stress:

$$\tau_{xy} = \tau_{yx} \qquad \text{symmetry} \tag{112}$$



Let us place the surface element at the corner.



The shearing stresses produce twisting moments which are in the opposite direction:

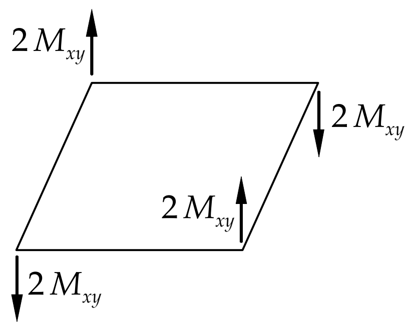
$$M_{xy}^{[k-1]} = -M_{yx}^{[k]} \tag{113}$$

Therefore, the boundary condition at the corner becomes:

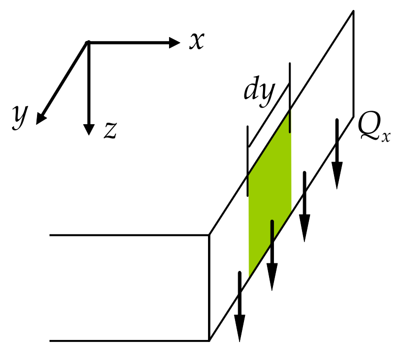
$$M_{tn}\ \delta w|_{beginning}^{end} = \left(M_{xy}^{[k-1]} - M_{yx}^{[k]}\right)\ \delta w = 2\ M_{xy}\ \delta w = 0 \tag{114}$$

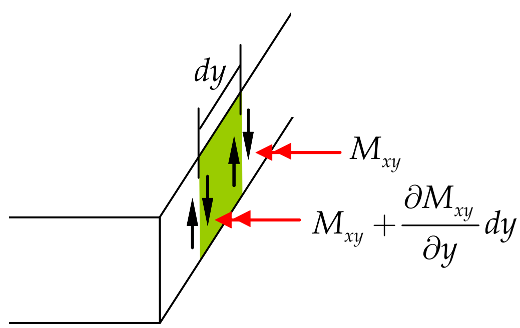
$$\boxed{F_{corner} = 2\ M_{xy}} \tag{115}$$

For the Entire Plate



Interpretation of the Effective Shear V_x





Equilibrium reads:

$$Q_x \ dy + \left(M_{xy} + \frac{\partial M_{xy}}{\partial y} \ dy \right) - M_{xy} \tag{116}$$

$$= \left(Q_x + \frac{\partial M_{xy}}{\partial y} \right) \ dy$$

$$= V_x \ dy$$

$$V_x = Q_x + \frac{\partial M_{xy}}{\partial y} \tag{117}$$

3.2 Bending-Membrane Theory of Plates

3.2.1 Total Potential Energy

The total potential energy of the system Π reads:

$$\Pi = U_b + U_m - V_b - V_m \tag{118}$$

where U_b is the bending strain energy, U_m is the membrane strain energy, V_b is the potential energy of external loading causing flexural response, and V_m is the potential energy of external loading causing membrane response.

Membrane Strain Energy The membrane strain energy reads:

$$U_m = \frac{1}{2} \int_S N_{\alpha\beta}\ \varepsilon^{\circ}_{\alpha\beta}\ dS \tag{119}$$

where

$$\varepsilon^{\circ}_{\alpha\beta} = \frac{1}{2}\left(u_{\alpha},_{\beta} + u_{\beta},_{\alpha}\right) + \frac{1}{2} w,_{\alpha}\ w,_{\beta} \tag{120}$$

Potential Energy of External Forces Evaluation of Boundary Terms

Normal in-plane loading, N_{nn}

$$\int_{\Gamma} \bar{N}_{nn}\ u_n\ dl \tag{121}$$

where u_n is normal in-plane displacement.

Shear in-plane loading, N_{tt}

$$\int_{\Gamma} \bar{N}_{tn}\ u_t\ dl \tag{122}$$

where u_t is shear component of the displacement vector.

Potential Energy of External Forces

$$V_m = \int_{\Gamma} \bar{N}_{nn}\ u_n\ dl + \int_{\Gamma} \bar{N}_{tn}\ u_t\ dl \tag{123}$$

3.2.2 First Variation of the Total Potential Energy

The first variation of the total potential energy reads:

$$\delta\Pi = (\delta U_b - \delta V_b) + (\delta U_m - \delta V_m) \tag{124}$$

The first parenthesis represent the terms considered already in the bending theory of plates. All we have to do is to evaluate the term in the second parenthesis. Here, the first variation of the membrane energy reads:

$$\delta U_m = \int_S N_{\alpha\beta}\ \delta\lambda_{\alpha\beta}\ dS \tag{125}$$

where

$$\delta\lambda_{\alpha\beta} = \frac{1}{2}\left(\delta u_{\alpha},_{\beta} + \delta u_{\beta},_{\alpha}\right) + \frac{1}{2}\left(\delta w,_{\alpha}\ w,_{\beta} + \delta w,_{\beta}\ w,_{\alpha}\right) \tag{126}$$

Because of the symmetry of the tensor of membrane forces:

$$N_{\alpha\beta} = N_{\beta\alpha} \tag{127}$$

by using the characteristics of dummy indices we obtain:

$$N_{\alpha\beta}\ \delta u_{\beta},_{\alpha} = N_{\beta\alpha}\ \delta u_{\beta},_{\alpha} = N_{\alpha\beta}\ \delta u_{\alpha},_{\beta} \tag{128}$$

Now, the first variation of the membrane strain energy reads:

$$\begin{aligned}\delta U_m &= \int_S \left\langle N_{\alpha\beta}\ \left[\frac{1}{2}\left(\delta u_{\alpha},_{\beta} + \delta u_{\beta},_{\alpha}\right) + \frac{1}{2}\left(\delta w,_{\alpha}\ w,_{\beta} + \delta w,_{\beta}\ w,_{\alpha}\right)\right]\right\rangle\ dS \\ &= \int_S \left(N_{\alpha\beta}\ \delta u_{\alpha},_{\beta} + N_{\alpha\beta}\ w,_{\beta}\ \delta w,_{\alpha}\right)\ dS\end{aligned} \tag{129}$$

Note that the displacement vector has now three components:

$$\{u_{\alpha},\ w\} \tag{130}$$

so that there are three independent variations:

$$\{\delta u_{\alpha},\ \delta w\} \tag{131}$$

We expect those to end up with three independent equations of equilibrium. The *first* term of δU_m reads:

$$\begin{aligned}\int_S N_{\alpha\beta}\ \delta u_{\alpha},_{\beta}\ dS &= \int_S \left(N_{\alpha\beta}\ \delta u_{\alpha}\right),_{\beta}\ dS - \int_S N_{\alpha\beta},_{\beta}\ \delta u_{\alpha}\ dS \\ &= \int_\Gamma N_{\alpha\beta}\ \delta u_{\alpha}\ n_{\beta}\ dl - \int_S N_{\alpha\beta},_{\beta}\ \delta u_{\alpha}\ dS \\ &= \int_\Gamma N_{\gamma\delta}\ \delta u_{\gamma}\ n_{\delta}\ dl - \int_S N_{\alpha\beta},_{\beta}\ \delta u_{\alpha}\ dS \\ &= \int_\Gamma N_{\gamma 1}\ \delta u_{\gamma}\ dl - \int_S N_{\alpha\beta},_{\beta}\ \delta u_{\alpha}\ dS \\ &= \int_\Gamma N_{\gamma n}\ \delta u_{\gamma}\ dl - \int_S N_{\alpha\beta},_{\beta}\ \delta u_{\alpha}\ dS \\ &= \int_\Gamma \left(N_{nn}\ \delta u_n + N_{tn}\ \delta u_t\right)\ dl - \int_S N_{\alpha\beta},_{\beta}\ \delta u_{\alpha}\ dS\end{aligned} \tag{132}$$

The *second* term of δU_m reads:

$$\begin{aligned}
\int_S N_{\alpha\beta}\ w,_\beta\ \ \delta w,_\alpha\ \ dS &= \int_S (N_{\alpha\beta}\ w,_\beta\ \ \delta w)\ ,_\alpha\ \ dS - \int_S (N_{\alpha\beta}\ w,_\beta\ )\ ,_\alpha\ \ \delta w\ dS \\
&= \int_\Gamma N_{\alpha\beta}\ w,_\beta\ \ \delta w\ n_\alpha\ dl - \int_S (N_{\alpha\beta}\ w,_\beta\ )\ ,_\alpha\ \ \delta w\ dS \\
&= \int_\Gamma N_{\gamma\delta}\ w,_\delta\ \ \delta w\ n_\gamma\ dl - \int_S (N_{\alpha\beta}\ w,_\beta\ )\ ,_\alpha\ \ \delta w\ dS \\
&= \int_\Gamma N_{1\delta}\ w,_\delta\ \ \delta w\ dl - \int_S (N_{\alpha\beta}\ w,_\beta\ )\ ,_\alpha\ \ \delta w\ dS \\
&= \int_\Gamma (N_{11}\ w,_1 + N_{12}\ w,_2\ )\ \ \delta w\ dl - \int_S (N_{\alpha\beta}\ w,_\beta\ )\ ,_\alpha\ \ \delta w\ dS \\
&= \int_\Gamma (N_{nn}\ w,_n + N_{nt}\ w,_t\ )\ \ \delta w\ dl - \int_S (N_{\alpha\beta}\ w,_\beta\ )\ ,_\alpha\ \ \delta w\ dS
\end{aligned} \tag{133}$$

Now, the variation of external work reads:

$$\delta V_m = \int_\Gamma \bar{N}_{nn}\ \delta u_n\ dl + \int_\Gamma \bar{N}_{tn}\ \delta u_t\ dl \tag{134}$$

3.2.3 Equilibrium Equation and Boundary Conditions

The contribution of the term $(\delta U_m - \delta V_m)$ then becomes:

$$\begin{aligned}
\delta\,(U_m - V_m) &= \int_\Gamma (N_{nn}\ \delta u_n + N_{tn}\ \delta u_t)\ \ dl - \int_S N_{\alpha\beta},_\beta\ \ \delta u_\alpha\ dS \\
&\quad + \int_\Gamma (N_{nn}\ w,_n + N_{nt}\ w,_t\ )\ \ \delta w\ dl - \int_S (N_{\alpha\beta}\ w,_\beta\ )\ ,_\alpha\ \ \delta w\ dS \\
&\quad - \int_\Gamma \bar{N}_{nn}\ \delta u_n\ dl - \int_\Gamma \bar{N}_{tn}\ \delta u_t\ dl \\
&= - \int_S N_{\alpha\beta},_\beta\ \ \delta u_\alpha\ dS + \int_\Gamma \left(N_{nn} - \bar{N}_{nn}\right)\ \delta u_n\ dl + \int_\Gamma \left(N_{tn} - \bar{N}_{tn}\right)\ \delta u_t\ dl \\
&\quad - \int_S (N_{\alpha\beta}\ w,_\beta\ )\ ,_\alpha\ \ \delta w\ dS + \int_\Gamma (N_{nn}\ w,_n + N_{nt}\ w,_t\ )\,\delta w\ dl
\end{aligned} \tag{135}$$

The first three integrals involve independent variations of u_α, i.e. δu_α or $\{\delta u_n,\ \delta u_t\}$. This gives us two independent equations of equilibrium in the plane of the plate:

Equation of Equilibrium I

$$\boxed{N_{\alpha\beta},_\beta = 0 \qquad \text{on } S} \tag{136}$$

and two additional boundary conditions:

Boundary Conditions I

$$\begin{array}{llll} N_{nn} - \bar{N}_{nn} = 0 & \text{or} & \delta u_n = 0 & \text{on } \Gamma \\ N_{tn} - \bar{N}_{tn} = 0 & \text{or} & \delta u_t = 0 & \text{on } \Gamma \end{array} \tag{137}$$

The remaining two integrals involve variation in the out-of-plane displacement δw and thus should be combined with the equation of equilibrium and boundary conditions governing the flexural response. The terms involving surface integral should be added to the equation of equilibrium:

Equation of Equilibrium II

$$M_{\alpha\beta},_{\alpha\beta} + (N_{\alpha\beta}\ w,_{\beta}),_{\alpha} + q = 0 \qquad \text{on } S \tag{138}$$

where the second term in the left hand is the new term arising from the finite rotation.

The term with the line integral should be added to the corresponding term involving variation δw:

$$\int_{\Gamma} \left(V_n + N_{nn}\ w,_n + N_{nt}\ w,_t - \bar{V}_n\right)\ \delta w\ dl = 0 \tag{139}$$

The generalized boundary conditions reads:

Boundary Conditions II- (A)

$$V_n + N_{nn}\ w,_n + N_{nt}\ w,_t - \bar{V}_n = 0 \qquad \text{or} \qquad \delta w = 0 \quad \text{on } \Gamma \tag{140}$$

where the second and third terms in the left hand side of the first equation are the new terms arising from the finite rotation.

If the boundaries of the plate are kept undeformed $w,_t = 0$ (simply supported or clamped plate), then the boundary condition is satisfied:

$$V_n + N_{nn}\ w,_n - \bar{V}_n = 0 \qquad \text{or} \qquad \delta w = 0 \,\text{on } \Gamma \tag{141}$$

Physically, the additional terms represent the contribution of the axial force to the vertical equilibrium. Using the in-plane equilibrium, $N_{\alpha\beta},_{\beta} = 0$, the out-of-plane equilibrium can be transformed to the form:

$$M_{\alpha\beta},_{\alpha\beta} + N_{\alpha\beta},_{\alpha}\ w,_{\beta} + N_{\alpha\beta}\ w,_{\alpha\beta} + q = 0 \qquad \text{on } S \tag{142}$$

Equation of Equilibrium II'

$$M_{\alpha\beta},_{\alpha\beta} + N_{\alpha\beta}\ w,_{\alpha\beta} + q = 0 \qquad \text{on } S \tag{143}$$

which is called as the von Karman equation. Note that $N_{\alpha\beta}$ is related through the Hook's law with the gradient of the in-plane displacement u_α, i.e. $N_{\alpha\beta} = N_{\alpha\beta}\,(u_\alpha)$. Therefore, the new term $N_{\alpha\beta}\ w,_{\alpha\beta}$ represents in fact coupling between in-plane and out-out-plane deformation.

To make derivation complete, the final boundary conditions which do not changed from the bending theory of plate are presented:

Boundary Conditions II- (B)

$$\begin{array}{llll} M_{nn} - \bar{M}_{nn} = 0 & \text{or} & \delta w,_n = 0 & \text{on } \Gamma \\ M_{nt} = 0 & \text{or} & \delta w = 0 & \text{at corner points of the contour } \Gamma \end{array} \tag{144}$$

4 General Theories of Plate

4.1 Bending Theory of Plates

4.1.1 Derivation of the Plate Bending Equation

Then, groups of equations!

- Equilibrium

$$M_{\alpha\beta},_{\alpha\beta} + q = 0 \quad \text{on} \;\; S \tag{145}$$

- Geometry

$$\kappa_{\alpha\beta} = -w,_{\alpha\beta} \tag{146}$$

- Elasticity

$$M_{\alpha\beta} = D \;\; [(1-\nu) \;\; \kappa_{\alpha\beta} + \nu \;\kappa_{\gamma\gamma} \;\delta_{\alpha\beta}] \tag{147}$$

Eliminating curvature $\kappa_{\alpha\beta}$ between Eq. (146) and (147), we obtain:

$$M_{\alpha\beta} = -D \;\; [(1-\nu) \;\; w,_{\alpha\beta} + \nu \; w,_{\gamma\gamma} \;\; \delta_{\alpha\beta}] \tag{148}$$

Substituting Eq. (148) into Eq. (145) reads:

$$\begin{aligned} -D \;\; [(1-\nu) \;\; w,_{\alpha\beta} + \nu \; w,_{\gamma\gamma} \;\; \delta_{\alpha\beta}]\,,_{\alpha\beta} + q = 0 \\ -D \;\; [(1-\nu) \;\; w,_{\alpha\beta\alpha\beta} + \nu \; w,_{\gamma\gamma\alpha\beta} \;\; \delta_{\alpha\beta}] + q = 0 \end{aligned} \tag{149}$$

Note that the components of the Kronecker "$\delta_{\alpha\beta}$" tensor are constant and thus are not subjected to differentiation:

$$\delta_{\alpha\beta} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \quad \text{or} \quad \delta_{\alpha\beta} = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{cases} \tag{150}$$

Also, note that only these components:

$$\Box_{\alpha\beta} \; \delta_{\alpha\beta} = \Box_{\alpha\alpha} \tag{151}$$

survive in the matrix multiplication for which $\alpha = \beta$. Therefore, Eq. (149) now reads:

$$-D \;\; [(1-\nu) \;\; w,_{\alpha\beta\alpha\beta} + \nu \; w,_{\gamma\gamma\alpha\alpha}] + q = 0 \tag{152}$$

Because "$\gamma\gamma$" are "dummy" indices, they can be replaced by any other indices, for example "$\beta\beta$."

$$-D \;\; [(1-\nu) \;\; w,_{\alpha\beta\alpha\beta} + \nu \; w,_{\beta\beta\alpha\alpha}] + q = 0 \tag{153}$$

The order of differentiation does not matter:

$$w,_{\alpha\beta\alpha\beta} = w,_{\alpha\alpha\beta\beta} = w,_{\beta\beta\alpha\alpha}$$

Thus, two terms in Eq. (153) can now be added to give the plate bending equation:

$$D\ w_{,\alpha\alpha\beta\beta} = q \qquad \text{for} \ \ \alpha, \beta = 1, 2 \tag{154}$$

Here, the index notation can be expended:

$$\begin{aligned} w_{,\alpha\alpha\beta\beta} &= w_{,11\beta\beta} + w_{,22\beta\beta} \\ &= w_{,1111} + w_{,2211} + w_{,1122} + w_{,2222} \\ &= w_{,1111} + 2\ w_{,1122} + w_{,2222} \end{aligned} \tag{155}$$

Now, letting "$1 \to x$", "$2 \to y$" leads:

$$D\ \left(\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial x^2} + \frac{\partial^4 w}{\partial y^4} \right) = q\,(x, y) \tag{156}$$

Alternative notation can be:

$$\boxed{D\ \nabla^4 w = q} \tag{157}$$

where Laplacian $\nabla^2 w$ reads:

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \tag{158}$$

and bi-Laplican $\nabla^4 w$ reads:

$$\begin{aligned} \nabla^4 w &= \nabla^2 \left(\nabla^2 w \right) \\ &= \frac{\partial^2}{\partial x^2} \left(\nabla^2 w \right) + \frac{\partial^2}{\partial y^2} \left(\nabla^2 w \right) \\ &= \frac{\partial^2}{\partial x^2} \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + \frac{\partial^2}{\partial y^2} \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) \\ &= \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial x^2} + \frac{\partial^4 w}{\partial y^4} \end{aligned} \tag{159}$$

4.1.2 Reduction to a System of Two Second Order Equations

Denote

$$D\ w_{,\alpha\alpha} = -M \tag{160}$$

Then, from the equilibrium equation:

$$\left[D\ w_{,\alpha\alpha} \right]_{,\beta\beta} = q \tag{161}$$

we obtain a system of two linear partial differential equations of the second order:

$$\boxed{\begin{aligned} & M_{,\beta\beta} = -q \\ & D\ w_{,\alpha\alpha} = -M \end{aligned}} \tag{162}$$

or

$$\begin{cases} \frac{\partial^2 M}{\partial x^2} + \frac{\partial^2 M}{\partial y^2} = -q \\ D \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) = -M \end{cases} \tag{163}$$

What is "M" ? Let us calculate $M_{\alpha\alpha}$:

$$\begin{aligned} M_{\alpha\alpha} &= M_{11} + M_{22} \\ &= D \ [(1-\nu) \ \kappa_{11} + \nu \ (\kappa_{11} + \kappa_{22}) \ \delta_{11}] \\ &\quad + D \ [(1-\nu) \ \kappa_{22} + \nu \ (\kappa_{11} + \kappa_{22}) \ \delta_{22}] \\ &= D \ [(1+\nu) \ (\kappa_{11} + \kappa_{22})] \\ &= D \ (1+\nu) \ \kappa_{\alpha\alpha} \end{aligned} \tag{164}$$

or

$$\frac{M_{\alpha\alpha}}{1+\nu} = D \ \kappa_{\alpha\alpha} = -D \ w_{,\alpha\alpha} = M \tag{165}$$

Therefore,

$$\begin{aligned} M_{\alpha\alpha} &= M \ (1+\nu) \\ &= D \ \kappa_{\alpha\alpha} \ (1+\nu) \end{aligned} \tag{166}$$

Now, moment sum reads:

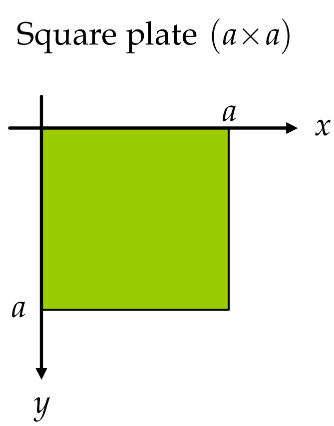
$$\boxed{M = D \ \kappa_{\alpha\alpha}} \tag{167}$$

and in expanded notation it reads:

$$M = D \ [\kappa_{xx} + \kappa_{yy}] \tag{168}$$

4.1.3 Exercise 1: Plate Solution

Consider a simply supported plate.



Boundary Condition General boundary condition reads:

$$\begin{aligned} \left(M_{nn} - \bar{M}_{nn}\right)\ w_{,n} = 0 \quad & \text{on } \Gamma \\ \left(V_n - \bar{V}_n\right)\ w = 0 \quad & \text{on } \Gamma \end{aligned} \tag{169}$$

$$\begin{aligned} \bar{M}_{nn} = 0 \qquad \Rightarrow M_{nn} = 0 \quad & \text{on } \Gamma \\ w = 0 \quad & \text{on } \Gamma \end{aligned} \tag{170}$$

$$\begin{aligned} w = 0 \quad & \text{at} \quad x = 0 \text{ and } x = a\ ,\ 0 \le y \le a \\ w = 0 \quad & \text{at} \quad y = 0 \text{ and } y = a\ ,\ 0 \le x \le a \end{aligned} \tag{171}$$

$$\begin{aligned} M_{xx} = 0 \quad & \text{at} \quad x = 0 \text{ and } x = a\ ,\ 0 \le y \le a \\ M_{yy} = 0 \quad & \text{at} \quad y = 0 \text{ and } y = a\ ,\ 0 \le x \le a \end{aligned} \tag{172}$$

Loading Condition Assume for simplicity the sinusoidal load distribution:

$$q\left(x, y\right) = q_0\ \sin\left(\frac{\pi\ x}{a}\right)\ \sin\left(\frac{\pi\ y}{a}\right) \tag{173}$$

where q_0 is a pressure intensity.

Solution of Problem The solution of the form

$$w\left(x, y\right) = w_0\ \sin\left(\frac{\pi\ x}{a}\right)\ \sin\left(\frac{\pi\ y}{a}\right) \tag{174}$$

satisfy both the boundary conditions and the governing equations (see below).

