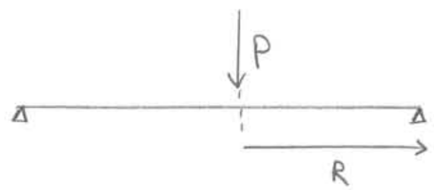
Recitation 6: Energy Methods

Example 1

A circular plate simply supported at its periphery is loaded in the center with point force, P.



Find the center deflection, w_0.

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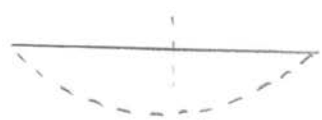
① Assume a deflection shape: $w(r) = w_0 \left[1 - \left(\frac{r}{R}\right)^2\right]$



② Calculate U: $U = \int_S \widetilde{U} \mathrm{d}A$
For a plate, $\widetilde{U}_b = \frac{D}{2}[(\kappa_{11} + \kappa_{22})^2 - 2(1-\nu)(\kappa_{11}\kappa_{22} - \kappa_{12}^2)]$ (eqn. 4.112)
In our case,

$$\kappa_{11} = \kappa_r = -\frac{\partial^2 w}{\partial r^2} \tag{6.1}$$

$$\kappa_{22} = \kappa_\theta = -\frac{1}{r}\frac{\partial w}{\partial r} \tag{6.2}$$

$$\kappa_{12} = \kappa_{r\theta} = 0 \tag{6.3}$$

$$\frac{\partial w}{\partial r} = -2w_0 \frac{r}{R^2}, \quad \frac{\partial^2 w}{\partial r^2} = -\frac{2w_0}{R^2} \tag{6.4}$$

So

$$\left.\begin{aligned} \kappa_r &= -\frac{\partial^2 w}{\partial r^2} = \frac{2w_0}{R^2} \\ \kappa_\theta &= -\frac{1}{r}\frac{\partial w}{\partial r} = \frac{2w_0}{R^2} \end{aligned}\right\} \kappa_r = \kappa_\theta \tag{6.5}$$

$$\begin{aligned} \widetilde{U}_b &= \frac{D}{2}[(\kappa_r + \kappa_\theta)^2 - 2(1-\nu)\kappa_r\kappa_\theta] \\ &= \frac{D}{2}[4\kappa_r^2 - 2(1-\nu)\kappa_r^2] = D(1+\nu)\kappa_r^2 = 40(1+\nu)\frac{w_o^2}{R^4} \end{aligned} \tag{6.6}$$

$$
\begin{aligned}
U = \int_S \widetilde{U}_b \mathrm{d}A &= \int_0^{2\pi} \int_0^R 4D(1+\nu)\frac{w_0^2}{R^4} r \mathrm{d}r \mathrm{d}\theta \\
&= 4D(1+\nu)\frac{w_0^2}{R^4} \cdot 2\pi \frac{R^2}{2} = 4\pi D(1+\nu)\frac{w_0^2}{R^2}
\end{aligned} \tag{6.7}
$$

③ Calculate work of external forces, W

$$W = P w_0 \tag{6.8}$$

④ Calculate $\Pi = U - W$: $\Pi = 4\pi D(1+\nu)\frac{w_0^2}{R^2} - P w_0$

⑤ Find $w_0(P)$ by setting $\frac{\partial \Pi}{\partial w_0} = 0$:

$$8\pi D(1+\nu)\frac{w_0}{R^2} - P = 0 \tag{6.9}$$

$$\rightarrow \boxed{w_0 = \frac{PR^2}{8\pi D(1+\nu)}} \tag{6.10}$$

Example 2

Same plate as Example 1, but with uniform pressure load, p. Find center deflection, and compare to the exact solution in Lecture 7.