Plate Bending Equation Substituting Eq. (173) and (174) into the plate bending equation (156), one gets:

$$\begin{aligned} \left\{D\ w_0\left[\left(\frac{\pi}{a}\right)^4 + 2\ \left(\frac{\pi}{a}\right)^4 + \left(\frac{\pi}{a}\right)^4\right] - q_0\right\} \sin\left(\frac{\pi\ x}{a}\right)\ \sin\left(\frac{\pi\ y}{a}\right) = 0 \\ \left\{4\ D\ w_0\ \left(\frac{\pi}{a}\right)^4 - q_0\right\} \sin\left(\frac{\pi\ x}{a}\right)\ \sin\left(\frac{\pi\ y}{a}\right) = 0 \end{aligned} \tag{175}$$

In order to satisfy the above equation for all values of x and y, the coefficient in the bracket must vanish. This gives:

$$w_0 = \frac{q_0}{4\ D}\left(\frac{a}{\pi}\right)^4 \tag{176}$$

where $D = \left(Eh^3\right) / \left[12\left(1 - \nu^2\right)\right]$.

Bending Moments The various bending moments are given by:

$$\begin{aligned} M_{xx} &= -D\ \left[\frac{\partial^2 w}{\partial x^2} + \nu\ \frac{\partial^2 w}{\partial y^2}\right] = D\ \left(1 + \nu\right)\ \left(\frac{\pi}{a}\right)^2\ w_0\ \sin\left(\frac{\pi\ x}{a}\right)\ \sin\left(\frac{\pi\ y}{a}\right) \\ M_{yy} &= -D\ \left[\frac{\partial^2 w}{\partial y^2} + \nu\ \frac{\partial^2 w}{\partial x^2}\right] = D\ \left(1 + \nu\right)\ \left(\frac{\pi}{a}\right)^2\ w_0\ \sin\left(\frac{\pi\ x}{a}\right)\ \sin\left(\frac{\pi\ y}{a}\right) \\ M_{xy} &= -D\ \left(1 - \nu\right)\ \frac{\partial^2 w}{\partial x \partial y} = -D\ \left(1 - \nu\right)\ \left(\frac{\pi}{a}\right)^2\ w_0\ \cos\left(\frac{\pi\ x}{a}\right)\ \cos\left(\frac{\pi\ y}{a}\right) \end{aligned} \tag{177}$$

Shear Components The shear components Q_x and Q_y are:

$$Q_x = \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} \qquad (178)$$
$$Q_y = \frac{\partial M_{yy}}{\partial y} + \frac{\partial M_{xy}}{\partial x}$$

Now, using the previously obtained bending moments, we get the shear components in the interior of the plate:

$$Q_x = 2\ D\ \left(\frac{\pi}{a}\right)^3\ w_0\ \cos\left(\frac{\pi\ x}{a}\right)\ \sin\left(\frac{\pi\ y}{a}\right) \qquad (179)$$
$$Q_x = 2\ D\ \left(\frac{\pi}{a}\right)^3\ w_0\ \sin\left(\frac{\pi\ x}{a}\right)\ \cos\left(\frac{\pi\ y}{a}\right)$$

Effective Shear Components Next, let us computer the effective shear components:

$$V_x = Q_x + \frac{\partial M_{xy}}{\partial y} \qquad (180)$$
$$V_y = Q_y + \frac{\partial M_{xy}}{\partial x}$$

Using the previous results, we get:

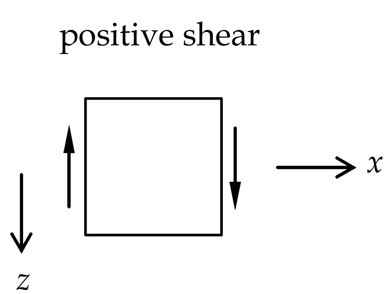
$$V_x = (3-\nu)\ D\ \left(\frac{\pi}{a}\right)^3\ w_0\ \cos\left(\frac{\pi\ x}{a}\right)\ \sin\left(\frac{\pi\ y}{a}\right) \qquad (181)$$
$$V_y = (3-\nu)\ D\ \left(\frac{\pi}{a}\right)^3\ w_0\ \sin\left(\frac{\pi\ x}{a}\right)\ \cos\left(\frac{\pi\ y}{a}\right)$$

We now need to evaluate the effective shear on the boundaries:

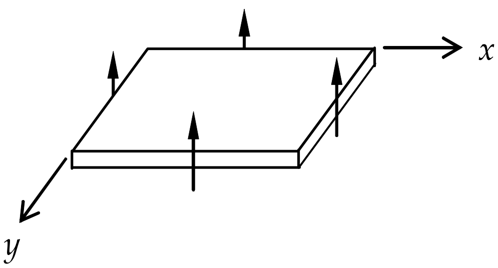
	$V_x/\left[(3-\nu)\ D\ \left(\frac{\pi}{a}\right)^3\right]$	$V_y/\left[(3-\nu)\ D\ \left(\frac{\pi}{a}\right)^3\right]$
$x=0$	$w_0\ \sin\left(\frac{\pi\ y}{a}\right)$	0
$x=a$	$-w_0\ \sin\left(\frac{\pi\ y}{a}\right)$	0
$y=0$	0	$w_0\ \sin\left(\frac{\pi\ x}{a}\right)$
$y=a$	0	$-w_0\ \sin\left(\frac{\pi\ x}{a}\right)$

(182)

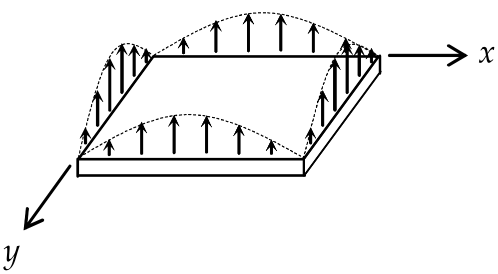
Because our sign convention is:



in our case, shear along the boundary is:



From the above results, we can plot the shear distribution:



Force Balance Integrating the effective shear along the boundary, we get:

$$R = \int_L V_n \, dx_t = 4 \int_0^a V_x|_{x=0} \; dy = 4 \; (3-\nu) \; D \; \left(\frac{\pi}{a}\right)^3 \; w_0 \int_0^a \sin\left(\frac{\pi \; y}{a}\right) \; dy \quad (183)$$

Then, the reduction force due to effective shear on boundaries reads:

$$\boxed{R = 2 \; (3-\nu) \; q_0 \; \left(\tfrac{a}{\pi}\right)^2} \quad (184)$$

Now, let us complete the total load acting on the plate:

$$P = \int_S q\,(x,y) \;\; dS = \int_0^a \int_0^a q_0 \; \sin\left(\frac{\pi \; x}{a}\right) \; \sin\left(\frac{\pi \; y}{a}\right) \; dx \; dy \quad (185)$$

Then, the total external load acting on the plate reads:

$$\boxed{P = 4 \; q_0 \; \left(\tfrac{a}{\pi}\right)^2} \quad (186)$$

Notice that R and P do not balance! We did not include the corner forces. These are given by:

$$(F_{corner})_{x_0,y_0} = 2 \;\; (M_{xy})|_{x=x_0,y=y_0} \quad (187)$$

Because of the symmetry, all four forces are equal. So, compute the corner force at $x = y = 0$, $(F_{corner})_{0,0}$:

$$(F_{corner})_{0,0} = 2 \left[\cos\left(\frac{\pi\ x}{a}\right) \ \cos\left(\frac{\pi\ y}{a}\right)\right]\Big|_{0,0} \\ = -\ 2\ D\ \ (1-\nu)\ \left(\frac{\pi}{a}\right)^2\ w_0 \tag{188}$$

Now, the vertical force balance is satisfied:

$$\boxed{R + 4\ F_{corner} = P} \tag{189}$$

$$2\ \ (3-\nu)\ \ q_0\ \left(\frac{a}{\pi}\right)^2 -\ 8\ D\ \ (1-\nu)\ \left(\frac{\pi}{a}\right)^2\ w_0 = 4\ q_0\ \left(\frac{a}{\pi}\right)^2 \\ 2\ \ (3-\nu)\ \ q_0\ \left(\frac{a}{\pi}\right)^2 -\ 2\ \ (1-\nu)\ \ q_0\ \left(\frac{\pi}{a}\right)^2 = 4\ q_0\ \left(\frac{a}{\pi}\right)^2 \tag{190}$$

4.1.4 Exercise 2: Comparison between Plate and Beam Solution

Plate Solution For a square simply supported plate under loading $q_{plate}\,(x, y)$ given by:

$$q_{plate}\,(x, y) = (q_0)_{plate}\ \sin\left(\frac{\pi\ x}{a}\right)\ \sin\left(\frac{\pi\ y}{a}\right) \tag{191}$$

we found that the plate deflection is:

$$w_{plate}\,(x, y) = (w_0)_{plate}\ \sin\left(\frac{\pi\ x}{a}\right)\ \sin\left(\frac{\pi\ y}{a}\right) \tag{192}$$

with:

$$(w_0)_{plate} = \frac{(q_0)_{plate}}{4\ D}\left(\frac{a}{\pi}\right)^4 \\ = \frac{3\ \ (1-\nu^2)\ \ (q_0)_{plate}}{E\ h^3}\left(\frac{a}{\pi}\right)^4 \tag{193}$$

For the plate, the total load is given by:

$$P_{plate} = \int_0^a \int_0^a (q_0)_{plate}\ \sin\left(\frac{\pi\ x}{a}\right)\ \sin\left(\frac{\pi\ y}{a}\right)\ dy\ dx \\ = 4\ \ (q_0)_{plate}\ \left(\frac{a}{\pi}\right)^2 \tag{194}$$

Wide Beam Solution For a wide beam under line loading given by:

$$q_{beam}(x,y) = (q_0)_{beam} \sin\left(\frac{\pi x}{a}\right) \tag{195}$$

we need to compute the central deflection $(w_0)_{beam}$ from:

$$E\ I\ w''''_{beam} = q_{beam}(x) \tag{196}$$

where $I = ah^3/12$. Assuming the deflection $w_{beam}(x)$:

$$w_{beam}(x) = (w_0)_{beam} \sin\left(\frac{\pi x}{a}\right) \tag{197}$$

we get:

$$E\ I \left(\frac{\pi}{a}\right)^4 (w_0)_{beam} \sin\left(\frac{\pi x}{a}\right) = (q_0)_{beam} \sin\left(\frac{\pi x}{a}\right) \tag{198}$$

Thus,

$$\begin{aligned}(w_0)_{beam} &= \frac{(q_0)_{beam}}{E\ I}\left(\frac{a}{\pi}\right)^4 \\ &= \frac{12\ (q_0)_{beam}}{E\ a\ h^3}\left(\frac{a}{\pi}\right)^4\end{aligned} \tag{199}$$

Now, let us compute the total forces:

$$\begin{aligned}P_{beam} &= \int_0^a (q_0)_{beam} \sin\left(\frac{\pi x}{a}\right) dx \\ &= 2\ (q_0)_{beam} \frac{a}{\pi}\end{aligned} \tag{200}$$

Comparison For both total forces to be equal, we need to have:

$$P_{plate} = P_{beam} \tag{201}$$

$$4\ (q_0)_{plate} \left(\frac{a}{\pi}\right)^2 = 2\ (q_0)_{beam} \frac{a}{\pi}$$

$$(q_0)_{beam} = 2\ (q_0)_{plate} \frac{a}{\pi} \tag{202}$$

With a concentrated load, the beam deflection now becomes:

$$\begin{aligned}(w_0)_{beam} &= \frac{(q_0)_{beam}}{E\ I}\left(\frac{a}{\pi}\right)^4 \\ &= \frac{24}{\pi}\frac{(q_0)_{plate}}{E\ h^3}\left(\frac{a}{\pi}\right)^4\end{aligned} \tag{203}$$

We now can compute the ratio of central deflections:

$$\begin{aligned}\alpha &= \frac{(w_0)_{plate}}{(w_0)_{beam}} = \frac{\frac{3\ (1-\nu^2)\ (q_0)_{plate}}{E\ h^3}\left(\frac{a}{\pi}\right)^4}{\frac{24}{\pi}\frac{(q_0)_{plate}}{E\ h^3}\left(\frac{a}{\pi}\right)^4} \\ &= \frac{\pi}{8}\left(1-\nu^2\right) \simeq 0.36\end{aligned} \tag{204}$$

The above equation means that under the same total load, a plate is three times stiffer than a wide beam. The ratio α will vary slightly depending on the load distribution (sinusoidal, uniform, concentrated load, etc.).

4.1.5 Exercise 3: Finite Difference Solution of the Plate Bending Problem

Governing Equations read:

$$\begin{aligned} \nabla^2 M &= -q \\ \nabla^2 w &= -\tfrac{M}{D} \end{aligned} \tag{205}$$

or in the component notation they read:

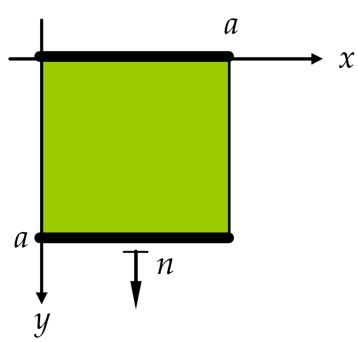
$$\begin{aligned} \tfrac{\partial^2 M}{\partial x^2} + \tfrac{\partial^2 M}{\partial y^2} &= -q \\ \tfrac{\partial^2 w}{\partial x^2} + \tfrac{\partial^2 w}{\partial y^2} &= -\tfrac{M}{D} \end{aligned} \tag{206}$$

where M is the moment sum defined by:

$$M = \frac{M_{\alpha\alpha}}{1+\nu} = D\ \kappa_{\alpha\alpha} \tag{207}$$

Case of Simply Supported Plate The boundary condition of a simple supported plate reads:

$$\begin{aligned} w &= 0 \qquad \text{on} \ \ \Gamma \\ M_{nn} &= 0 \qquad \text{on} \ \ \Gamma \end{aligned} \tag{208}$$



For sides parallel to x-axis (thick lines), one gets:

$$M_{nn} = M_{yy} = 0 \tag{209}$$

$$w = 0 \quad \rightarrow \quad \frac{dw}{dx} = 0 \quad \rightarrow \quad \frac{d^2 w}{dx^2} = 0 \quad \rightarrow \quad \kappa_{xx} = 0 \tag{210}$$

From the general constitutive equations,

$$\begin{aligned} M_{yy} &= D\ \ [\kappa_{yy} + \nu\ \kappa_{xx}] \\ 0 &= D\ \ [\kappa_{yy} + \nu \cdot 0] \qquad \rightarrow \qquad \kappa_{yy} = 0 \end{aligned} \tag{211}$$

Therefore,

$$M = D\ \ [\kappa_{\alpha\alpha} + \kappa_{\beta\beta}] = D\ \ [0 + 0] = 0 \tag{212}$$

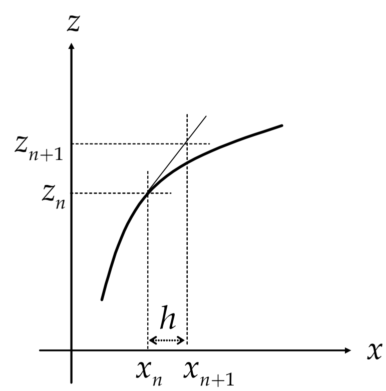
Similar derivation can be performed for two edges parallel to y-axis. Then, $M = 0$. It can be concluded that for a simply supported plate the following boundary conditions hold:

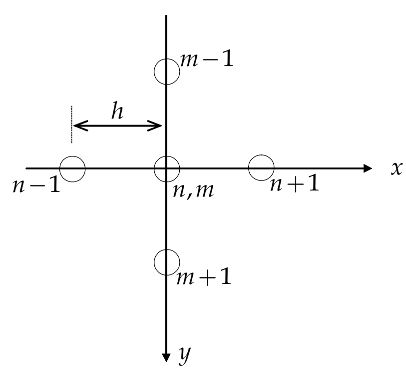
$$\begin{cases} \frac{\partial^2 M}{\partial x^2} + \frac{\partial^2 M}{\partial y^2} = -q & \text{in } S \\ M = 0 & \text{on } \Gamma \end{cases} \tag{213}$$

$$\begin{cases} \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{M}{D} & \text{in } S \\ w = 0 & \text{on } \Gamma \end{cases} \tag{214}$$

Therefore, the above two boundary value problems are uncoupled.

The Finite Difference Technique An approximation to the first and second derivatives.





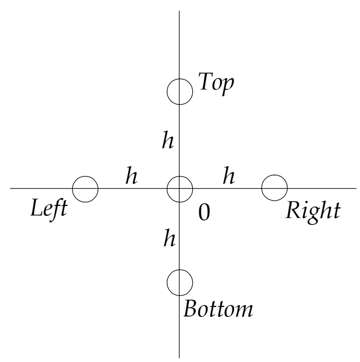
$$\left.\frac{dz}{dx}\right|_n^{backward} \simeq \frac{z_n - z_{n-1}}{h} \tag{215}$$

$$\left.\frac{dz}{dx}\right|_{n+1}^{forward} \simeq \frac{z_{n+1} - z_n}{h}$$

$$\left.\frac{d^2z}{dx^2}\right|_n = \frac{d}{dx}\left[\frac{dz}{dx}\right] \tag{216}$$
$$= \frac{\left(\frac{dz}{dx}\right)_{n+1} - \left(\frac{dz}{dx}\right)_n}{h}$$
$$= \frac{\frac{z_{n+1}-z_n}{h} - \frac{z_n-z_{n-1}}{h}}{h}$$
$$= \frac{z_{n+1} - 2\ z_n + z_{n-1}}{h^2}$$

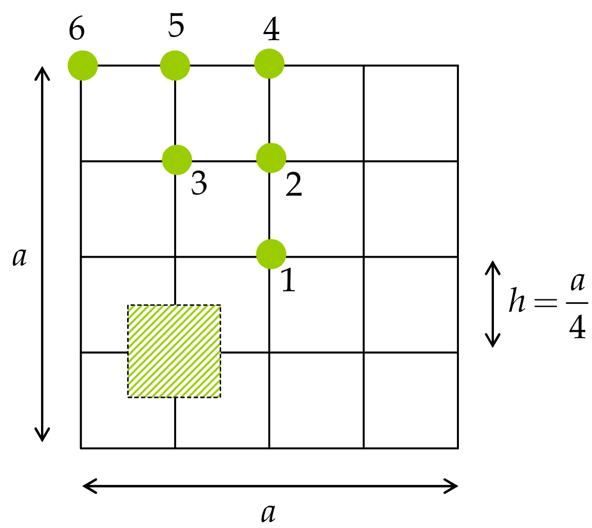
$$\left.\frac{d^2z}{dy^2}\right|_m = \frac{z_{m+1} - 2\ z_m + z_{m-1}}{h^2} \tag{217}$$

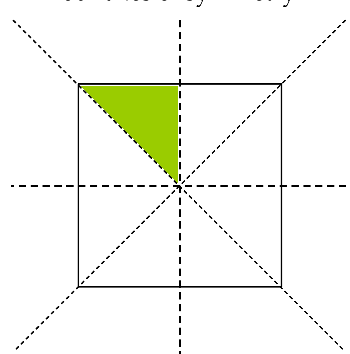
$$\nabla^2 z = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \tag{218}$$
$$= \frac{1}{h^2}\left(z_{n+1} - 2\ z_n + z_{n-1} + z_{m+1} - 2\ z_m + z_{m-1}\right)$$



$$\nabla^2 z = \frac{1}{h^2}\left(z_T + z_B + z_L + z_R - 4\ z_0\right) \tag{219}$$

Divide the plate into sixteen identical squares and distinguish six representative nodes: three in the interior and three at the boundary. Because of symmetry, it is enough to consider only an eighth of the plate.





Determination of Moment For each interior point (1, 2, 3), we write equation $\nabla^2 M = -q$. For each boundary point (4, 5, 6), we write boundary condition $M = 0$ (uniform pressure).

$$
\begin{aligned}
&\text{Point 1:} && 4\ M_2 - 4\ M_1 = -\frac{q\ a^2}{16} \\
&\text{Point 2:} && M_1 + M_4 + 2\ M_3 - 4\ M_2 = -\frac{q\ a^2}{16} \\
&\text{Point 3:} && 2\ M_5 + 2\ M_2 - 4\ M_3 = -\frac{q\ a^2}{16} \\
&\text{Point 4:} && M_4 = 0 \\
&\text{Point 5:} && M_5 = 0 \\
&\text{Point 6:} && M_6 = 0
\end{aligned}
\tag{220}
$$

Substituting three last equations of Eq. (220) into the first three equations of Eq. (220), one ends up with the following system of linear algebraic equations:

$$\begin{cases} 4\ M_2 - 4\ M_1 = -\frac{q\ a^2}{16} \\ M_1 + 2\ M_3 - 4\ M_2 = -\frac{q\ a^2}{16} \\ 2\ M_2 - 4\ M_3 = -\frac{q\ a^2}{16} \end{cases} \tag{221}$$

whose solution is:

$$\begin{aligned} M_1 &= \frac{9}{128}\ q\ a^2 \\ M_2 &= \frac{7}{128}\ q\ a^2 \\ M_3 &= \frac{11}{256}\ q\ a^2 \end{aligned} \tag{222}$$

At the plate center, $M_{xx} = M_{yy}$ so that:

$$M = \frac{M_{xx} + M_{yy}}{1+\nu} = \frac{2\ M_{xx}}{1+\nu} \tag{223}$$

$$M_{xx} = \frac{1}{2}\ (1+\nu)\ M \tag{224}$$

At the center, $M = M_1$, thus,

$$\begin{aligned} M_{xx} &= \frac{1}{2}\ (1+\nu)\ M_1 \\ &= \frac{1}{2}\ (1+\nu)\ \frac{9}{128}\ q\ a^2 \\ &= 0.0457\ q\ a^2 \end{aligned} \tag{225}$$

This is 4.6% less than the exact solution which is $(M_{xx})_{exact} = 0.0479\ q\ a^2$ from the text book.

Determination of Deflection For each interior point (1, 2, 3), we write equation $\nabla^2 w = -M/D$. For each boundary point (4, 5, 6), we write boundary condition $w = 0$.

$$\begin{aligned} &\text{Point 1:} && 4\ w_2 - 4\ w_1 = -\frac{M_1}{D}\frac{a^2}{16} = -\left(\frac{9}{128}\frac{qa^2}{D}\right)\ \frac{a^2}{16} \\ &\text{Point 2:} && w_1 + w_4 + 2\ w_3 - 4\ w_2 = -\frac{M_2}{D}\frac{a^2}{16} = -\left(\frac{7}{128}\ \frac{q\ a^2}{D}\right)\ \frac{a^2}{16} \\ &\text{Point 3:} && 2\ w_5 + 2\ w_2 - 4\ w_3 = -\frac{M_3}{D}\frac{a^2}{16} = -\left(\frac{11}{256}\ \frac{q\ a^2}{D}\right)\ \frac{a^2}{16} \\ &\text{Point 4:} && w_4 = 0 \\ &\text{Point 5:} && w_5 = 0 \\ &\text{Point 6:} && w_6 = 0 \end{aligned} \tag{226}$$

Similarly,

$$\begin{cases} 4\ w_2 - 4\ w_1 = -\frac{9}{2048}\frac{qa^4}{D} \\ w_1 + 2\ w_3 - 4\ w_2 = -\frac{7}{2048}\ \frac{q\ a^4}{D} \\ 2\ w_2 - 4\ w_3 = -\frac{11}{4096}\ \frac{q\ a^4}{D} \end{cases} \tag{227}$$

Finally, the finite difference solution is:

$$w_1 = \frac{33}{8196}\frac{q\ a^4}{D} = 0.00403\ \frac{q\ a^4}{D} \tag{228}$$
$$w_2 = \frac{3}{1024}\frac{q\ a^4}{D} = 0.00293\ \frac{q\ a^4}{D}$$
$$w_3 = \frac{35}{16384}\frac{q\ a^4}{D} = 0.00214\ \frac{q\ a^4}{D}$$

On the other hand, the exact deflection of the center point is:

$$(w_1)_{exact} = 0.00416\ \frac{q\ a^4}{D} \tag{229}$$

Thus, the error of the finite different solution is 3.1%.

4.2 Membrane Theory of Plates

4.2.1 Plate Membrane Equation

Assume that the bending rigidity is zero, $D = 0$. The plate becomes now a membrane.

- Equilibrium of in-plane equation

$$N_{\alpha\beta},_{\alpha} = 0 \qquad \text{on } S \tag{230}$$

  Equilibrium of out-of-plane equation

$$N_{\alpha\beta}\ w,_{\alpha\beta} + q = 0 \qquad \text{on } S \tag{231}$$

- Strain-displacement relation

$$\varepsilon^{\circ}_{\alpha\beta} = \frac{1}{2}\left(u_{\alpha},_{\beta} + u_{\alpha},_{\beta}\right) + \frac{1}{2}\ w,_{\alpha}\ \ w,_{\beta} \tag{232}$$

- Constitutive equation

$$N_{\alpha\beta} = C\ \left[(1-\nu)\ \varepsilon^{\circ}_{\alpha\beta} + \nu\ \varepsilon^{\circ}_{\gamma\gamma}\ \delta_{\alpha\beta}\right] \tag{233}$$

  where $C = Eh/\left(1-\nu^2\right)$.

This is a non-linear system of equation which is difficult to solve. Note that corresponding system of equation for the plate bending was linear.

4.2.2 Plate Equation for the Circular Membrane

Cylindrical coordinate system is composed of u_r, u_θ, $u_z = w$.

- Equilibrium of in-plane equation

$$r\frac{\partial N_{rr}}{\partial r} + N_{rr} - N_{\theta\theta} = 0 \qquad \text{on} \quad S \tag{234}$$

Equilibrium of out-of-plane equation

$$\frac{\partial}{\partial r}\left[N_{rr}\ \frac{\partial w}{\partial r}\ r\right] + r\ q = 0 \qquad \text{on} \quad S \tag{235}$$

- Strain-displacement relation

$$\begin{aligned} \lambda_{rr} &= \frac{\partial u_r}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 \\ \lambda_{\theta\theta} &= \frac{u_r}{r} \end{aligned} \tag{236}$$

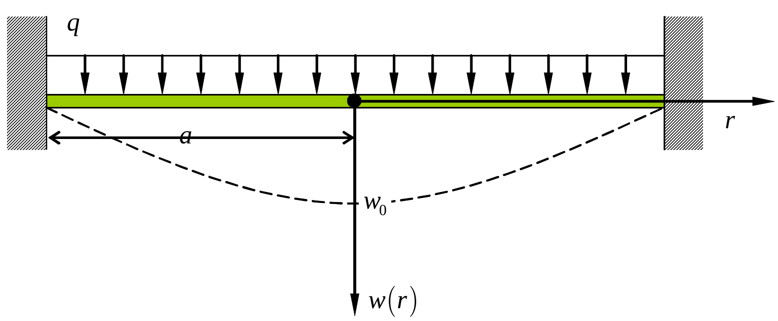
- Constitutive equation

$$\begin{aligned} N_{rr} &= C\ \ [\lambda_{rr} + \nu\ \lambda_{\theta\theta}] \\ N_{\theta\theta} &= C\ \ [\lambda_{\theta\theta} + \nu\ \lambda_{rr}] \end{aligned} \tag{237}$$

where $C = Eh/\left(1 - \nu^2\right)$.

4.2.3 Example: Approximation Solution for the Clamped Membrane

Consider a circular plate with the clamped support.



Membrane Solution From the symmetry and clamped boundary condition, the radial displacement u_r reads:

$$u_r\,(r=0)=0 \qquad (238)$$
$$u_r\,(r=a)=0$$

Thus, as a first approximation, it is appropriate to assume:

$$u_r \equiv 0 \qquad \text{for} \;\; 0 \le r \le a \qquad (239)$$

Then, the hoop strain vanishes:

$$\varepsilon_{\theta\theta} = 0$$

Now, the radial force and the radial strain component become:

$$N_{rr} = C\;\varepsilon_{rr} \qquad (240)$$

$$\varepsilon_{rr} = \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 \qquad (241)$$

With the assumption $u_r = 0$, the in-plane equilibrium equation can not be satisfied.

Consider out-of-plane equilibrium equation only. Substituting Eq. (240) and (241) into Eq. (235), one gets:

$$\frac{\partial}{\partial r}\left[\frac{C}{2}\left(\frac{\partial w}{\partial r}\right)^2\;\frac{\partial w}{\partial r}\;r\right] = -\,r\;q \quad \text{ on } \;S \qquad (242)$$

Integrating both sides once with respect to r reads:

$$\frac{C}{2}\;r\;\left(\frac{\partial w}{\partial r}\right)^3 = -\,\frac{r^2\;q}{2} + c_1 \qquad (243)$$

At the center of the membrane, the slope should be zero. Thus, one gets:

$$\frac{\partial w}{\partial r} = 0 \qquad \text{at } r = 0 \qquad (244)$$
$$\Rightarrow c_1 = 0$$

Then, Eq. (243) can be written:

$$\frac{\partial w}{\partial r} = -\sqrt[3]{\frac{q\;r}{C}} \qquad (245)$$

Integrating the above equation again reads:

$$w = -\frac{3}{4}\;\sqrt[3]{\frac{q}{C}}\;r^{4/3} + c_2 \qquad (246)$$

The integration constant c_2 can be determined from the zero deflection condition at the clamped edge:

$$\begin{aligned} w = 0 \qquad & \text{at } r = a \\ \Rightarrow c_2 = \frac{3}{4} \sqrt[3]{\frac{q}{C}} \, a^{4/3} & \end{aligned} \tag{247}$$

Recalling the definition of the axial rigidity $C = Eh/\left(1-\nu^2\right)$, Eq. (246) can be put into a final form:

$$\begin{aligned} \frac{w}{a} &= \frac{3}{4} \sqrt[3]{\frac{(1-\nu^2)\ q\ a}{E\ h}} \left[1 - \left(\frac{r}{a}\right)^{4/3}\right] \\ &\simeq 0.73 \sqrt[3]{\frac{q\ a}{E\ h}} \left[1 - \left(\frac{r}{a}\right)^{4/3}\right] \end{aligned} \tag{248}$$

In particular, the central deflection $w\,(r=0) = w_0$ is related to the load intensity by:

$$\frac{w_0}{a} = 0.73 \sqrt[3]{\frac{q\ a}{E\ h}} \tag{249}$$

Bending Solution It is interesting to compare the bending and membrane response of the clamped circular plate. From the page 55 of *Theory of Plates and Shells (2nd Ed.)* by Timoshenko and Woinowsky-Krieger, the central deflection of the plate is linearly related to the loading intensity:

$$\begin{aligned} \frac{w_0}{a} &= \frac{q\ a^3}{64\ D} \\ &= \frac{3\ \left(1-\nu^2\right)\ q}{16\ E} \left(\frac{a}{h}\right)^3 \\ &\simeq 0.17 \frac{q}{E} \left(\frac{a}{h}\right)^3 \end{aligned} \tag{250}$$

Assume that $a/h = 10$, then Eq. (250) yields:

$$\frac{w_0}{a} = 17 \frac{q\ a}{E\ h} \tag{251}$$

Comparison A comparison of the bending and membrane solution is shown in the next figure.

It is seen that a transition from the bending to membrane response occurs at $w_0/a = 0.15$ which corresponds to $w_0 = 1.5\ h$. When the plate deflection reach approximately plate thickness, the membrane action takes over the bending action in a clamped plate. If the plate is not restrained from axial motion, then the assumption $u_r = 0$ is no longer valid, and a separate solution must be developed.

4.3 Buckling Theory of Plates

4.3.1 General Equation of Plate Buckling

- Equilibrium of in-plane equation

$$N_{\alpha\beta},_{\alpha} = 0 \qquad \text{on} \;\; S \tag{252}$$

  Equilibrium of out-of-plane equation

$$M_{\alpha\beta},_{\alpha\beta} + N_{\alpha\beta}\ w,_{\alpha\beta} + q = 0 \quad \text{on} \;\; S \tag{253}$$

- Strain-displacement relation

$$\begin{aligned} \varepsilon^{\circ}_{\alpha\beta} &= \frac{1}{2}\left(u_{\alpha},_{\beta} + u_{\alpha},_{\beta}\right) + \frac{1}{2}\ w,_{\alpha}\ w,_{\beta} \\ \kappa_{\alpha\beta} &= -\ w,_{\alpha\beta} \end{aligned} \tag{254}$$

- Constitutive equation for axial force and axial strain

$$N_{\alpha\beta} = C\ \left[(1-\nu)\ \varepsilon^{\circ}_{\alpha\beta} + \nu\ \varepsilon^{\circ}_{\gamma\gamma}\ \delta_{\alpha\beta}\right] \tag{255}$$

  where $C = Eh/\left(1-\nu^2\right)$, and another one for moment and curvature

$$M_{\alpha\beta} = D\ \left[(1-\nu)\ \kappa_{\alpha\beta} + \nu\ \kappa_{\gamma\gamma}\ \delta_{\alpha\beta}\right] \tag{256}$$

  where $D = Eh^3/\left[12\left(1-\nu^2\right)\right]$.

By combining Eq. (254) and (256), one gets:

$$M_{\alpha\beta},_{\alpha\beta} = -D\ w,_{\alpha\alpha\beta\beta} \tag{257}$$

Substituting Eq. (257) into Eq. (253) leads:

$$-D\ w,_{\alpha\alpha\beta\beta} + N_{\alpha\beta}\ w,_{\alpha\beta} + q = 0 \quad \text{on} \quad S \tag{258}$$

The buckling problem is specified by:

$$q \equiv 0 \tag{259}$$

Now, changing signs leads the general out-of-plane equation for the buckling of the plates:

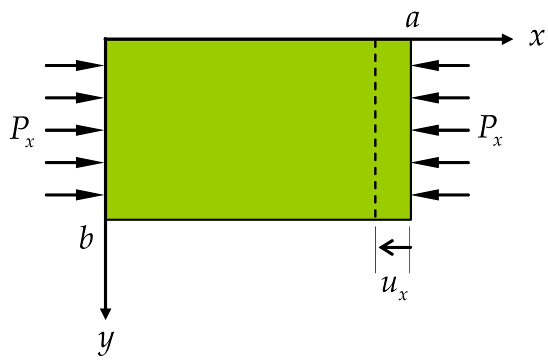
$$\boxed{D\ w,_{\alpha\alpha\beta\beta} - N_{\alpha\beta}\ w,_{\alpha\beta} = 0} \tag{260}$$

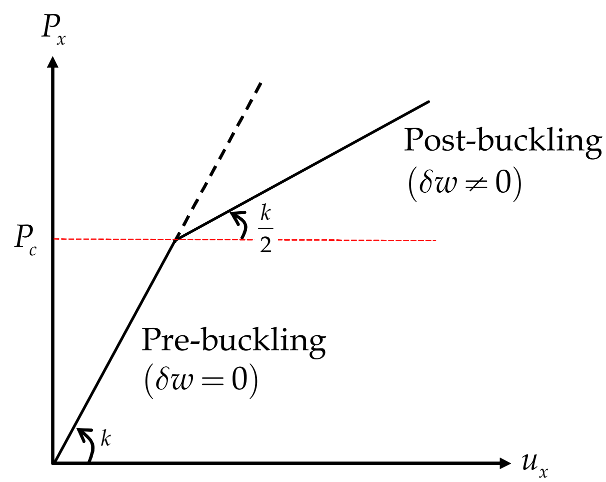
where the second term in the left hand is non-linear due to $N_{\alpha\beta}$ which should be obtained from:

$$\boxed{N_{\alpha\beta},_{\alpha} = 0} \tag{261}$$

4.3.2 Linearized Buckling Equation of Rectangular Plates

The nonlinear buckling equation can be separated into two linear equations: one for in-plane equation for $N_{\alpha\beta}$ and another one for w.





Pre-Buckling Problem Recall that:

$$\varepsilon^{\circ}_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right) + \frac{1}{2}\ \frac{\partial w}{\partial x_\alpha}\ \frac{\partial w}{\partial x_\beta} \tag{262}$$

$$N_{\alpha\beta} = \frac{Eh}{1-\nu^2}\left[(1-\nu)\ \varepsilon^{\circ}_{\alpha\beta} + \nu\ \varepsilon^{\circ}_{\gamma\gamma}\ \delta_{\alpha\beta}\right] \tag{263}$$

In the pre-buckling problem, the linear equilibrium equations are obtained by omitting the nonlinear terms in the governing equations Eq. (260) and (261). The resulting equations are now:

$$D\ w_{,\alpha\alpha\beta\beta} = 0$$

$$N_{\alpha\beta},_{\beta} = 0$$

For the pre-buckling trajectory, $\delta w = 0$, one gets the equilibrium equation:

$$N_{\alpha\beta},_{\beta} = 0 \tag{264}$$

where

$$N_{\alpha\beta} = \frac{Eh}{1-\nu^2}\left[(1-\nu)\ \varepsilon^\circ_{\alpha\beta} + \nu\ \varepsilon^\circ_{\gamma\gamma}\ \delta_{\alpha\beta}\right] \tag{265}$$

$$\varepsilon^\circ_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right) \tag{266}$$

and boundary condition:

$$\left(N_{nn} - \bar{N}_{nn}\right)\ \delta u_n = 0 \qquad \text{on } \Gamma \tag{267}$$

Here, it is assumed that the unknown membrane force tensor $N_{\alpha\beta}$ is equal to the similar quantity known from the pre-buckling solution $N^\circ_{\alpha\beta}$:

$$N_{\alpha\beta} = -N^\circ_{\alpha\beta}$$

where the compressive pre-buckling membrane force are defined as positive.

Post-Buckling Problem Now, the governing equation for buckling of plates reads:

$$\boxed{D\ \nabla^4 w\ +\ N^\circ_{\alpha\beta}\ w_{,\alpha\beta}\ =\ q} + \begin{array}{c}\text{Boundary}\\ \text{Conditions}\end{array} \tag{268}$$

where membrane force tensor in the pre-buckling solution $N^\circ_{\alpha\beta}$ is defined as:

$$N^\circ_{\alpha\beta} = \lambda\ \tilde{N}_{\alpha\beta} = \lambda\ \begin{vmatrix}\tilde{N}_{xx} & \tilde{N}_{xy}\\ \tilde{N}_{yx} & \tilde{N}_{yy}\end{vmatrix} \tag{269}$$

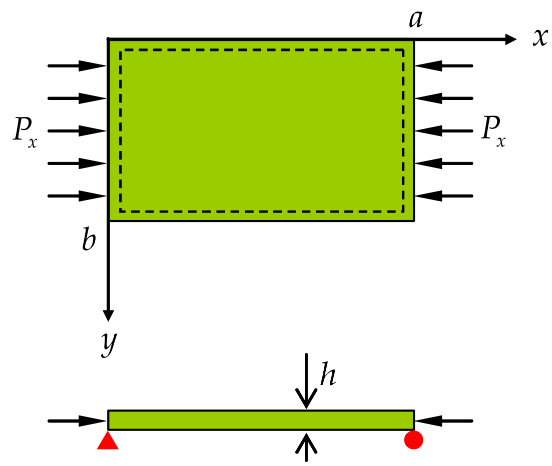
where $\bar{N}_{\alpha\beta}$ is the known direction from the pre-buckling analysis, and η is unknown load amplitude. Now, the nonlinear buckling equation becomes a linear eigenvalue problem:

$$\boxed{D\ \nabla^4 w + \lambda\ \tilde{N}_{\alpha\beta}\ w_{,\alpha\beta} = 0} \tag{270}$$

where $\lambda \geq 0$ is eigenvalues, and w is eigenfunctions.

4.3.3 Analysis of Rectangular Plates Buckling

Simply Supported Plate under In-Plane Compressive Loading Consider a plate simply supported on four edges. The plate is subjected to an in-plane compressive load P_x uniformly distributed along the edges $x = [0, a]$.



From equilibrium equations, one gets:

$$N^\circ_{\alpha\beta} = \begin{vmatrix} N^\circ_{xx} \; N^\circ_{xy} \\ N^\circ_{yx} \; N^\circ_{yy} \end{vmatrix} = \lambda \begin{vmatrix} \tilde{N}_{xx} \; \tilde{N}_{xy} \\ \tilde{N}_{yx} \; \tilde{N}_{yy} \end{vmatrix} = \frac{P_x}{b} \begin{vmatrix} 1 \, 0 \\ 0 \, 0 \end{vmatrix} \tag{271}$$

Introducing Eq. (271) into Eq. (268) leads:

$$D \ \nabla^4 w + \frac{P_x}{b} \ w_{,xx} = 0 \tag{272}$$

Boundary condition for this simply supported plate are written as:

$$\begin{aligned} w &= 0 \qquad \text{on} \ \ \Gamma \\ M_{nn} &= 0 \qquad \text{on} \ \ \Gamma \end{aligned} \tag{273}$$

where the moment components read:

$$\begin{aligned} M_{xx} &= -D \ \ (w_{,xx} + \nu \ w_{,yy}) = 0 \\ M_{yy} &= -D \ \ (w_{,yy} + \nu \ w_{,xx}) = 0 \end{aligned} \tag{274}$$

Thus, one gets:

$$\begin{aligned} w = w_{,xx} = 0 \qquad &\text{on} \quad x = [0, a] \\ w = w_{,yy} = 0 \qquad &\text{on} \quad x = [0, b] \end{aligned} \tag{275}$$

Equation (272) is a constant-coefficient equation, and a solution of the following form:

$$w = c_1 \ \sin\left(\frac{m \ \pi \ x}{a}\right) \sin\left(\frac{n \ \pi \ y}{b}\right) \qquad \text{for} \ \ m, n = 1, 2 \tag{276}$$

satisfies both the differential equation and the boundary conditions. Introduction into Eq. (272) gives:

$$D\ \left[\left(\frac{m\ \pi}{a}\right)^4 + 2\left(\frac{m\ \pi}{a}\right)^2\left(\frac{n\ \pi}{b}\right)^2 + \left(\frac{n\ \pi}{b}\right)^4\right] - \frac{P_x}{b}\left(\frac{m\ \pi}{a}\right)^2 = 0 \tag{277}$$

$$\Rightarrow \quad \frac{P_x}{b} = D\ \left(\frac{\pi\ a}{m}\right)^2\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2 \tag{278}$$

where for the discrete values of P_x Eq. (272) has nontrivial solutions. The critical load can be determined by the smallest eigenvalue, i.e. $n = 1$ for all values of a:

$$\begin{aligned} \frac{P_x}{b} &= D\ \left(\frac{\pi\ a}{m}\right)^2\left[\left(\frac{m}{a}\right)^2 + \left(\frac{1}{b}\right)^2\right] \\ &= \frac{\pi^2\ D}{b^2}\left(\frac{a\ b}{m}\right)^2\ \left[\left(\frac{m}{a}\right)^2 + \left(\frac{1}{b}\right)^2\right] \\ &= \frac{\pi^2\ D}{b^2}\ \left(\frac{m\ b}{a} + \frac{a}{m\ b}\right)^2 \end{aligned} \tag{279}$$

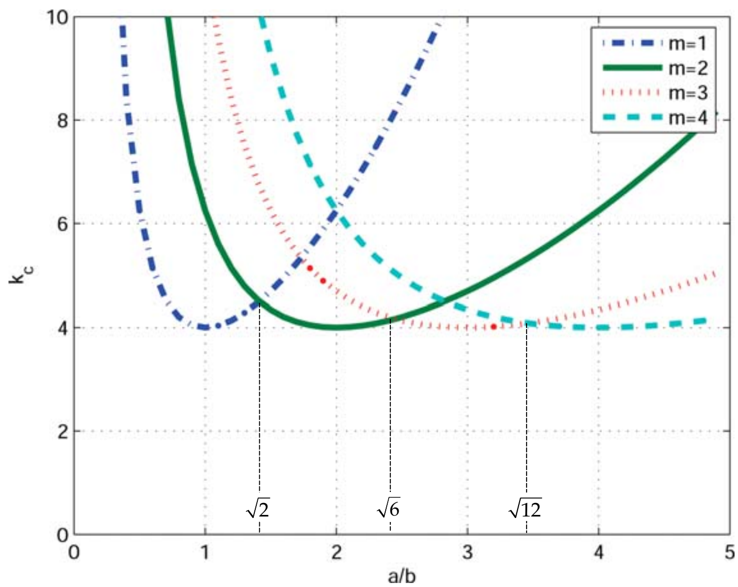
Now, the critical load $(P_x)_{cr}$ can be written as:

$$\boxed{(P_x)_{cr} = k_c\frac{\pi^2\ D}{b}} \tag{280}$$

where

$$k_c = \left(\frac{m\ b}{a} + \frac{a}{m\ b}\right)^2 \tag{281}$$

where coefficient k_c is a function of aspect ratio a/b and wavelength parameter m.



For a given a/b, m may be chosen to yield the smallest eigenvalue. In order to minimize k_c in Eq. (281), treating m as a continuous variable produces:

$$\frac{\partial k_c}{\partial m} = 2\left(\frac{m\ b}{a} + \frac{a}{m\ b}\right)\left(\frac{b}{a} - \frac{a}{b\ m^2}\right) = 0 \tag{282}$$

where the first bracket can not be zero, so the second bracket should be zero:

$$\Rightarrow \qquad \frac{b}{a} - \frac{a}{b}\frac{1}{m^2} = 0 \tag{283}$$

Now, one gets:

$$\boxed{\begin{array}{c} m = \dfrac{a}{b} \\ k_c = 4 \end{array}} \tag{284}$$

Here, this is valid when a/b is integer and when considering a very long plates.

Transition from m to $m+1$ half-waves occurs when the two corresponding curves have equal ordinates, i.e. from Eq. (281):

$$k_c|_m = k_c|_{m+1} \tag{285}$$

$$\begin{aligned} &\Rightarrow \quad \frac{m\ b}{a} + \frac{a}{m\ b} = \frac{(m+1)\ b}{a} + \frac{a}{(m+1)\ b} \\ &\Rightarrow \quad \frac{a}{b} = \sqrt{m\ (m+1)} \end{aligned} \tag{286}$$

$$\boxed{\frac{a}{b} = \sqrt{m\ (m+1)}} \tag{287}$$

Example 1 *For* $m = 1$, $a/b = \sqrt{2}$

Example 2 *For a very large* m*, i.e. a very long plate,* $a/b \simeq m$*. Thus,* $k_c = 4$ *is now independent of* m*.*

A very long plate buckles in half-waves, whose lengths approach the width of the plate:

$$w = c_1\ \sin\left(\frac{\pi\ x}{b}\right) \sin\left(\frac{n\ \pi\ y}{b}\right)$$

Thus, the buckled plate subdivides approximately into squares.