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① Deflected shape should be the same: $w(r) = w_0 \left[1 - \left(\frac{r}{R}\right)^2\right]$

② $U = 4\pi D(1+\nu)\frac{w_0^2}{R^2}$

③ $W = \int_S p w \mathrm{d}A$

$$
\begin{aligned}
&= \int_0^{2\pi} \int_0^R p w_0 \left[1 - \left(\frac{r}{R}\right)^2\right] r \mathrm{d}r \mathrm{d}\theta \\
&= 2\pi p w_0 \left(\frac{r^2}{2} - \frac{r^4}{4R^2}\right)\Bigg|_0^R = 2\pi p w_0 \left(\frac{R^2}{2} - \frac{R^2}{4}\right) = \frac{\pi}{2} p w_0 R^2
\end{aligned}
$$

④ $\Pi = U - W = 4\pi D(1+\nu)\frac{w_0^2}{R^2} - \frac{\pi}{2} p w_0 R^2$

⑤ $\frac{\partial \Pi}{\partial w_0} = 0 \rightarrow 8\pi D(1+\nu)\frac{w_0}{R^2} - \frac{\pi}{2} p R^2 = 0 \rightarrow \boxed{w_0 = \frac{pR^4}{16D(1+\nu)}}$

Exact soln:

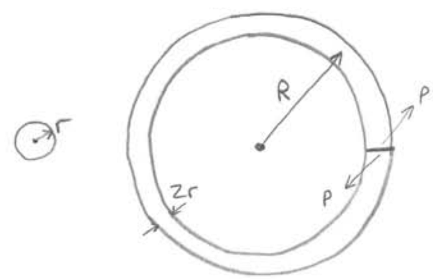
$$w(r) = \frac{pR^2}{64D}\left[\left(\frac{r}{R}\right)^4 - 2\left(\frac{r}{R}\right)^2\frac{3+\nu}{1+\nu} + \frac{5+\nu}{1+\nu}\right] \quad \text{(eqn. 7.28)} \tag{6.11}$$

$$w_0 = w(r=0) = \frac{pR^4}{64D}\left(\frac{5+\nu}{1+\nu}\right) \tag{6.12}$$

$$\frac{w_{0,\text{approx.}}}{w_{0,\text{exact}}} = \frac{\left(\dfrac{pR^4}{16D(1+\nu)}\right)}{\left(\dfrac{pR^4(5+\nu)}{64D(1+\nu)}\right)} = \frac{64}{16(5+\nu)} = \frac{4}{\underbrace{5.3}_{\nu=0.3}} \simeq \boxed{0.75} \tag{6.13}$$

Example 3

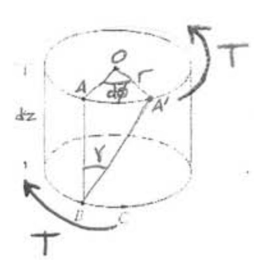
Find the out-of-plane force required to open a keyring.



Compare to the required in-plane force for the same opening.

This is a combined *bending-torsion* problem.

Introduction to torsion



— Torque, T, causes rotation $d\Phi$

$$\alpha \overset{\text{def}}{=} \frac{d\Phi}{dz} \quad (\text{like } \kappa = \frac{d\theta}{dx}) \tag{6.14}$$

Shear modulus,

$$G = \frac{E}{2(1+\nu)} \tag{6.15}$$

Polar moment of inertia,

$$J = I_p \overset{\text{def}}{=} \int_A r^2 \mathrm{d}A \tag{6.16}$$

For a solid circle, $J = \dfrac{\pi r^4}{2}$

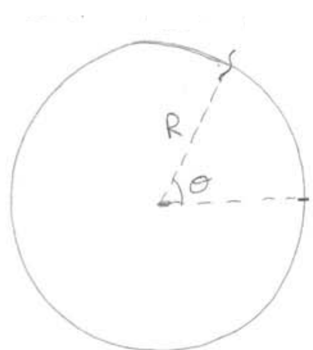
$$T = GJ\alpha \quad \text{(like bending, } M = EI\kappa\text{)} \tag{6.17}$$

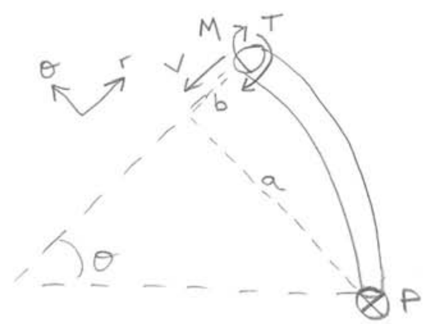
Strain energy,

$$U = \int_L \frac{1}{2} T\alpha \mathrm{d}z \quad \text{(like } U = \int_L \frac{1}{2} M\kappa \mathrm{d}x\text{)} \tag{6.18}$$

$$\alpha = \frac{T}{GJ} \quad \Rightarrow \quad U = \int_L \frac{T^2}{2GJ} \mathrm{d}z \tag{6.19}$$

In our keyring problem:





$a = R\sin\theta$

$b = R(1-\cos\theta)$

$$\begin{aligned} M_\theta &= T = Pb = PR(1-\cos\theta) \\ M_r &= M_b = Pa = PR\sin\theta \end{aligned} \tag{6.20}$$

$$\begin{aligned} U &= \int_L \frac{T^2}{2GJ}\mathrm{d}s + \int_L \frac{M_b^2}{2EI}\mathrm{d}s \qquad (\mathrm{d}s = R\mathrm{d}\theta) \\ &= \int_0^{2\pi} \left(\frac{T^2}{2GJ} + \frac{M_b^2}{2EI} \right) R\mathrm{d}\theta \\ &= \frac{P^2R^3}{2} \int_0^{2\pi} \left[\frac{(1-\cos\theta)^2}{GJ} + \frac{\sin^2\theta}{EI} \right] \mathrm{d}\theta \\ &= \frac{P^2R^3}{2} \int_0^{2\pi} \left(\frac{1 - 2\cos\theta + \cos^2\theta}{GJ} + \frac{\sin^2\theta}{EI} \right) \mathrm{d}\theta \\ &= \frac{P^2R^3}{2} \left(\frac{2\pi}{GJ} + \frac{\pi}{GJ} + \frac{\pi}{EI} \right) = \frac{\pi P^2R^3}{2} \Big(\underbrace{\frac{3}{GJ}}_{\text{Torsion}} + \underbrace{\frac{1}{EI}}_{\text{Bending}} \Big) \end{aligned} \tag{6.21}$$

Which contribution is larger?

$$G = \frac{E}{2(1+\nu)} = \frac{E}{2.6} \tag{6.22}$$

$$J = \frac{\pi r^4}{2}, \quad I = \frac{\pi r^4}{4} \quad \rightarrow \quad J = 2I \tag{6.23}$$

So $\frac{3}{GJ} = \frac{3(2.6)}{E(2I)} \simeq \frac{4}{EI} \Rightarrow$ *torsion is ∼4x bending*

CAstigliano:

$$w_0 = \frac{\partial U}{\partial P} = \pi P R^3 \left[\frac{3(2.6)}{2EI} + \frac{1}{EI}\right] \tag{6.24}$$

$$\rightarrow \boxed{P = \frac{EIw_0}{4.9\pi R^3}} \tag{6.25}$$

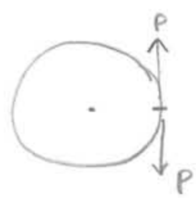
In-plane force:

Bending only:

$$M_b = PR(1-\cos\theta)$$

$$U = \int_0^{2\pi} \frac{M_b^2}{2EI} R\mathrm{d}\theta$$

$$= \frac{P^2R^3}{2EI}\int_0^{2\pi}(1 - 2\cos\theta + \cos^2\theta)\mathrm{d}\theta = \frac{P^2R^3}{2EI}(2\pi + \pi) = \frac{3\pi P^2 R^3}{2EI}$$



$$w_0 = \frac{\partial U}{\partial P} = \frac{3\pi P R^3}{EI} \tag{6.26}$$

$$\longrightarrow \boxed{P = \frac{EIw_0}{3\pi R^3}} \tag{6.27}$$

Compare:

$$\frac{P_{\text{out-of-plane}}}{P_{\text{in-plane}}} = \frac{\left(\frac{EIw_o}{4.9\pi R^3}\right)}{\left(\frac{EIw_o}{3\pi R^3}\right)} = \frac{3}{4.9} \simeq \boxed{0.6} \tag{6.28}$$