Various Boundary Conditions of Plate under In-Plane Compressive Loading The critical buckling load reads:

$$(P_x)_{cr} = k_c \frac{\pi^2\ D}{b}$$

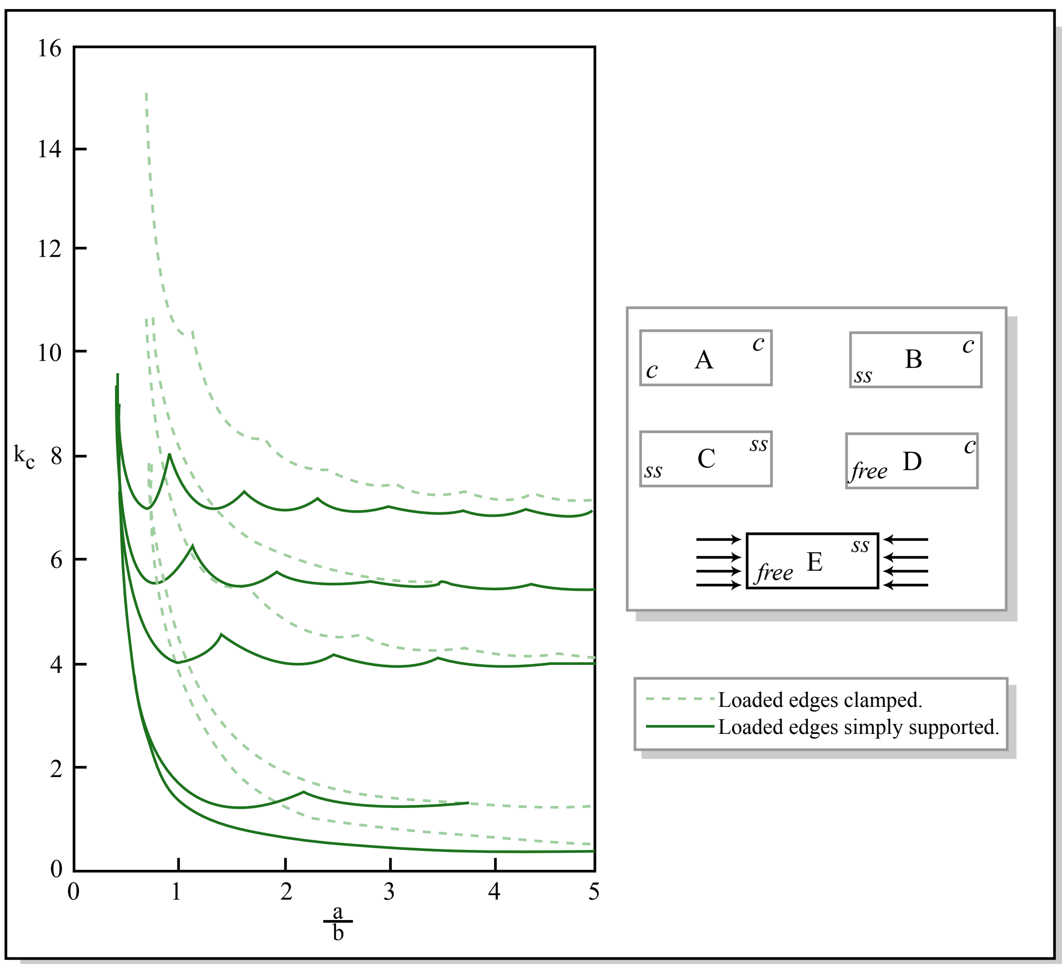
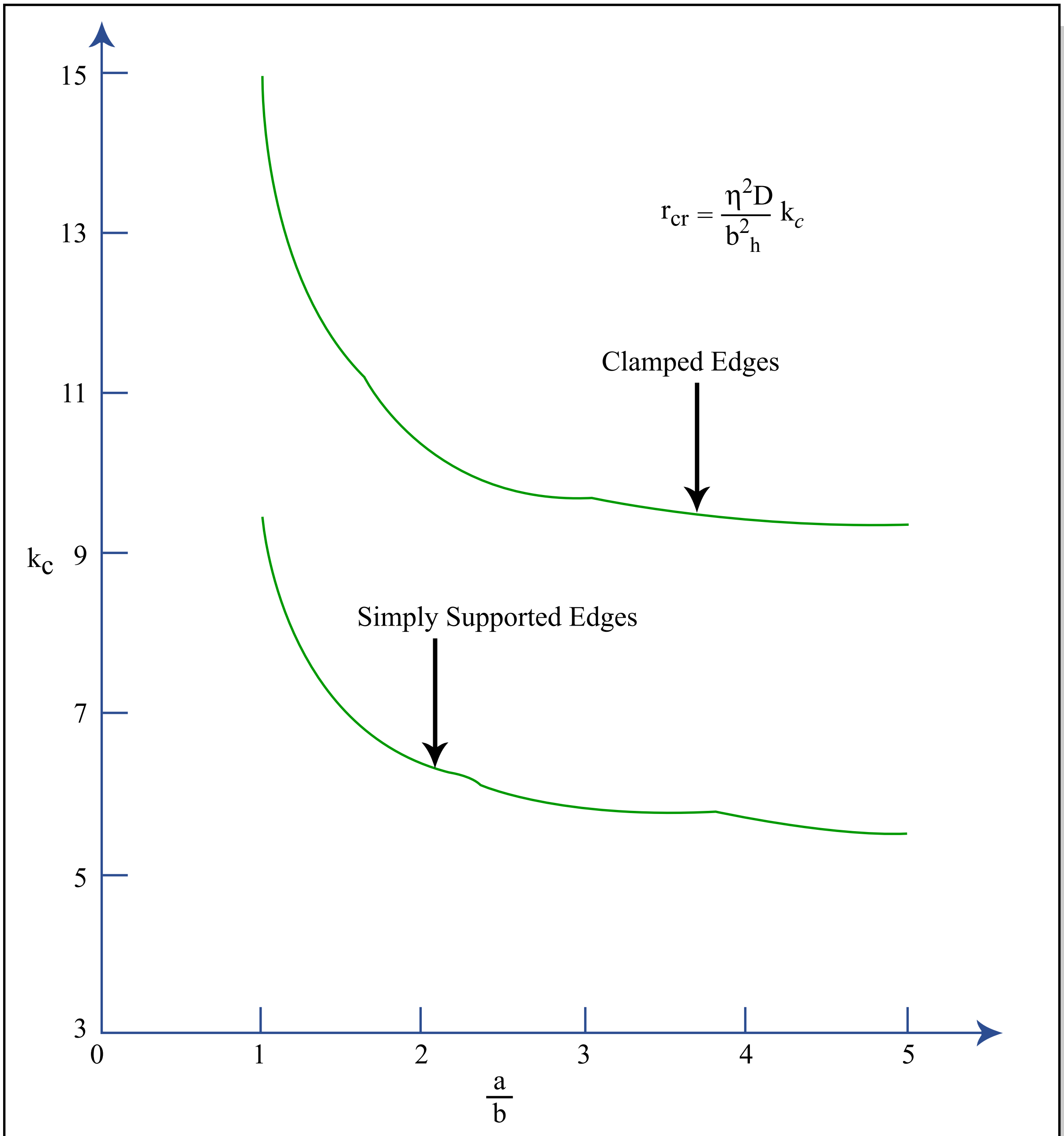


Figure by MIT OCW.

Influence of boundary conditions on the buckling coefficients of plates subjected to in-plane compressive loading

Various Boundary Conditions of Plate under In-Plane Shear Loading The critical buckling load per unit length reads:

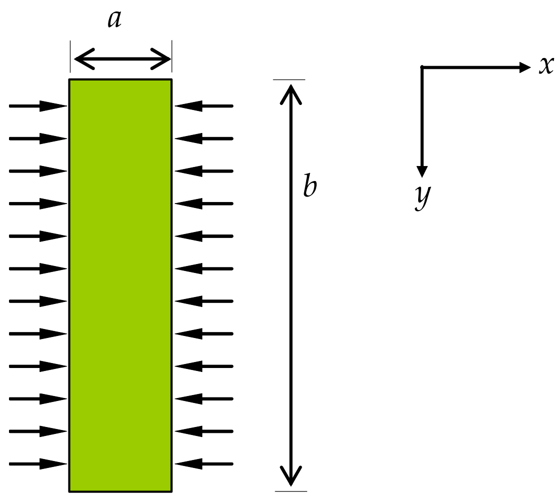
$$(N_{xy})_{cr} = k_c \frac{\pi^2 \ D}{b^2}$$



Critical values of shear stress for plates subjected to in-plane loading.

Figure by MIT OCW.

Limiting Case: Wide Plates Consider a wide plate for which $a/b \ll 1$. From the diagram, we see that if $a/b < 1$, then m is set to be equal to unity, i.e. just one wavelength in the x-direction.



The buckling formula thus becomes:

$$\begin{aligned}(N_x)_{cr} &= k_c|_{m=1} \ \frac{\pi^2 \ D}{b^2} \\ &= \frac{\pi^2 \ D}{b^2} \ \left(\frac{b}{a} + \frac{a}{b}\right)^2 \\ &= \frac{\pi^2 \ D}{a^2} \ \left(\frac{a}{b}\right)^2 \left(\frac{b}{a} + \frac{a}{b}\right)^2 \\ &= \frac{\pi^2 \ D}{a^2} \ \left[1 + \left(\frac{a}{b}\right)^2\right]^2\end{aligned} \tag{288}$$

If $a/b \ll 1$, then the second term in the bracket can be neglected so that the buckling load per unit length becomes:

$$\boxed{(N_x)_{cr} = \frac{\pi^2 D}{a^2}}$$

which is called Sezawa's formula for wide plates.

Example 3 *Here, relative merits of stiffening a large panel are investigated in the longitudinal or in the transverse direction. It is assumed that the stiffeners provide for a simply supported boundary conditions.*
Consider the case of longitudinal stiffeners.

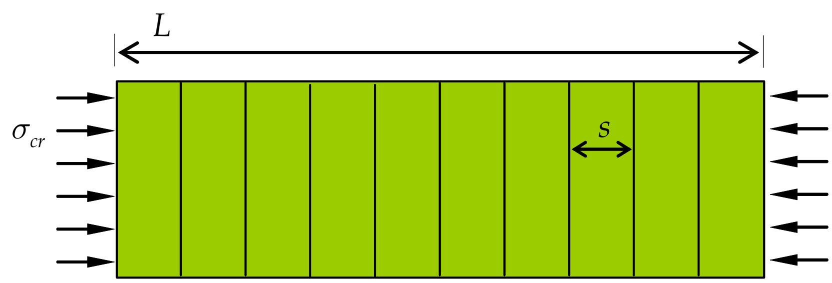
From the von Karman formula, the buckling load per unit length for each divided part reads:

$$(N_x)_{cr} = \frac{4\pi^2 D}{s^2}$$

Now, the buckling stress can be calculated:

$$(\sigma_{cr})_{longitudinal} = \frac{(N_x)_{cr}}{h} = \frac{4\pi^2 D}{s^2 h}$$

Consider the case of transverse stiffeners.



From the Sezawa's formula, the buckling load per unit length along the loaded edges reads:

$$(N_x)_{cr} = \frac{\pi^2 D}{s^2}$$

Now, the buckling stress can be calculated:

$$(\sigma_{cr})_{transverse} = \frac{(N_x)_{cr}}{h} = \frac{\pi^2 D}{s^2 h}$$

Thus, it is concluded that

$$\boxed{(\sigma_{cr})_{longitudinal} = 4 \ (\sigma_{cr})_{transverse}}$$

This shows advantages of longitudinal stiffeners over transverse stiffeners.

4.3.4 Derivation of Raleigh-Ritz Quotient

Recall the total potential energy of system and other corresponding definitions:

$$\Pi = (U_b - V_b) + (U_m - V_m) \tag{289}$$

where each term for buckling problems will be discussed in the following.

Term Relating to Plate Bending Response In the buckling problem, the work done by external load causing bending response considered as zero:

$$V_b = 0 \tag{290}$$

The bending energy can be expressed:

$$\begin{aligned} U_b &= \frac{1}{2} \int_S M_{\alpha\beta}\ \kappa_{\alpha\beta}\ dS \\ &= \frac{D}{2} \int_S\ [(1-\nu)\ \ \kappa_{\alpha\beta} + \nu\ \kappa_{\gamma\gamma}\ \delta_{\alpha\beta}]\ \ \kappa_{\alpha\beta}\ dS \\ &= \frac{D}{2} \int_S\ \left[(1-\nu)\ \ \kappa_{\alpha\beta}\kappa_{\alpha\beta} + \nu\ \ (\kappa_{\gamma\gamma})^2\right]\ dS \\ &= \frac{D}{2} \int_S\ \left[(1-\nu)\ \ (\kappa_{11}\kappa_{11} + \kappa_{12}\kappa_{12} + \kappa_{21}\kappa_{21} + \kappa_{22}\kappa_{22}) + \nu\ \ (\kappa_{11} + \kappa_{22})^2\right]\ dS \\ &= \frac{D}{2} \int_S\ \left\{(\kappa_{11} + \kappa_{22})^2 - 2\ \ (1-\nu)\left[\kappa_{11}\ \kappa_{22} - (\kappa_{12})^2\right]\right\}\ dS \end{aligned} \tag{291}$$

Here, the term in the square bracket is called Gaussian curvature:

$$\kappa_{11}\ \kappa_{22} - (\kappa_{12})^2 = \kappa_I\ \kappa_{II} \tag{292}$$

where κ_I and κ_{II} are the principal curvatures. For plates with straight edges, Gaussian curvature vanishes, so one gets:

$$U_b = \frac{D}{2} \int_S\ (\kappa_{11} + \kappa_{22})^2\ \ dS \tag{293}$$

The integrand of the above equation can be written in terms of the transverse displacement:

$$\begin{aligned} (\kappa_{11} + \kappa_{22})^2 &= \left(-\nabla^2 w\right)^2 \\ &= \nabla^2 w\ \nabla^2 w \end{aligned} \tag{294}$$

Now, the bending energy reads:

$$U_b = \frac{D}{2} \int_S\ \nabla^2 w\ \nabla^2 w\ dS \tag{295}$$

Term Relating to Plate Membrane Response The work done by external load causing membrane response reads:

$$V_m = \int_\Gamma \bar{N}_{nn}\ u_n\ dl + \int_\Gamma \bar{N}_{tn}\ u_t\ dl \tag{296}$$

In the buckling problem, the axial force $N^\circ_{\alpha\beta} = \lambda \tilde{N}_{\alpha\beta}$ is determined from the pre-buckling solution and is considered as constant, so the membrane energy reads:

$$\begin{aligned} U_m &= -\lambda \int_S \tilde{N}_{\alpha\beta}\ \varepsilon^\circ_{\alpha\beta}\ dS \\ &= -\lambda \int_S \tilde{N}_{\alpha\beta}\ \left[\frac{1}{2}\left(u_\alpha,_\beta + u_\alpha,_\beta\right) + \frac{1}{2}\ w,_\alpha\ w,_\beta\right]\ dS \\ &= -\lambda \int_S \tilde{N}_{\alpha\beta}\ u_\alpha,_\beta\ dS - \frac{\lambda}{2}\int_S \tilde{N}_{\alpha\beta}\ w,_\alpha\ w,_\beta\ dS \end{aligned} \tag{297}$$

Here, the first term can be extended in a similar way shown in Eq. (132):

$$\begin{aligned} -\lambda \int_S \tilde{N}_{\alpha\beta}\ u_\alpha,_\beta\ dS &= -\lambda \int_\Gamma \left(\tilde{N}_{nn}\ u_n + \tilde{N}_{tn}\ u_t\right)\ dl \\ &\quad -\lambda \int_S \tilde{N}_{\alpha\beta},_\beta\ u_\alpha\ dS \\ &= -\lambda \int_\Gamma \left(\tilde{N}_{nn}\ u_n + \tilde{N}_{tn}\ u_t\right)\ dl \end{aligned} \tag{298}$$

where the in-plane equilibrium is applied:

$$-\lambda\ \tilde{N}_{\alpha\beta},_\beta = 0 \tag{299}$$

Now, the membrane energy can be expressed:

$$U_m = -\lambda \int_\Gamma \left(\tilde{N}_{nn}\ u_n + \tilde{N}_{tn}\ u_t\right)\ dl - \frac{\lambda}{2}\int_S \tilde{N}_{\alpha\beta}\ w,_\alpha\ w,_\beta\ dS \tag{300}$$

Thus, the term relating to membrane response can be summarized:

$$\begin{aligned} U_m - V_m &= -\lambda \int_\Gamma \left(\tilde{N}_{nn}\ u_n + \tilde{N}_{tn}\ u_t\right)\ dl - \frac{\lambda}{2}\int_S \tilde{N}_{\alpha\beta}\ w,_\alpha\ w,_\beta\ dS \\ &\quad - \int_\Gamma \bar{N}_{nn}\ u_n\ dl - \int_\Gamma \bar{N}_{tn}\ u_t\ dl \\ &= \int_\Gamma \left(-\lambda\ \tilde{N}_{nn} - \bar{N}_{nn}\right)\ u_n\ dl - \int_\Gamma \left(-\lambda\ \tilde{N}_{tn} - \bar{N}_{tn}\right)\ u_t\ dl \\ &\quad -\frac{\lambda}{2}\int_S \tilde{N}_{\alpha\beta}\ w,_\alpha\ w,_\beta\ dS \\ &= -\frac{\lambda}{2}\int_S \tilde{N}_{\alpha\beta}\ w,_\alpha\ w,_\beta \end{aligned} \tag{301}$$

where the boundary conditions on Γ are applied.

Total Potential Energy and Its Variations Now, one gets the total potential energy:

$$\begin{aligned}\Pi &= (U_b - V_b) + (U_m - V_m) \\ &= \frac{D}{2}\int_S \nabla^2 w\ \nabla^2 w\ dS - \frac{\lambda}{2}\int_S \tilde{N}_{\alpha\beta}\ w,_{\alpha}\ \ w,_{\beta}\ dS\end{aligned} \tag{302}$$

The first variation of the potential energy can be obtained:

$$\begin{aligned}\delta\Pi &= \frac{D}{2}\int_S \left[\delta\left(\nabla^2 w\right)\ \nabla^2 w + \nabla^2 w\ \delta\left(\nabla^2 w\right)\right]\ dS \\ &\quad -\frac{\lambda}{2}\int_S \tilde{N}_{\alpha\beta}\ \left(\delta w,_{\alpha}\ w,_{\beta} + w,_{\alpha}\ \delta w,_{\beta}\right)\ dS \\ &= D\int_S \nabla^2 w\ \delta\left(\nabla^2 w\right)\ dS - \lambda\int_S \tilde{N}_{\alpha\beta}\ w,_{\alpha}\ \ \delta w,_{\beta}\ dS \\ &= D\int_S \nabla^2 w\ \nabla^2 \delta w\ dS - \lambda\int_S \tilde{N}_{\alpha\beta}\ w,_{\alpha}\ \ \delta w,_{\beta}\ dS\end{aligned} \tag{303}$$

where $\tilde{N}_{\alpha\beta}$ is considered as constant under the variation. Similarly, the second variation of the potential energy reads:

$$\begin{aligned}\delta^2\Pi &= D\int_S \delta\left(\nabla^2 w\right)\ \nabla^2 \delta w\ dS - \lambda\int_S \tilde{N}_{\alpha\beta}\ \delta w,_{\alpha}\ \ \delta w,_{\beta}\ dS \\ &\quad D\int_S \nabla^2 \delta w\ \nabla^2 \delta w\ dS - \lambda\int_S \tilde{N}_{\alpha\beta}\ \delta w,_{\alpha}\ \ \delta w,_{\beta}\ dS\end{aligned} \tag{304}$$

Raleigh-Ritz Quotient Application of the Trefftz condition for invertability, $\delta^2\Pi = 0$, determines the load intensity:

$$\lambda = \frac{D\int_S \nabla^2 \delta w\ \nabla^2 \delta w\ dS}{\int_S \tilde{N}_{\alpha\beta}\ \delta w,_{\alpha}\ \ \delta w,_{\beta}\ dS} \tag{305}$$

Here, choose a trial function for w:

$$w = A\ \phi \tag{306}$$

where A is the undetermined magnitude, and $\phi = \hat{\phi}(x, y)$ is a normalized shape function. Then, the variation of the trial function reads:

$$\delta w = \delta A\ \phi \tag{307}$$

Now, the load intensity reads:

$$\begin{aligned}\lambda &= \frac{D\int_S\ \nabla^2\left(\delta A\ \phi\right)\ \nabla^2\left(\delta A\ \phi\right)\ dS}{\int_S \tilde{N}_{\alpha\beta}\ \left(\delta A\ \phi\right),_{\alpha}\ \left(\delta A\ \phi\right),_{\beta}\ dS} \\ &= \frac{D\int_S\ \delta A\ \nabla^2\phi\ \delta A\ \nabla^2\phi\ dS}{\int_S \tilde{N}_{\alpha\beta}\ \delta A\ \phi,_{\alpha}\ \ \delta A\ \phi,_{\beta}\ dS} \\ &= \frac{D\int_S\ \nabla^2\phi\ \nabla^2\phi\ dS}{\int_S \tilde{N}_{\alpha\beta}\ \phi,_{\alpha}\ \ \phi,_{\beta}\ dS}\end{aligned} \tag{308}$$

The Raleigh-Ritz quotient is defined as:

$$\boxed{\lambda = \frac{D \int_S \ \nabla^2\phi \ \nabla^2\phi \ dS}{\int_S \tilde{N}_{\alpha\beta} \ \phi_{,\alpha} \ \ \phi_{,\beta} \ \ dS}} \tag{309}$$

Example 4 *As a special case, consider 1-D case:*

$$\tilde{N}_{\alpha\beta} = \begin{vmatrix} 1 \, 0 \\ 0 \, 0 \end{vmatrix}$$

then, the Raleigh-Ritz quotient becomes:

$$\lambda = \frac{D \int_S \ \nabla^2\phi \ \nabla^2\phi \ dS}{\int_S (\phi_{,x})^2 \ \ dS}$$

Example 5 *Similarly, consider 2-D compression case:*

$$\tilde{N}_{\alpha\beta} = \delta_{\alpha\beta} = \begin{vmatrix} 1 \, 0 \\ 0 \, 1 \end{vmatrix}$$

then, the Raleigh-Ritz quotient becomes:

$$\begin{aligned} \lambda &= \frac{D \int_S \ \nabla^2\phi \ \nabla^2\phi \ dS}{\int_S \phi_{,\alpha} \ \ \phi_{,\alpha} \ \ dS} \\ &= \frac{D \int_S \ \nabla^2\phi \ \nabla^2\phi \ dS}{\int_S \left[(\phi_{,x})^2 + (\phi_{,y})^2\right] \ \ dS} \end{aligned}$$

4.3.5 Ultimate Strength of Plates

The onset of buckling stress σ_{cr} does not necessarily means the total collapse of the plate. Usually, there is redistribution of stresses, and the plate takes additional load until the ultimate strength σ_u is reached.

Von Karman Analysis of the Effective Width For a simply supported plate, the buckling load is:

$$\begin{aligned} P_{cr} &= \frac{4\pi^2 D}{b} \\ &= \frac{4\pi^2 E}{12\,(1-\nu^2)} \frac{h^3}{b} \end{aligned} \tag{310}$$

and the corresponding buckling stress is:

$$\begin{aligned} \sigma_{cr} &= \frac{P_{cr}}{bh} \\ &= \frac{4\pi^2 E}{12\,(1-\nu^2)} \left(\frac{h}{b}\right)^2 \\ &= \frac{\pi^2 E}{3\,(1-\nu^2)} \left(\frac{h}{b}\right)^2 \end{aligned} \tag{311}$$

The normalization of buckling stress by the yield stress reads:

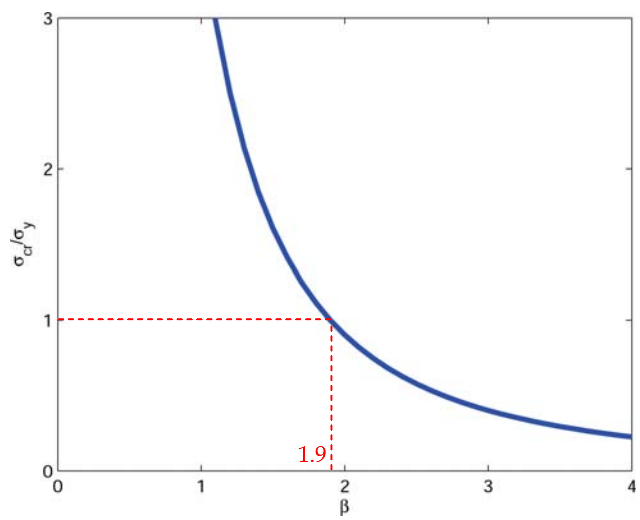
$$\frac{\sigma_{cr}}{\sigma_y} = \frac{\pi^2}{3\left(1-\nu^2\right)} \frac{E}{\sigma_y} \left(\frac{h}{b}\right)^2 \tag{312}$$
$$= \frac{(1.9)^2}{\frac{\sigma_y}{E}\left(\frac{b}{h}\right)^2}$$

$$\boxed{\frac{\sigma_{cr}}{\sigma_y} = \left(\frac{1.9}{\beta}\right)^2} \tag{313}$$

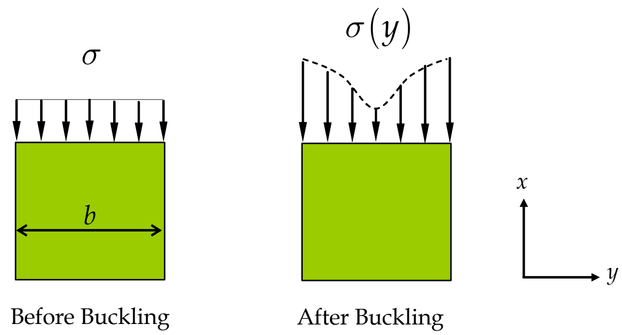
where β is a non-dimensional parameter defined by:

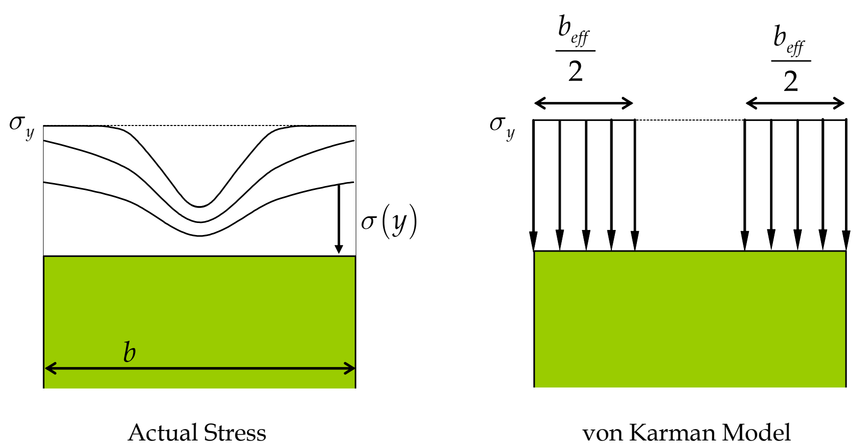
$$\beta = \sqrt{\frac{\sigma_y}{E}}\frac{b}{h} \tag{314}$$

The relation between the normalized buckling stress versus β is plotted in the next figure.



On further loading the plate beyond σ_{cr}, a greater proportion of the load is taken by the regions of the plate near the edge. Von Karman assumed that these edge regions, each of the width $b_{eff}/2$, carry the stress up to the yield while the center is stress free.





The edge zones are at yield, i.e. $\sigma_{cr}/\sigma_y = 1$, but the width of the effective portion of the plate is unknown:

$$\frac{\sigma_{cr}}{\sigma_y} = \frac{(1.9)^2}{\frac{\sigma_y}{E}\left(\frac{b_{eff}}{h}\right)^2} = 1 \tag{315}$$

from which one obtains:

$$\boxed{b_{eff} = 1.9\ h\sqrt{\frac{E}{\sigma_y}}} \tag{316}$$

Taking, for example, $E/\sigma_y = 900$ for mild steel, one gets:

$$b_{eff} = 1.9\sqrt{900} = 57h \tag{317}$$

This is somehow high, but there is not much difference from the empirically determined values of $b_{eff} = 40h \sim 50h$.

The total force at the point of ultimate load is:

$$P = \sigma_y \ b_{eff} \ h \tag{318}$$

Now, the average ultimate stress can be calculated:

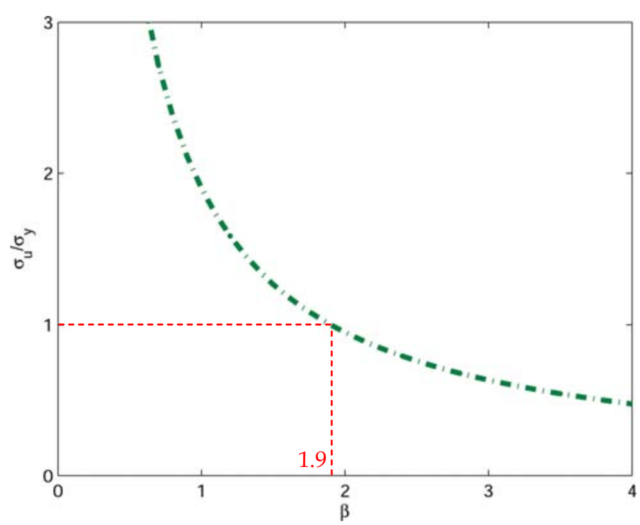
$$\begin{aligned} \sigma_u &= \frac{P}{bh} \\ &= \sigma_y \frac{b_{eff}}{b} \\ &= 1.9 \ \frac{h}{b} \sqrt{E \sigma_y} \end{aligned} \tag{319}$$

The average ultimate stress can be normalized by the yield stress:

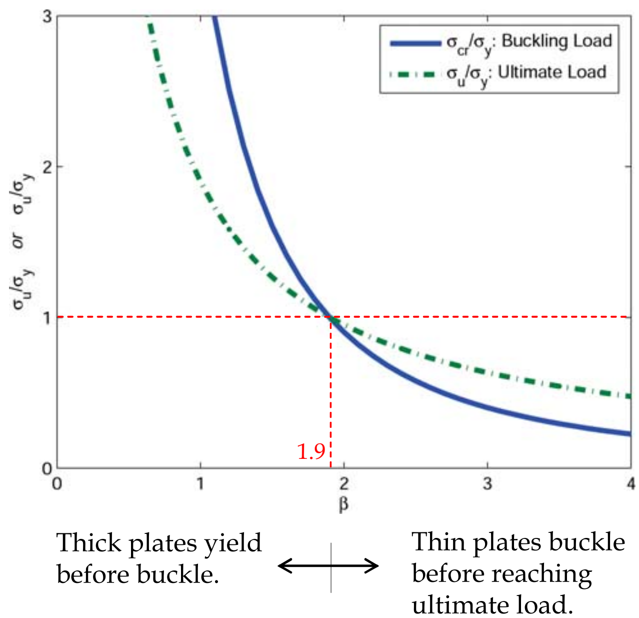
$$\frac{\sigma_u}{\sigma_y} = 1.9 \frac{\frac{h}{b} \sqrt{E \sigma_y}}{\sigma_y} \tag{320}$$

$$\boxed{\frac{\sigma_u}{\sigma_y} = \frac{1.9}{\beta}} \tag{321}$$

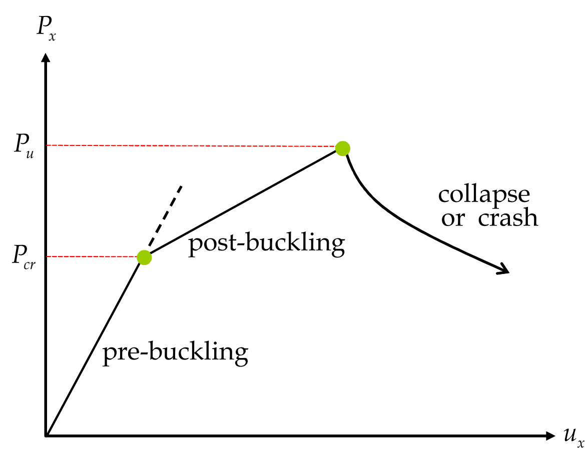
The average ultimate stress is plotted with respect to β in the next figure.

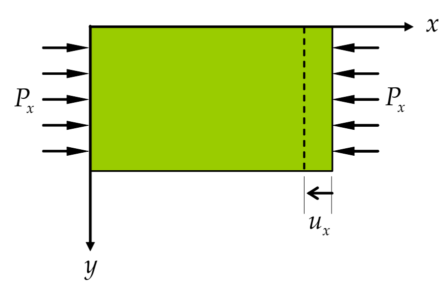


Comparison of the ultimate and buckling load solution is shown in the next figure.



Under the uniaxial loading, the relation between an applied load and the corresponding displacement is schematically shown all the way to collapse in the next figure.





Empirical Formulas

- Foulkner correction

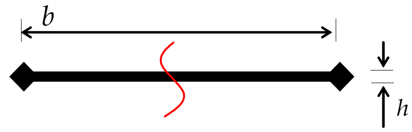
$$\frac{\sigma_u}{\sigma_y} = \frac{2}{\beta} - \frac{1}{\beta^2} \tag{322}$$

- Gerard (Handbook of elastic stability)

$$\frac{\sigma_u}{\sigma_y} = 0.56\left(\frac{gh^2}{A}\sqrt{\frac{E}{\sigma_y}}\right)^{0.85} \tag{323}$$

where g is the sum of the number of cuts and the number of flanges after the cuts, A is the cross sectional area $A = bh$, and the coefficients 0.56 and 0.85 are empirical constants.

Example 6 *Consider a plate which has one cut and two flanges.*



Then,

$$g = 1 + 2 = 3$$

Now,

$$\begin{aligned}\frac{\sigma_u}{\sigma_y} &= 0.56\left(\frac{3h^2}{bh}\sqrt{\frac{E}{\sigma_y}}\right)^{0.85}\\ &= 1.42\left(\frac{h}{b}\sqrt{\frac{E}{\sigma_y}}\right)^{0.85}\\ &= \frac{1.42}{\beta^{0.85}}\end{aligned}$$

Modifications in Codes In the original von Karman formula, the effective width ratio reads:

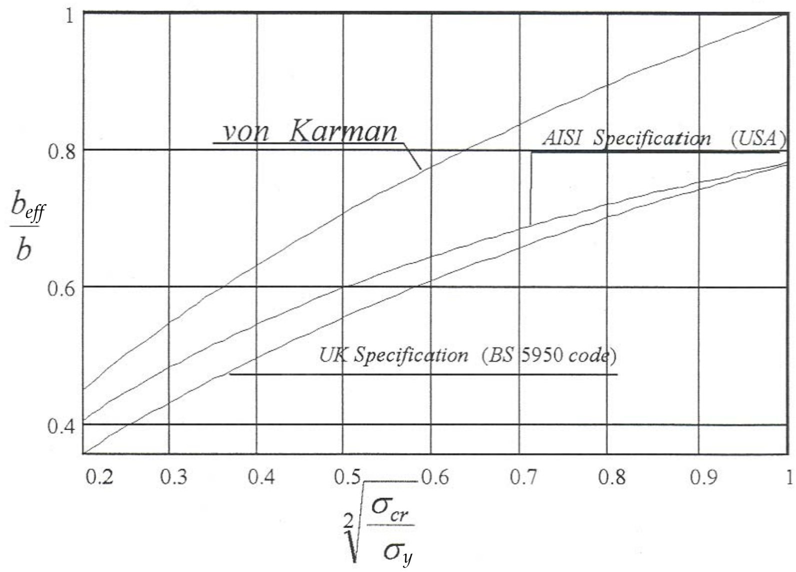
$$\frac{b_{eff}}{b} = \sqrt{\frac{\sigma_{cr}}{\sigma_y}} \tag{324}$$

In the ANSI specification, imperfection is considered:

$$\frac{b_{eff}}{b} = \sqrt{\frac{\sigma_{cr}}{\sigma_y}}\left(1 - 0.218\sqrt{\frac{\sigma_{cr}}{\sigma_y}}\right) \tag{325}$$

In UK specification, the effective width ratio is defined as:

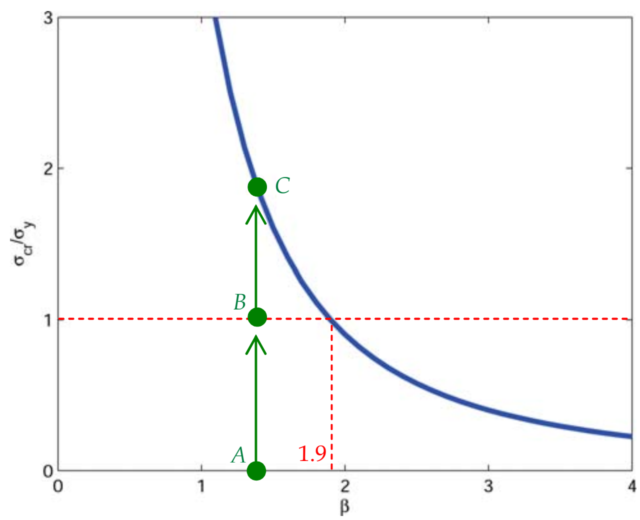
$$\frac{b_{eff}}{b} = \left[1 + 14\left(\sqrt{\frac{\sigma_{cr}}{\sigma_y}} - 0.35\right)^4\right]^{-0.2} \tag{326}$$



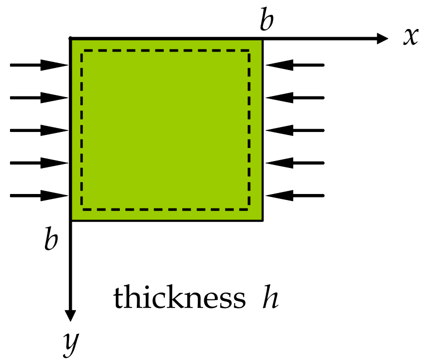
A comparison of theoretical (von Karman) and experimental predictions for the effective width in compressed steel plates

4.3.6 Plastic Buckling of Plates

Stocky plates with low b/h ratio will yield before buckling at the point B. After additional load, the plate will deform plastically on the path BC until conditions are met for the plate to buckle in the plastic range.



Stowell's Theory for the Buckling Strain Stowell developed the theory of plastic buckling for simply-supported square plates loaded in one direction.

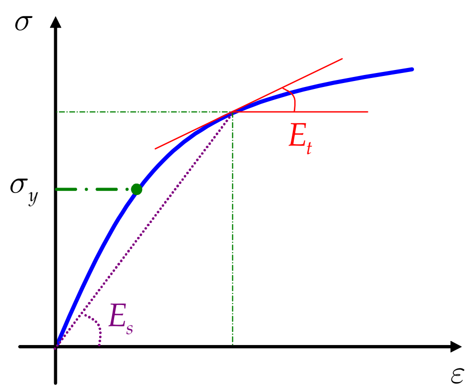


The critical buckling strain ε_{cr} was derived by him in the form:

$$\varepsilon_{cr} = \frac{\pi^2}{9}\left(\frac{h}{b}\right)^2\left[2 + \sqrt{1 + 3\left(E_t/E_s\right)}\right] \tag{327}$$

where the tangent modulus E_t and the secant modulus E_s are defined by:

$$E_t = \frac{d\sigma}{d\varepsilon} \quad ; \quad E_s = \frac{\sigma}{\varepsilon} \tag{328}$$



For the materials obeying the power hardening low:

$$\sigma = \sigma_r\left(\frac{\varepsilon}{\varepsilon_r}\right)^n \tag{329}$$

where σ_r and ε_r are the reference stress and strain. Now, the tangent and secant modulus are:

$$E_t = \frac{d\sigma}{d\varepsilon} = n\frac{\sigma_r}{\varepsilon_r}\left(\frac{\varepsilon}{\varepsilon_r}\right)^{n-1} \tag{330}$$
$$E_s = \frac{\sigma}{\varepsilon} = \frac{\sigma_r}{\varepsilon_r}\left(\frac{\varepsilon}{\varepsilon_r}\right)^{n-1}$$

Substituting these expression back into the buckling equation (327), one gets:

$$\varepsilon_{cr} = \frac{\pi^2}{9}\left(\frac{h}{b}\right)^2 \left(2+\sqrt{1+3n}\right) \tag{331}$$

The exponent n varies usually between $n = 0$ (perfectly plastic material) and $n = 1$ (elastic material). This makes the coefficient $\left(2+\sqrt{1+3n}\right)$ vary in the range $3 \sim 4$. For a realistic value of $n = 0.3$, the buckling strain becomes:

$$\varepsilon_{cr} = 3.7\left(\frac{h}{b}\right)^2 \tag{332}$$

Having determined ε_{cr}, the corresponding buckling stress is calculated from the power law.

Approximate Solution for the Buckling Strain Consider an elastic plane stress relation:

$$\begin{aligned}\sigma_{xx} &= \frac{E}{1-\nu^2}\left(\varepsilon_{xx}+\nu\varepsilon_{yy}\right) \\ \sigma_{yy} &= \frac{E}{1-\nu^2}\left(\varepsilon_{yy}+\nu\varepsilon_{xx}\right)\end{aligned} \tag{333}$$

Using the solution for the pre-buckling state, i.e. $\sigma_{xx} = \sigma_{cr}$ and $\sigma_{yy} = 0$ leads:

$$\begin{aligned}\sigma_{cr} &= \frac{E}{1-\nu^2}\left(\varepsilon_{xx}+\nu\varepsilon_{yy}\right) \\ 0 &= \frac{E}{1-\nu^2}\left(\varepsilon_{yy}+\nu\varepsilon_{xx}\right)\end{aligned} \tag{334}$$

from which one gets:

$$\begin{aligned}\varepsilon_{yy} &= -\nu\varepsilon_{xx} \\ \sigma_{cr} &= E\varepsilon_{xx} = E\varepsilon_{cr}\end{aligned} \tag{335}$$

The critical elastic buckling stress is:

$$\sigma_{cr} = k_c\frac{\pi^2 E}{12\left(1-\nu^2\right)}\left(\frac{h}{b}\right)^2 \tag{336}$$

So, the corresponding critical elastic buckling strain read:

$$\varepsilon_{cr} = k_c\frac{\pi^2}{12\left(1-\nu^2\right)}\left(\frac{h}{b}\right)^2 \tag{337}$$

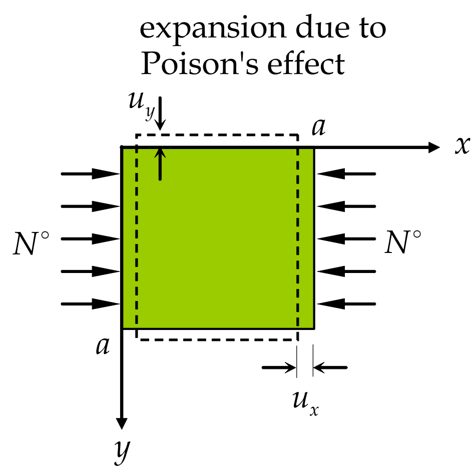
Equation (337) is more general than a similar expression Eq. (327) given by the Stowell theory because it applies to all type of boundary conditions. At the same time, Stowell's equation was derived only for the simply supported boundary conditions. In particular, for $k_c = 4$, Eq. (337) predicts:

$$\varepsilon_{cr} = 3.6 \left(\frac{h}{b}\right)^2 \tag{338}$$

which should be compared with the coefficient 3.7 of Eq. (332) in the Stowell's theory. For a plastic material or very high hardening exponent, the prediction of both method are much closer.

4.3.7 Exercise 1: Effect of In-Plane Boundary Conditions, $\delta w = 0$

No Constraint in In-Plane Displacement Consider no constraint in in-plane displacement in y-direction, $N^\circ_{yy} = 0$.



The membrane force tensor reads:

$$N^\circ_{\alpha\beta} = \begin{vmatrix} N^\circ & 0 \\ 0 & 0 \end{vmatrix} \tag{339}$$

From the constitutive equation, one gets:

$$\begin{aligned} N_{yy} &= \frac{Eh}{1-\nu^2}\left[\varepsilon^\circ_{yy} + \nu\ \varepsilon^\circ_{xx}\right] = 0 = -N^\circ_{yy} \\ &\Longrightarrow \varepsilon^\circ_{yy} = -\nu\ \varepsilon^\circ_{xx} \end{aligned} \tag{340}$$

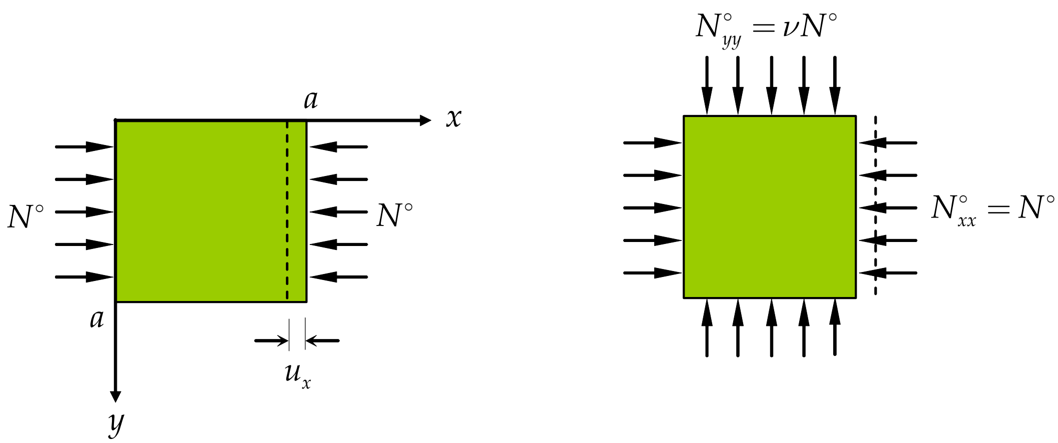
By applying the geometric equation between strain and the displacement and considering $\delta w = 0$, here, one gets the relation between u_x and u_y:

$$\frac{\partial u_y}{\partial y} = -\nu \frac{\partial u_x}{\partial x} \tag{341}$$

Integrating both sides over the plate length leads:

$$\int_0^a \frac{\partial u_y}{\partial y} dy = -\nu \int_0^a \frac{\partial u_x}{\partial x} dx \qquad (342)$$
$$u_y = -\nu \; u_x$$

Constraint in In-Plane Displacement Consider a plate fully constrained in the y-direction, $u_y = 0$.



Consequently, one also gets:

$$\varepsilon^\circ_{yy} = \frac{\partial u_y}{\partial y} = 0 \qquad (343)$$

Under the uniform compression, ε°_{xx} reads:

$$\varepsilon^\circ_{xx} = \frac{\partial u_x}{\partial x} = \frac{u_x}{a} \qquad (344)$$

From the constitutive relation, the membrane forces reads:

$$N_{xx} = \frac{Eh}{1-\nu^2} \; \left[\varepsilon^\circ_{xx} + \nu \; \varepsilon^\circ_{yy}\right] = \frac{Eh}{1-\nu^2} \; \frac{u_x}{a} = -N^\circ_{xx} \qquad (345)$$
$$N_{yy} = \frac{Eh}{1-\nu^2} \; \left[\varepsilon^\circ_{yy} + \nu \; \varepsilon^\circ_{xx}\right] = \frac{Eh}{1-\nu^2} \; \nu \; \frac{u_x}{a} = -N^\circ_{yy}$$

Finally, the membrane force tensor reads:

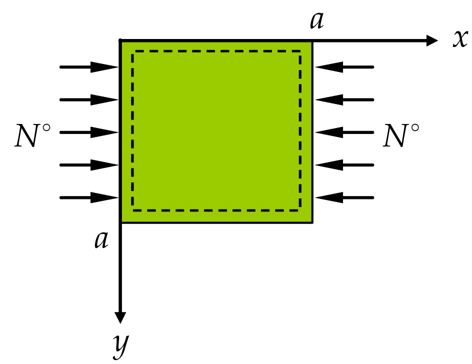
$$N^\circ_{\alpha\beta} = \begin{vmatrix} N^\circ_{xx} & 0 \\ 0 & N^\circ_{yy} \end{vmatrix} = \lambda \; \begin{vmatrix} 1 & 0 \\ 0 & \nu \end{vmatrix} \qquad (346)$$

where

$$\lambda = \frac{Eh}{1-\nu^2} \; \frac{u_x}{a} \qquad (347)$$

4.3.8 Exercise 2: Raleigh-Ritz Quotient for Simply Supported Square Plate under Uniaxial Loading

Consider a simply supported square plate subjected to uniform compressive load in the x-direction.



Then, the membrane force tensor reads:

$$N^{\circ}_{\alpha\beta} = N^{\circ} \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = \lambda \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \tag{348}$$

The plate will deform into a dish, so for the trial function take the following:

$$\phi = \hat{\phi}\,(x, y) = \sin\left(\frac{\pi\ x}{a}\right)\sin\left(\frac{\pi\ y}{a}\right) \tag{349}$$

Now, in order to obtain the load intensity, we first calculate $\phi_{,x}$ and $\nabla^2\phi$:

$$\begin{aligned} \phi_{,x} &= \frac{\pi}{a}\cos\left(\frac{\pi\ x}{a}\right)\sin\left(\frac{\pi\ y}{a}\right) \\ \phi_{,y} &= \frac{\pi}{a}\sin\left(\frac{\pi\ x}{a}\right)\cos\left(\frac{\pi\ y}{a}\right) \end{aligned} \tag{350}$$

$$\begin{aligned} \phi_{,xx} &= -\left(\frac{\pi}{a}\right)^2\sin\left(\frac{\pi\ x}{a}\right)\sin\left(\frac{\pi\ y}{a}\right) \\ \phi_{,yy} &= -\left(\frac{\pi}{a}\right)^2\sin\left(\frac{\pi\ x}{a}\right)\sin\left(\frac{\pi\ y}{a}\right) \end{aligned} \tag{351}$$

$$\begin{aligned} \nabla^2\phi &= \phi_{,xx} + \phi_{,yy} \\ &= -2\ \left(\frac{\pi}{a}\right)^2\sin\left(\frac{\pi\ x}{a}\right)\sin\left(\frac{\pi\ y}{a}\right) \end{aligned} \tag{352}$$

$$\nabla^2\phi\ \nabla^2\phi = 4\ \left(\frac{\pi}{a}\right)^4\sin^2\left(\frac{\pi\ x}{a}\right)\sin^2\left(\frac{\pi\ y}{a}\right) \tag{353}$$

Now, the Raleigh-Ritz quotient is calculated:

$$N^{\circ} = \lambda = \frac{D \int_S \; \nabla^2\phi \; \nabla^2\phi \; dS}{\int_S \left(\phi_{,x}\right)^2 \; dS} \tag{354}$$

$$= \frac{4D \; \left(\frac{\pi}{a}\right)^4}{\left(\frac{\pi}{a}\right)^2} \frac{\int_S \sin^2\left(\frac{\pi \; x}{a}\right) \sin^2\left(\frac{\pi \; y}{a}\right) \; dS}{\int_S \cos^2\left(\frac{\pi \; x}{a}\right) \sin^2\left(\frac{\pi \; y}{a}\right) \; dS}$$

$$= 4D \; \left(\frac{\pi}{a}\right)^2$$

$$\boxed{N^{\circ}_{cr} = \lambda = 4D\left(\frac{\pi}{a}\right)^2} \tag{355}$$

This is the classical buckling solution, and it is exact because of the right guess of the displacement field.

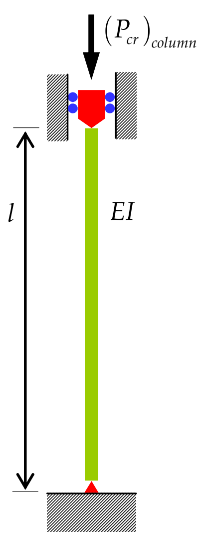
4.4 Buckling of Sections

4.4.1 Transition from Global and Local Buckling

Euler buckling load of a simply-supported column reads:

$$\left(P_{cr}\right)_{column} = \frac{\pi^2 EI}{l^2} \tag{356}$$

where I is the bending rigidity of the column. Consider a section column which is composed of several thin plates, then the Euler buckling load can be considered as a global buckling load of the column.

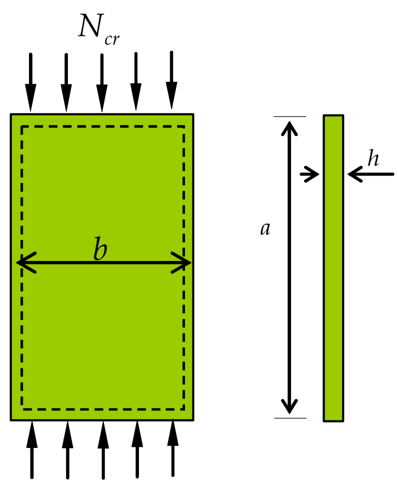


On the other hands, local buckling force of a simply-supported plate reads:

$$N_{cr} = \frac{k_c \pi^2 D}{b^2} \tag{357}$$

where $D = Eh^3/\left[12\left(1-\nu^2\right)\right]$ and $k_c = \left[(mb)/a + a/(mb)\right]^2$. Thus, the total local buckling load can be obtained:

$$(P_{cr})_{plate} = \frac{k_c \pi^2 D}{b} \tag{358}$$

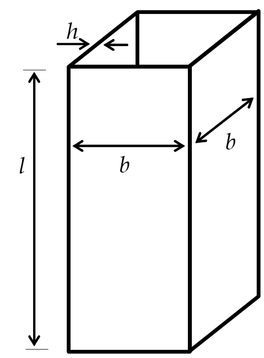


Transition from global to local buckling can be calculated by $(P_{cr})_{column} = (P_{cr})_{plate}$:

$$\begin{aligned} \pi^2 \frac{EI}{l^2} &= k_c \frac{\pi^2 D}{b} \\ &= \frac{k_c \pi^2 E}{12\left(1-\nu^2\right)} \frac{h^3}{b} \end{aligned} \tag{359}$$

$$\frac{I}{bh^3} = \frac{k_c}{12\left(1-\nu^2\right)} \left(\frac{l}{b}\right)^2 \tag{360}$$

Example 7 *Consider a square box column.*



Then, one obtains:

$$k_c = 4$$
$$I = \frac{2}{3} h b^3$$

Now, the global buckling load reads:

$$(P_{cr})_{column} = \pi^2 E \frac{I}{l^2} = \frac{2\pi^2 E}{3} \frac{hb^3}{l^2}$$

and the local buckling load from four plates can be calculated:

$$(P_{cr})_{four\ plates} = \frac{4 k_c \pi^2 E}{12\,(1-\nu^2)} \frac{h^3}{b}$$

Then, applying $(P_{cr})_{column} = (P_{cr})_{four\ plates}$, *one gets:*

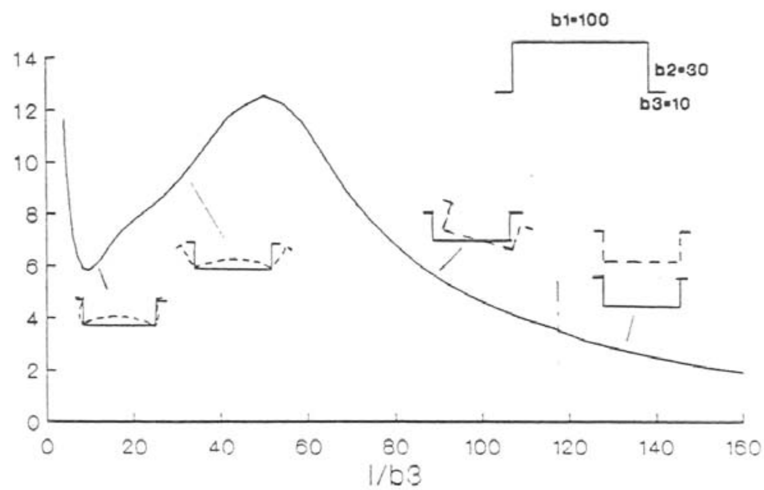
$$\left(\frac{b^2}{hl}\right)^2 \simeq 2.20$$

Thus, the local and global buckling loads become same when

$$b^2 \simeq 1.5hl$$

For example, if $b = 40h$, *then* $l = 60b$.

Transition from the local to global buckling for an open channel section with lips is shown in the figure below.



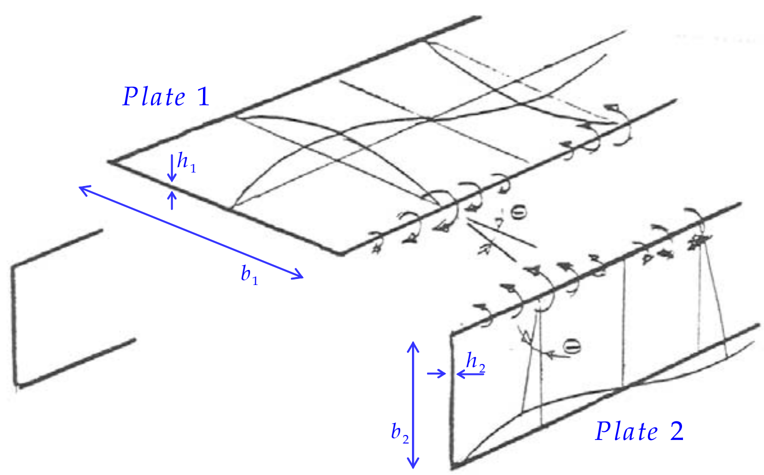
Buckling coefficients and modes for a hat section

4.4.2 Local Buckling

The remainder of this section deals only with the local buckling. Dividing both sides of Eq. (357) by the plate thickness b gives the expression of the buckling stress σ_{cr}:

$$\sigma_{cr} = \frac{N_{cr}}{h} = \frac{k_c \pi^2 E}{12\,(1-\nu^2)} \left(\frac{h}{b}\right)^2 \tag{361}$$

Consider two adjacent plates of a section of the prismatic column.



Compatibility and equilibrium conditions at junction of adjoining walls of a section

In general, there will be a restraining moment acting at the corner line between

Plate 1 and *Plate* 2. The buckling stresses for those two plates are:

$$(\sigma_{cr})_1 = \frac{k_1 \pi^2 E}{12(1-\nu^2)} \left(\frac{h_1}{b_1}\right)^2 \qquad (362)$$
$$(\sigma_{cr})_2 = \frac{k_2 \pi^2 E}{12(1-\nu^2)} \left(\frac{h_2}{b_2}\right)^2$$

Before buckling, stresses in the entire cross-section are the same. So, at the point of buckling, one gets:

$$(\sigma_{cr})_1 = (\sigma_{cr})_2 \qquad (363)$$

from which the buckling coefficient k_2 is relating to k_1:

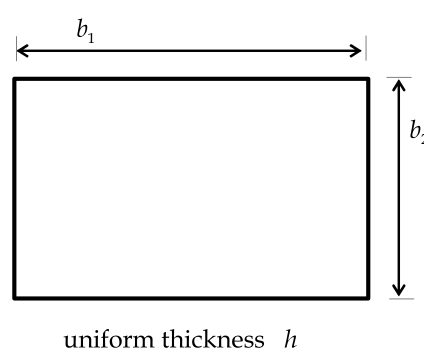
$$k_2 = k_1 \left(\frac{h_1}{h_2}\frac{b_2}{b_1}\right)^2 \qquad (364)$$

The total buckling load on the *angle* element is:

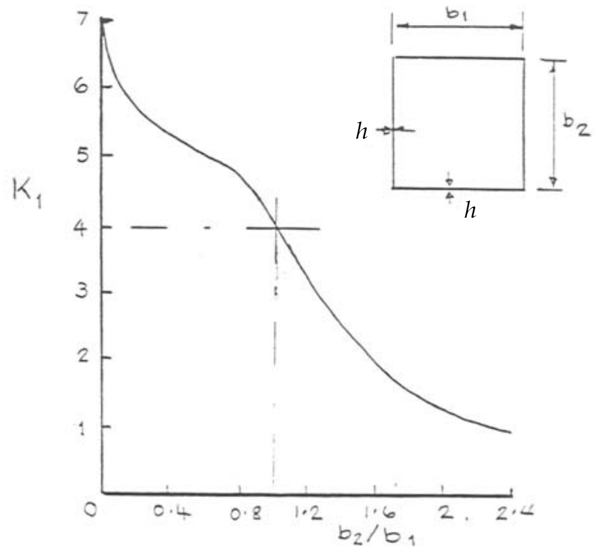
$$\begin{aligned} P_{cr} &= (\sigma_c)_1 h_1 b_1 + (\sigma_c)_2 h_2 b_2 \qquad (365) \\ &= \frac{\pi^2 E}{12(1-\nu^2)} \left[k_1 \left(\frac{h_1}{b_1}\right)^2 h_1 b_1 + k_2 \left(\frac{h_2}{b_2}\right)^2 h_2 b_2\right] \\ &= \frac{\pi^2 E}{12(1-\nu^2)} \left[k_1 \frac{(h_1)^3}{b_1} + k_1 \left(\frac{h_1}{h_2}\frac{b_2}{b_1}\right)^2 \frac{(h_2)^3}{b_2}\right] \\ &= \frac{\pi^2 E k_1}{12(1-\nu^2)} \left(\frac{h_1}{b_1}\right)^2 A \end{aligned}$$

where A is the sectional area of two plate $A = b_1 h_1 + b_2 h_2$. From this derivation, the conclusion is that only one buckling coefficient is needed to calculate the buckling load of the section consisting of several plates.

Determination of the buckling coefficient is a bit more complicated because of the existence of the edge bending moment. This can be illustrated in an example of a box column with a rectangular cross section with the same thickness h.

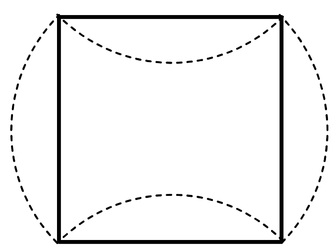


The wider flange will be ready to buckling first while the narrow plate is not ready to buckle. When the second plate buckles, the first plate would have buckled long before. Thus, there is an interaction between left plates, and a compromise must be established because left plates must buckle at the same time. The buckling coefficient as a function of the ratio b_2/b_1 is plotted in the figure next.

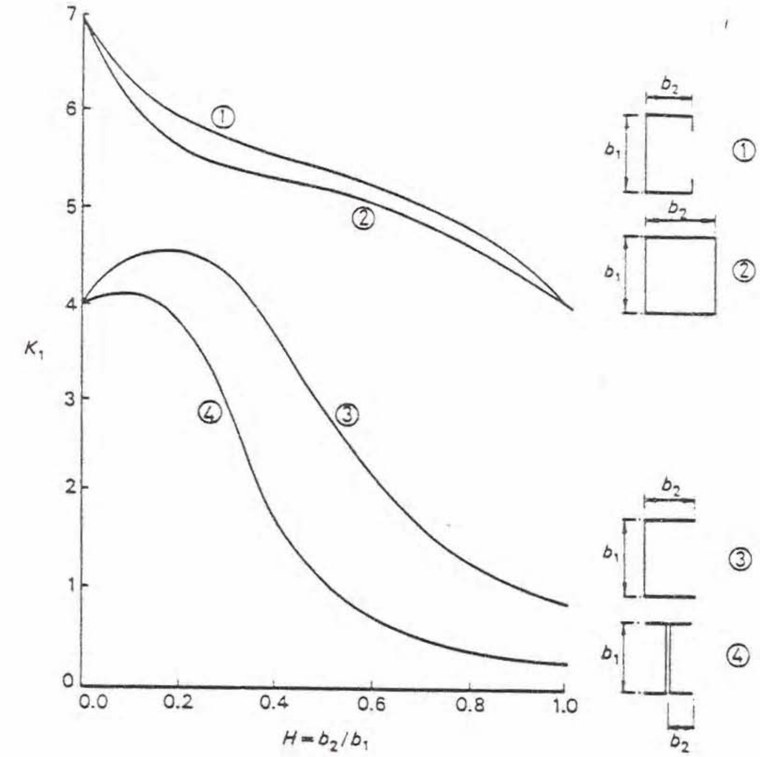


Buckling coefficients for box sections

In the limiting case of a square box ($b_1 = b_2 = b$), $k_1 = 4$ and the edge interactive moment between adjacent plates is zero.



Some useful graphs and formulas for typical sections are given next.



Curve 1: lipped channel

$$K_1 \simeq 7 - \frac{1.8H}{0.15 + H} - 1.43H^3.$$

Curve 2: box section

$$K_1 \simeq 7 - \frac{2H}{0.11 + H} - 1.2H^3.$$

Curve 3: plain channel

$$K_1 \simeq \frac{2}{\beta_0} + \frac{2 + 4.8H}{\beta_0^2}, \quad \text{where } \beta_0 = \sqrt{(1 + 15H^3)}.$$

Curve 4: I-section made from two channels fixed back-to-back

$$K_1 \simeq \frac{2}{\gamma_0} + \frac{2 + H}{\gamma_0^2}, \quad \text{where } \gamma_0 = \sqrt{(1 + 90H^4)}.$$

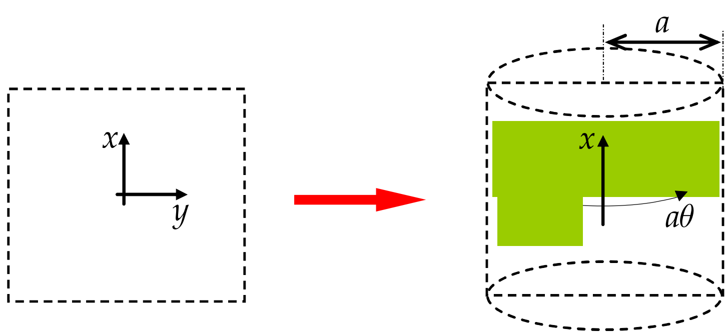
5 Buckling of Cylindrical Shells

5.1 Governing Equation for Buckling of Cylindrical Shells

The starting point of the analysis is the strain-displacement relation for plates:

$$\begin{aligned} \varepsilon^{\circ}_{\alpha\beta} &= \frac{1}{2}\left(u_{\alpha},_{\beta} + u_{\beta},_{\alpha}\right) + \frac{1}{2}\, w,_{\alpha}\, w,_{\beta} \\ \kappa_{\alpha\beta} &= -\, w,_{\alpha\beta} \end{aligned} \tag{366}$$

Consider a flat plate (x, y) and a segment of a cylinder $(x, a\theta)$, where a is the radius of a cylinder and θ is the hoop coordinate.



Can the strain-displacement relation for a cylinder be derived from similar relation for a flat plate?

$$x \rightarrow x \qquad ; \qquad dx \rightarrow dx \tag{367}$$

$$y \rightarrow a\theta \qquad ; \qquad \frac{\partial}{\partial y} \rightarrow \frac{1}{a}\frac{\partial}{\partial \theta} \tag{368}$$

Consider component by component. Use the notation:

$$\begin{aligned} u_1 &= u_x \rightarrow u \\ u_2 &= u_y \rightarrow v \\ u_3 &\quad\; \rightarrow w \end{aligned} \tag{369}$$

Then, one gets:

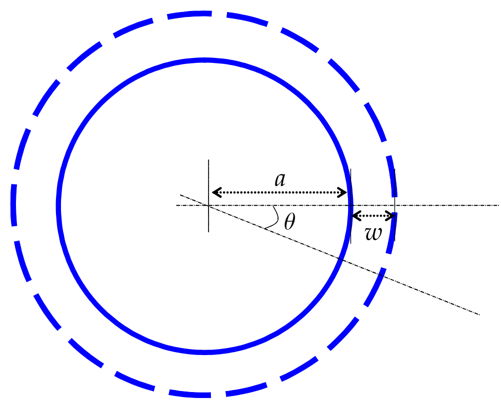
$$\varepsilon^{\circ}_{xx} = u,_{x} + \frac{1}{2}\left(w,_{x}\right)^2 \tag{370}$$

$$\varepsilon^{\circ}_{\theta\theta} = \frac{u,_{\theta}}{a} + \frac{1}{2}\left(\frac{w,_{\theta}}{a}\right)^2 + \frac{w}{a} \tag{371}$$

$$\varepsilon^{\circ}_{x\theta} = \frac{1}{2}\left(\frac{u,_{\theta}}{a} + u,_{x}\right) + \frac{1}{2} w,_{x}\frac{w,_{\theta}}{a} \tag{372}$$

There is a new term w/a in the expression for the hoop strain. The physical meaning of this new term becomes clear if we consider axisymmetric deformation with $v = 0$ and w independent of θ. Define the hoof strain as a relative change in the length of circumference when the original circle has a radius of "a" before deformation and a new circle has a radius of "$a + w$" after deformation:

$$\varepsilon^{\circ}_{\theta\theta} = \frac{2\pi (a + w) - 2\pi a}{2\pi a} = \frac{w}{a} \tag{373}$$



Mathematically, Eq. (370)-(372) can be derived from its counterparts by transforming the rectangular coordinate system into the curvilinear coordinate system.

$$\begin{aligned} x &= r \sin\theta \\ y &= r \cos\theta \end{aligned} \tag{374}$$

The step-by-step derivation can be found, for example, in the book by Y.C. Fung, "*First Course in the Continuum Mechanics.*" The expression for curvature are transformed in a similar way:

$$\begin{aligned} \kappa_{xx} &= w_{,xx} \\ \kappa_{\theta\theta} &= \frac{w_{,\theta\theta}}{a^2} \\ \kappa_{x\theta} &= \frac{w_{,x\theta}}{a} \end{aligned} \tag{375}$$

Using the variational approach explained in details for the plate problem, one can see that the only new term in the expression for $\delta\Pi = 0$ is

$$\int_S N_{\theta\theta} \frac{\delta w}{a} dS \tag{376}$$

Therefore, the new term should be added in the equation for out-of-plane equilibrium:

$$D\nabla^4 w + \frac{1}{a} N_{\theta\theta} - \left(N_{xx} w_{,xx} + \frac{2}{a} N_{x\theta} w_{,x\theta} + \frac{1}{a^2} N_{\theta\theta} w_{,\theta\theta} \right) = q \tag{377}$$

where

$$\nabla^4 w = w_{,xxxx} + \frac{2}{a^2} w_{,xx\theta\theta} + \frac{1}{a^4} w_{,\theta\theta\theta\theta} \tag{378}$$

The above equations are the nonlinear equilibrium equations for quasi-shallow cylindrical shells. The linear equilibrium equations are obtained by omission the nonlinear terms, i.e. terms in the parenthesis. The resulting equations are:

$$\begin{aligned} aN_{xx},_x + N_{x\theta},_\theta &= 0 \\ aN_{\theta\theta},_x + N_{\theta\theta},_\theta &= 0 \\ D\nabla^4 w + \frac{1}{a} N_{\theta\theta} &= q \end{aligned} \tag{379}$$

with

$$\begin{aligned} N_{xx} &= C\left(u,_x + \nu \frac{w}{a}\right) \\ N_{\theta\theta} &= C\left(\frac{w_0}{a} + \nu\ u,_x\right) \end{aligned} \tag{380}$$

where C is the axial rigidity, $C = Eh/\left(1-\nu^2\right)$. The pre-buckling solution should satisfy the system Eq. (379) and (380). These solutions will be denoted by $N_{\alpha\beta} = -N^\circ_{\alpha\beta}$.

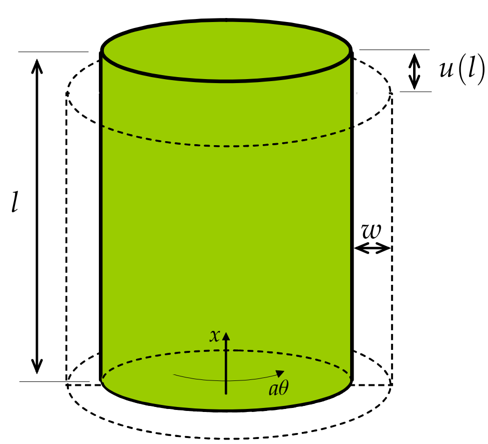
5.1.1 Special Case I: Cylinder under Axial Load P, $q = 0$

The membrane forces in the pre-buckling state are:

$$N^\circ_{\alpha\beta} = \frac{P}{2\pi a} \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \tag{381}$$

and the corresponding displacement field reads:

$$\begin{aligned} u\left(x\right) &= -\frac{P}{2\pi a h E}\ x \\ w &= \frac{\nu}{E}\frac{Pl}{2\pi a h} \end{aligned} \tag{382}$$



It is easy to prove that the above solution satisfies all field equation.

5.1.2 Special Case II: Cylinder under Lateral Pressure

$$N^{\circ}_{\alpha\beta} = N_{\theta\theta} \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} \tag{383}$$

where from the constitutive equations Eq. (380):

$$N_{\theta\theta} = Eh\frac{w}{a} \tag{384}$$

Substituting Eq. (384) into Eq. (379) leads the following linear forth order inhomogeneous ordinary differential equation for $w(x)$:

$$Dw'''' + \frac{Ehw}{a^2} = q \tag{385}$$

or in a dimensionless form:

$$\frac{d^4w}{dx^4} + 4\beta^4 w = \frac{q}{D} \tag{386}$$

where

$$\beta^4 = \frac{Eh}{a^2 D} = \frac{3\left(1-\nu^2\right)}{a^2 h^4} \tag{387}$$

The dimension of β is $[L^{-1}M^0T^0]$, so βx is dimensionless. There are four boundary conditions for a simply supported cylinder:

$$\begin{aligned} M_{xx} = w = 0 \qquad &\text{at} \quad x = 0 \\ M_{xx} = w = 0 \qquad &\text{at} \quad x = l \end{aligned} \tag{388}$$

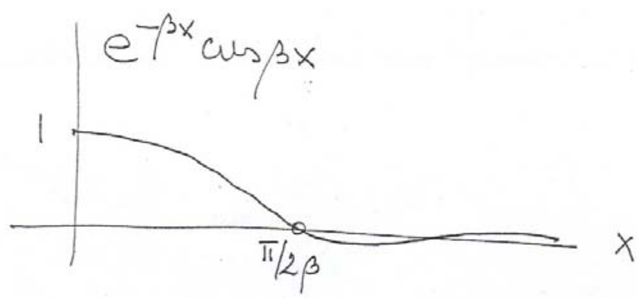
where one gets from the moment-curvature relation:

$$\begin{aligned} M_{xx} &= -D\frac{d^2w}{dx^2} \\ M_{\theta\theta} &= 0 \end{aligned} \tag{389}$$

The general solutions of the above boundary value problem is:

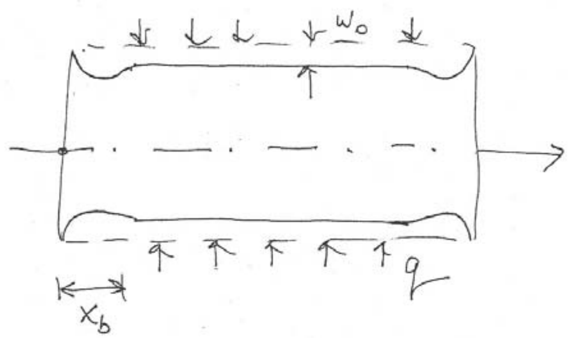
$$w(x) = e^{-\beta x}\left[c_1 \sin\beta x + c_2 \cos\beta x\right] + e^{\beta x}\left[c_3 \sin\beta x + c_4 \cos\beta x\right] - \frac{q}{D^4\beta^4} \tag{390}$$

The four integration constants can be found from the boundary conditions. A typical term of the solution is a rapidly decaying function of x.



It can be conducted that the curvature and bending is confined to a narrow boundary zone of the width $x_b = \pi / (2\beta)$. The remainder of the shell undergoes a uniform radial contraction:

$$w_0 = -\frac{q}{D\ 4\beta^4} \tag{391}$$



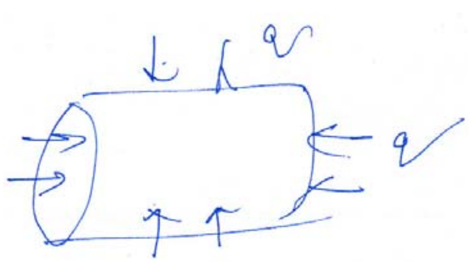
For the sake of simplicity, this localized bending can be neglected ($D \to 0$). Then, from Eq. (379), the hoop membrane force is related to the lateral pressure by $N_{\theta\theta} = qa$ and the pre-buckling solution is:

$$N^{\circ}_{\alpha\beta} = qa \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} \tag{392}$$

5.1.3 Special Case III: Hydrostatic Pressure

For a cylinder subjected to the hydrostatic pressure, the total axial compressive force is:

$$P = q\pi a^2 \tag{393}$$



The pre-buckling solution is:

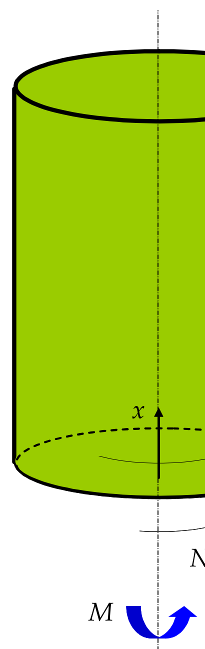
$$N^{\circ}_{\alpha\beta} = qa \begin{vmatrix} 1/2 & 0 \\ 0 & 1 \end{vmatrix} \tag{394}$$

which is a classical membrane stress state in a thin cylinder.

5.1.4 Special Case IV: Torsion of a Cylinder

The pre-buckling stress in a cylinder subjected to the total torque of the magnitude T is:

$$N^{\circ}_{\alpha\beta} = \frac{M}{2\pi a} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \tag{395}$$



5.2 Derivation of the Linearized Buckling Equation

We are now in the position to linearize the nonlinear buckling equation Eq. (377). It is assumed that the state of membrane forces does not change at the point of buckling from the pre-buckling value. Thus,

$$N_{\alpha\beta} = -N^{\circ}_{\alpha\beta} \tag{396}$$

and Eq. (377) becomes:

$$D\nabla^4 w + \frac{1}{a}N_{\theta\theta} + \left[N^{\circ}_{xx} w_{,xx} + \frac{2}{a} N^{\circ}_{x\theta} w_{,x\theta} + \frac{1}{a^2} N^{\circ}_{\theta\theta} w_{,\theta\theta} \right] = q \tag{397}$$

where the hoop membrane force $N_{\theta\theta}$, the second term in Eq. (397), depends linearly on three component of the displacement vector (u, v, w):

$$N_{\theta\theta} = C\left(\frac{1}{a} v_{,\theta} + \frac{w}{a} + \nu u_{,x} \right) \tag{398}$$

Therefore, the out-of-plane equilibrium equation is coupled with the in-plane displacement $(u,\ v)$ through the presence of the term $N_{\theta\theta}$. Note that in the plate buckling problem the in-plane and out-of-plane response was uncoupled. It is possible to eliminate the terms involving in-plane components using the full set of equilibrium and constitutive equations in the in-plane direction. By doing this, the order of the governing equation has to be raised by four to give eight:

$$D\nabla^8 w + \frac{1-\nu^2}{a^2} C w_{,xxxx} + \nabla^4 \left[N^{\circ}_{xx} w_{,xx} + \frac{2}{a} N^{\circ}_{x\theta} w_{,x\theta} + \frac{1}{a^2} N^{\circ}_{\theta\theta} w_{,\theta\theta} \right] = 0 \qquad (399)$$

The above equation is called the Donnell stability equation in the uncoupled form. Not that $w\,(x,\ \theta)$ in the above equation represents additional lateral deflection over and above those produced by the pre-buckling solution. The total deflection is a sum of the two.

5.3 Buckling under Axial Compression

5.3.1 Formulation for Buckling Stress and Buckling Mode

We are now in a position to develop solutions to the buckling equations, Eq. (399) for four different loading cases discussed in the previous section. Consider first Case I of a simply-supported cylindrical shell in which the pre-buckling solution is given by Eq. (381). In this case, Eq. (399) reduces to:

$$D\nabla^8 w + \frac{1-\nu^2}{a^2} C w_{,xxxx} + \frac{P}{2\pi a} \nabla^4 w_{,xx} = 0 \qquad (400)$$

The buckling deflection of the shell is assumed in the following form:

$$w\,(x,\theta) = c_1 \sin\left(\frac{m\pi x}{l}\right) \sin\,(n\theta) \qquad (401)$$

where c_1 is the magnitude, and the integer numbers $(m,\ n)$ denote the number of half-waves respectively in the axial and circumferential direction. The above deformation satisfies both simply-supported boundary conditions at the ends, $x = 0$ and $x = L$, and periodicity conditions along the circumference.

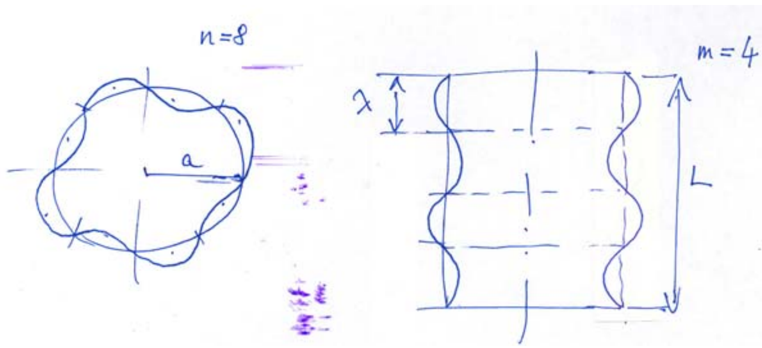
Here, the half-length of the buckling wave is defined:

$$\lambda = \frac{l}{m} \qquad (402)$$

It is convenient to introduce a dimensionless buckling number, $\bar{m}$:

$$\frac{m\pi x}{l} = \left(\frac{m\pi a}{l}\right) \frac{x}{a} = \bar{m} \frac{x}{a} \qquad (403)$$

$$\Rightarrow \bar{m} = \frac{m\pi a}{l}$$



Using the dimensionless buckling number, substituting the solution Eq. (401) into the governing equation Eq. (400) leads:

$$\left[\frac{D}{a^2}\left(\bar{m}^2+n^2\right)^4+\bar{m}^4\left(1-\nu^2\right)C-\frac{P}{2\pi a}\left(\bar{m}^2+n^2\right)^2\bar{m}^2\right]\frac{c_1}{a^6}\sin\left(\bar{m}\frac{x}{a}\right)\sin\left(n\theta\right)=0 \tag{404}$$

By setting the coefficient in the square bracket to zero, the critical buckling membrane force per unit length becomes:

$$\begin{aligned} N_{cr} &= \frac{P_{cr}}{2\pi a} \\ &= \frac{D}{a^2}\frac{\left(\bar{m}^2+n^2\right)^2}{\bar{m}^2}+\left(1-\nu^2\right)C\frac{\bar{m}^2}{\left(\bar{m}^2+n^2\right)^2} \end{aligned} \tag{405}$$

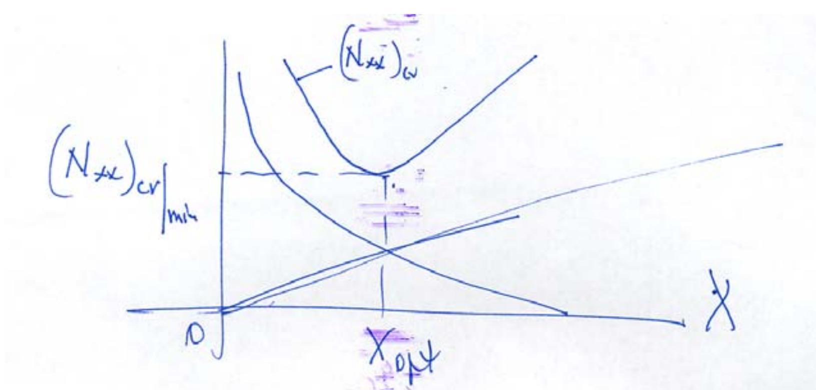
Here, by introducing the dimensionless parameter χ:

$$\chi=\frac{\left(\bar{m}^2+n^2\right)^2}{\bar{m}^2} \tag{406}$$

Eq. (405) reads:

$$N_{cr}=\frac{D}{a^2}\chi+\left(1-\nu^2\right)C\frac{1}{\chi} \tag{407}$$

The dependence of the buckling force on the parameter χ is shown in the figure below.



Treating χ as a continuous variable, one can find an analytical minimum:

$$\frac{dN_{cr}}{d\chi} = \frac{D}{a^2} - \left(1-\nu^2\right) C \frac{1}{\chi^2} = 0 \tag{408}$$

from which the optimum value of the parameter χ is:

$$\begin{aligned} \chi_{opt} &= \sqrt{\frac{(1-\nu^2)\, Ca^2}{D}} \\ &= \frac{a}{h}\sqrt{12\,(1-\nu^2)} \\ &\simeq 3.3\frac{a}{h} \end{aligned} \tag{409}$$

Introducing the expression for the optimum parameter χ into Eq. (407) and using definitions of bending and axial rigidity, one gets:

$$(N_{cr})_{\min} = \frac{E}{\sqrt{3\,(1-\nu^2)}}\frac{h^2}{a} \tag{410}$$

The buckling stress is obtained by dividing the critical membrane fore by the shell thickness h:

$$\begin{aligned} \sigma_{cr} &= \frac{(N_{cr})_{\min}}{h} \\ &= \frac{E}{\sqrt{3\,(1-\nu^2)}}\frac{h}{a} \end{aligned} \tag{411}$$

$$\boxed{\sigma_{cr} \simeq 0.605\; E\frac{h}{a}} \tag{412}$$

This is the classical solution for the buckling stress of a cylindrical shell subjected to axial compression. While the buckling load is unique and does not depend on $(\bar{m},\, n)$, the buckling mode is not unique as:

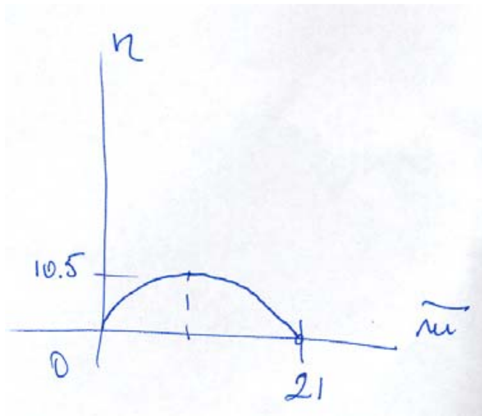
$$\chi_{opt} = \frac{\left(\bar{m}^2 + n^2\right)^2}{\bar{m}^2} = 3.3\frac{a}{h} \tag{413}$$

There are infinity of combinations of $\bar{m}$ and n that give the same expression.

Example 8 *Let* $a/h = 134$*, then*

$$\begin{aligned} \chi_{opt} &= \frac{\left(\bar{m}^2 + n^2\right)^2}{\bar{m}^2} = 3.3 \times 134 \simeq 442 \\ \Rightarrow &\quad \frac{\bar{m}^2 + n^2}{\bar{m}} \simeq 21 \end{aligned}$$

$$n = \sqrt{21\bar{m} - \bar{m}^2}$$



5.3.2 Buckling Coefficient and Batdorf Parameter

Let us introduce the dimensionless buckling stress or the buckling coefficient k_c:

$$k_c = \sigma_{cr} \frac{hl^2}{\pi^2 D} \tag{414}$$

Additionally, the Batdorf parameter Z is defined:

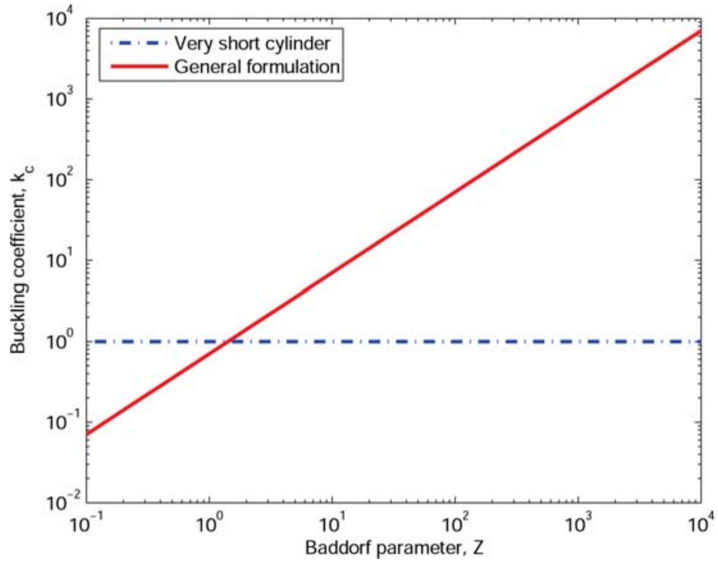
$$Z = \sqrt{1 - \nu^2} \frac{l^2}{ah} \tag{415}$$

Using Eq. (411), the buckling coefficient becomes:

$$\begin{aligned} k_c &= \frac{E}{\sqrt{3(1-\nu^2)}} \frac{h}{a} \times \frac{hl^2}{\pi^2 D} \\ &= \frac{12}{\sqrt{3}\pi^2} \sqrt{1-\nu^2} \frac{l^2}{ah} \\ &= \frac{12}{\sqrt{3}\pi^2} Z \end{aligned} \tag{416}$$

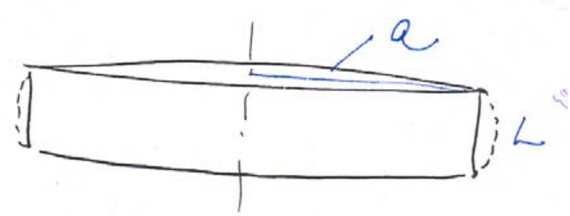
$$\boxed{k_c \simeq 0.702\ Z} \tag{417}$$

This relation between k_c and Z are shown in the figure below together with two limiting cases of very short and very long cylindrical shells. These two limiting cases are discussed below.



Limiting Cases: Short Cylinders, $a \gg l$ Consider a case of the following conditions:

$$a \gg l$$
$$\Rightarrow \quad \frac{a}{l} \to \infty \tag{418}$$



Then, it is natural to assume that the number of half-waves in the axial direction is unity:

$$m = 1$$

Consequently, one gets:

$$\bar{m} = \pi \frac{a}{l} \to \infty \tag{419}$$

Since $\bar{m}$ is much larger than n, one also gets:

$$\bar{m}^2 + n^2 \simeq \bar{m}^2 \tag{420}$$

Now, from Eq. (405), the membrane force reads:

$$\begin{aligned} N_{cr} &= \frac{P}{2\pi a} \\ &= \frac{D}{a^2}\bar{m}^2 + \left(1-\nu^2\right) C \frac{1}{\bar{m}^2} \\ &= \frac{\bar{m}^2 D}{a^2} \\ &= \frac{\pi^2 D}{l^2} \end{aligned} \tag{421}$$

The buckling stress reads:

$$\sigma_{cr} = \frac{N_{cr}}{h} = \frac{\pi^2 D}{hl^2}$$

Now, from the definition of the buckling coefficient, Eq. (414), the buckling coefficient for the very short cylindrical shells reads:

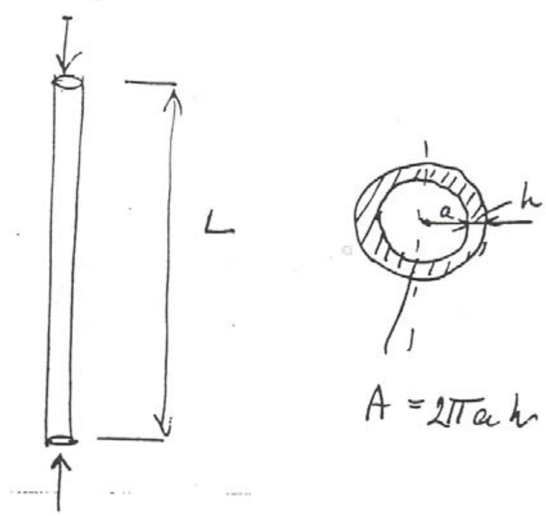
$$\boxed{k_c = 1}$$

This solution indicated in previous figure as the lower bound cut-off value. The upper bound cut-off values is given by the Euler buckling load.

Limiting Cases: Very Long Cylinders, $l \gg a$ The cylindrical shell is becoming the Euler column. The buckling load of the column is:

$$P_{cr} = \frac{\pi^2 EI}{l^2} \tag{422}$$

where $I = \pi a^3 h$ for cylinder sections.



The buckling stress reads:

$$\begin{aligned}\sigma_{cr} &= \frac{P_{cr}}{2\pi a h} \\ &= \frac{\pi^2}{2} E \left(\frac{a}{l}\right)^2\end{aligned} \tag{423}$$

For this very long cylindrical shells, the buckling coefficient can be written:

$$k_c = 6\left(1-\nu^2\right)\left(\frac{a}{h}\right)^2 \tag{424}$$

$$\boxed{k_c = 6\left(\frac{a}{l}\right)^4 Z^2} \tag{425}$$

From Eq. (411) and (423), the transition between the local shell buckling and global thin-walled column buckling occurs when

$$(\sigma_{cr})_{shell} = (\sigma_{cr})_{column} \tag{426}$$

which gives:

$$\frac{E}{\sqrt{3\left(1-\nu^2\right)}}\frac{h}{a} = \frac{\pi^2}{2} E \left(\frac{a}{l}\right)^2 \tag{427}$$

$$\Rightarrow \frac{hl^2}{a^3} = \frac{\pi^2\sqrt{3\left(1-\nu^2\right)}}{2} \simeq 8.15 \tag{428}$$

5.4 Buckling under Lateral Pressure

From the pre-buckling solution, Eq. (392), the governing equation Eq. (399) becomes:

$$D\nabla^8 w + \frac{1-\nu^2}{a^2} C w_{,xxxx} + \frac{q}{a}\nabla^4 w_{,\theta\theta} = 0 \tag{429}$$

Assuming the double sine buckling deflection function, similar to the case of axial compression, the governing equation becomes:

$$\left[\frac{D}{a^2}\left(\bar{m}^2+n^2\right)^4 + \bar{m}^4\left(1-\nu^2\right)C - qa\; n^2\left(\bar{m}^2+n^2\right)^2\right]\frac{c_1}{a^6}\sin\left(\bar{m}\frac{x}{a}\right)\sin\left(n\theta\right) = 0 \tag{430}$$

By setting the coefficient in the square bracket to zero, the equation for the buckling pressure becomes:

$$qa = \frac{D}{a^2}\frac{\left(\bar{m}^2+n^2\right)^2}{n^2} + \left(1-\nu^2\right)C\frac{\bar{m}^4}{n^2\left(\bar{m}^2+n^2\right)^2} \tag{431}$$

It can be shown that the smallest eigenvalue is obtained when $m = 1$ or $\bar{m} = \pi a/l$. With this observation, the solution becomes a function of the parameter n, or dimensionless parameter $\bar{n}$:

$$\bar{n} = \frac{nl}{\pi a} \tag{432}$$

Additionally, we define the dimensionless buckling pressure, $\bar{q}$:

$$\bar{q} = q\frac{l^2 a}{\pi^2 D} \tag{433}$$

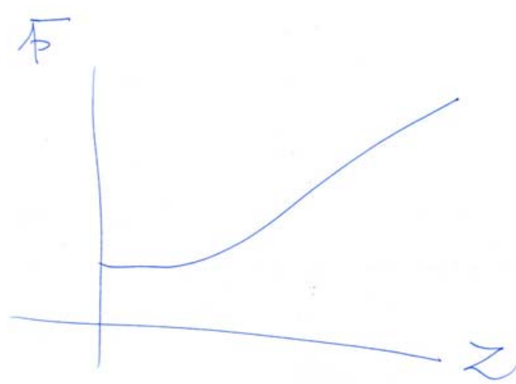
Now, substituting $\bar{m}$, $\bar{n}$ and $\bar{p}$ into Eq. (431) leads:

$$\bar{q} = \frac{\left(1+\bar{n}^2\right)^2}{\bar{n}^2} + \frac{1}{\bar{n}^2\left(1+\bar{n}^2\right)^2}\frac{\left(1-\nu^2\right)C}{a^2}\left(\frac{l}{a}\right)^2 \tag{434}$$

By introducing the Batdorf parameter Z, one gets:

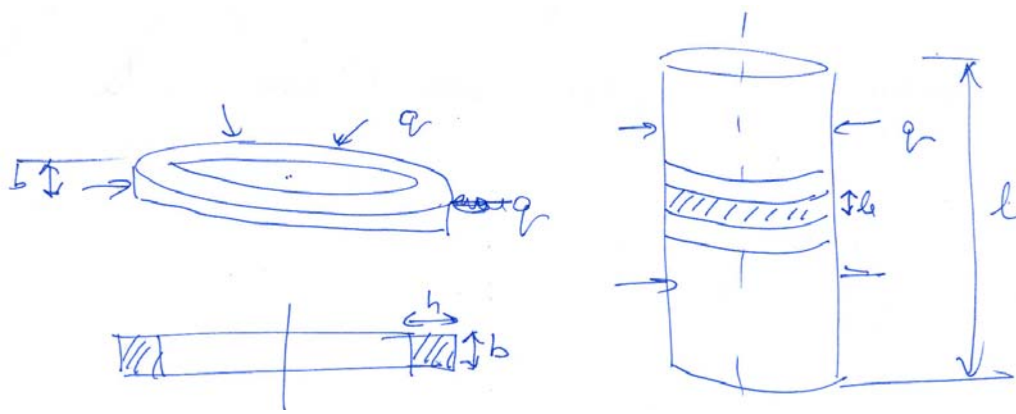
$$\bar{q} = \frac{\left(1+\bar{n}^2\right)^2}{\bar{n}^2} + \frac{1}{\bar{n}^2\left(1+\bar{n}^2\right)^2}\frac{12}{\pi}Z^2 \tag{435}$$

For any value of the geometrical parameter Z, there exists a preferred $\bar{n}$ which minimize the buckling pressure. Treating $\bar{n}$ as a continuous variable, the optimum $\bar{n}$ can be found analytically from $d\bar{p}/d\bar{n} = 0$. Substituting this back into Eq. (432), there will be a unique relation between the buckling pressure and the Batdorf parameter. The solution is shown graphically in the figure below.



Limiting Cases: Very Long Cylinders, $l \gg a$ In the limiting case of a long tube ($l \gg a$), one gets:

$$\bar{m} = m\pi\frac{a}{l} \rightarrow 0 \tag{436}$$



In this case, the last term in Eq. (431) vanishes and the buckling pressure becomes:

$$q = \frac{n^2\ D}{a^3} \tag{437}$$

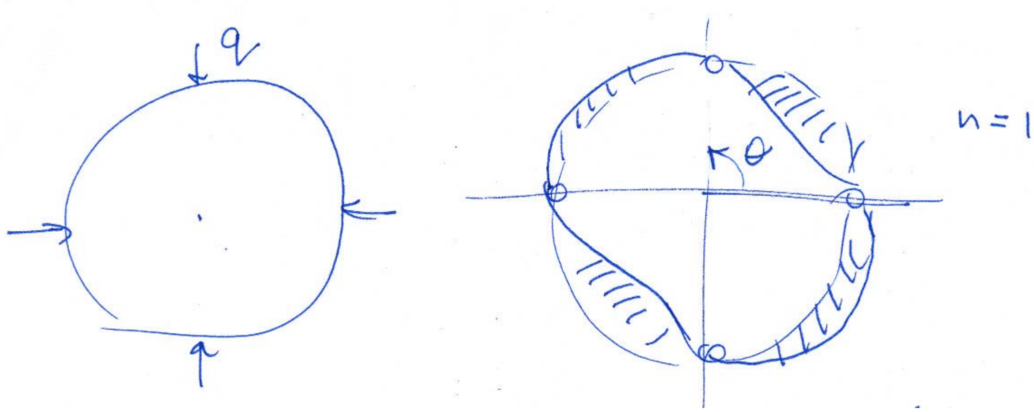
Imagine a long cylinder consisting of a change of ring, each of the height b. The moment of inertia of the ring along the axial axis reads:

$$I = \frac{bh^3}{12} \tag{438}$$

From Eq. (437) and (438), the intensity of the line load $Q = qb$ can be written:

$$\begin{aligned} Q &= qb \qquad (439)\\ &= \frac{n^2}{a^3}\frac{Eh^3}{12\,(1-\nu^2)}b \\ &= \frac{n^2 EI}{a^3\,(1-\nu^2)} \end{aligned}$$

The above approximation is due to Donnell. The smallest integer value is $n = 1$ which gives the following buckling mode.

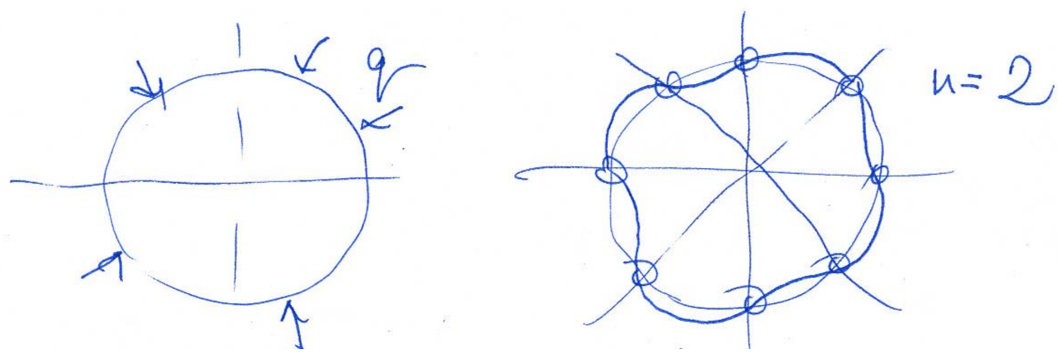


The Donnell solution should be compared with more exact solution of the ring buckling problem which take into account a more complex incremental displacement field with both $w(\theta)$ and $v(\theta)$. Here, a distinction should be made between the centrally directed pressure (as in all preceding analysis) and the field -pressure loading where pressure is always directed normal to the deformed surface.

In the later case, the ring buckling occurs at:

$$Q_c = \left(n^2 - 1\right) \frac{EI}{a^3} \tag{440}$$

where the smallest integer $n = 2$ so that $Q = 3EI/a^3$. The buckling mode has now eight nodal points rather than four.



The solution of centrally directed pressure loading case is:

$$Q_c = \frac{\left(n^2 - 1\right)^2}{n^2 - 2} \frac{EI}{a^3} \tag{441}$$

where again the smallest $n = 2$. Thus the smallest buckling load intensity is:

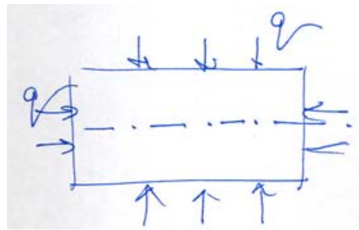
$$Q_c = 4.5 \frac{EI}{a^3} \tag{442}$$

It is seen that for a realistic value n=2, the present solution, Eq. (439) gives the buckling pressure between the two cases of field-pressure loading and centrally directed loading, $3 < 4 < 4.5$.

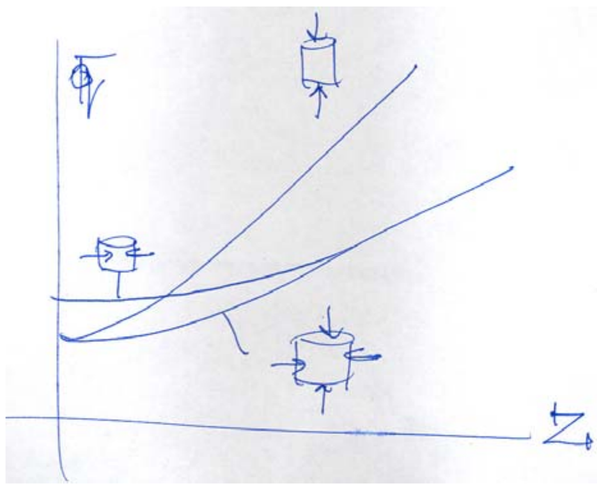
5.5 Buckling under Hydrostatic Pressure

Special case of the combined loading in which the total axial load P is:

$$P = \pi a^2 q \tag{443}$$



The pre-buckling solution is given by Eq. (394). The solution of the buckling equation still can be sought through the sinusoidal function, Eq. (401). The optimum solution can be found by a trial and error method varying parameters $\bar{m}$ and $\bar{n}$. A graphical representation of the solution is shown in figure below.

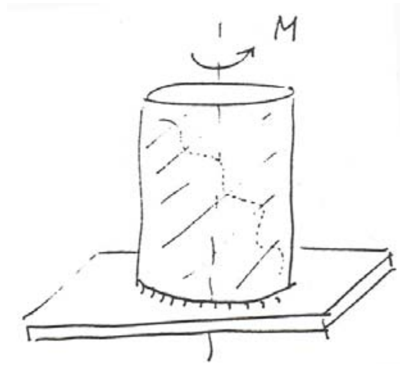


5.6 Buckling under Torsion

A twist applied to one end of a cylindrical shell process a twisting force $N_{x\theta}$:

$$N^{\circ}_{x\theta} = \frac{M}{2\pi a} \tag{444}$$

The other two components of the pre-buckling membrane forces vanish $N^{\circ}_{xx} = N^{\circ}_{\theta\theta} = 0$. Moreover, the force is constant.



Under these conditions, the governing equation reduces to:

$$D\nabla^8 w + \frac{1-\nu^2}{a^2} C \; w_{,xxxx} + \frac{2}{a} N^{\circ}_{x\theta} \; \nabla^4 w_{,x\theta} = 0 \tag{445}$$

In view of the presence of odd-ordered derivatives in the above equation, the separable form of the solution for $w\,(x,\theta)$, assumed previously, does not satisfy the equation.

Under torsional loading, the buckling deformation consists of circumferential waves that spiral around the cylindrical shell from one end to the other. Such waves can be represented by a deflection function of the form:

$$w\,(x,\theta) = C \sin\left(\bar{m}\frac{x}{a} - n\theta\right) \tag{446}$$

with

$$\bar{m} = \frac{m\pi a}{L} \tag{447}$$

where m and n are integers. The alone displacement field satisfy the differential equation and the periodical condition is the circumferential direction, but does not satisfy any commonly used boundary at the cylinder ends. Consequently, this simple examples can be used only for long cylinders.

For such cylinders, introduction of the additional displacement field into the governing equation, yields:

$$N^{\circ}_{x\theta} = \frac{\left(\bar{m}^2 + n^2\right)^2}{2\;\bar{m}\;n}\frac{D}{a^2} + \frac{\bar{m}^3}{2\left(\bar{m}^2 + n^2\right)^2\;n}\left(1 - v^2\right) C \tag{448}$$

For sufficiently long cylinders, the shell buckles in two circumferential waves, $n = 2$. Also, the term $\bar{m}^2$ is small compared with 4. Then, the approximate expression is:

$$N^{\circ}_{x\theta} = \frac{4}{\bar{m}}\frac{D}{a^2} + \frac{\bar{m}^3}{64}\left(1 - v^2\right) C \tag{449}$$

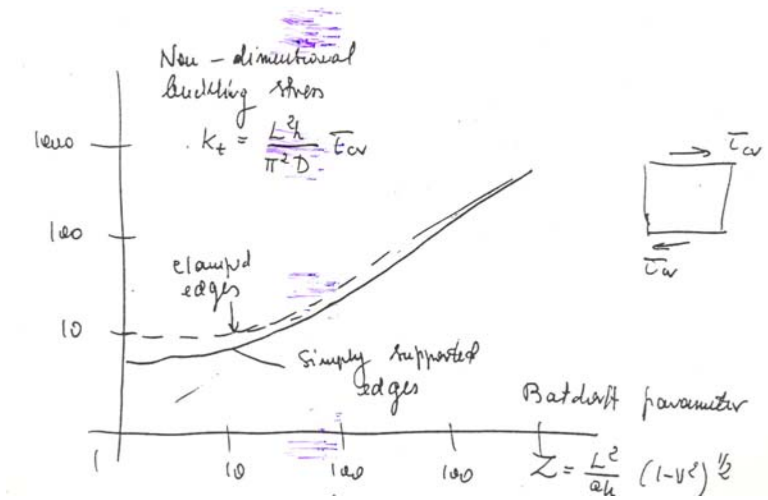
An analytical minimization of the alone experiment with repeat to $\bar{m}$ gives:

$$\bar{m}^4 = \frac{64}{9\left(1-\nu^2\right)}\left(\frac{h}{a}\right)^2 \tag{450}$$

Upon substitution, the final expression for the buckling force, or better, critical shear strain causing buckling is:

$$\tau_{cr} = \frac{N^{\circ}_{x\theta}}{h} = \frac{0.272\ E}{(1-\nu^2)^{3/4}} \left(\frac{h}{a}\right)^{3/2} \tag{451}$$

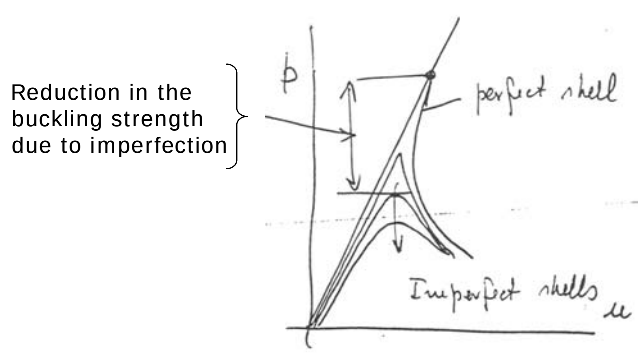
The above solution was given by Donnell. As noted, the above solution is invalid for short shells due to the difficulties in satisfying boundary condition. A more rigorous analytical-numerical solution is shown in the figure below.



As the radius of the shell approaches infinity, the critical stress coefficient for simply supported and clamped edge approaches respectively the value 5.35 and 8.98 corresponding to plates under the shear loading.

5.7 Influence of Imperfection and Comparison with Experiments

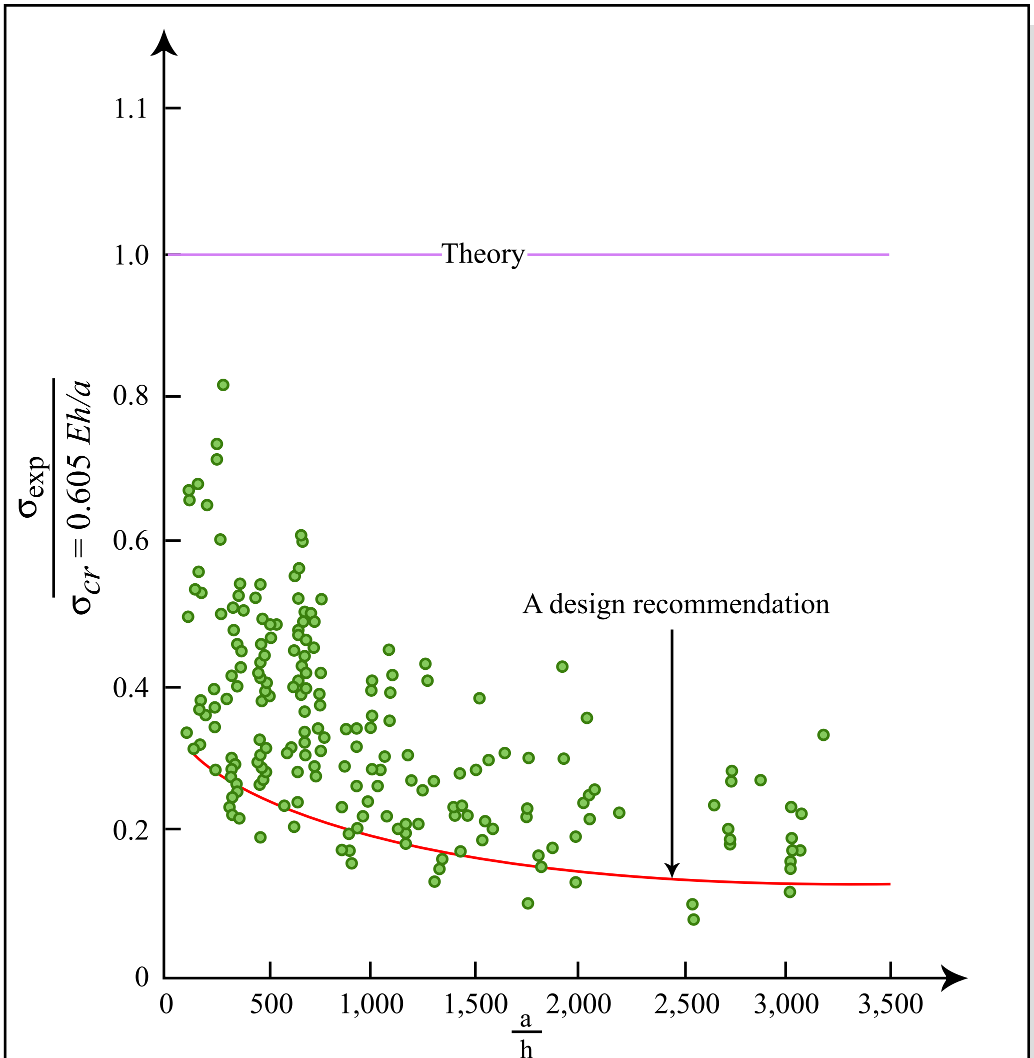
Because of the presence of unavoidable imperfection in real shells, the experimentally measured buckling load are much smaller than the ones found theoretically.



Comparison of theoretical and experimental results for four different type of loads:

- Axial compression
- Torsion
- Lateral pressure
- Hydrostatic pressure

are shown in the subsequent two pages. Note that the graphs were presented in *log* − *log* scale. Replotting the results for axially loaded plate yields the graph shown below.



Distribution of test data for cylinders subjected to axial compression.

Figure by MIT OCW.

The differences are shown to be very large. Design curves for cylindrical and other shells are based on the theoretical solution modified by empirical prediction factors called *"know-down factors"*. For example, the solid curve shown in the receding page is a *"90 percent probability curve."* For $a/h = 50$, the reduction factor predicted by this probability curve is 0.24. Thus, the theoretical solution $\sigma_{cr} = 0.605\ Eh/a$ should be multiplied by 0.24 for the design stress $\sigma = 0.15\ Eh/a$.

Cylindrical shells loaded in different way are been sensitive to imperfections and the resulting knock-down factors are smaller.

Most industrial organization establish their own design criteria. They are frequently loose-leaf and are continuously updated on the volume of the experimented evidence increase. The situation can be converted to the firmly established manual of steel constructions for the design of columns and beam. The reason is that column are not sensitive to imperfection as far as the ultimate strength is concerned.