# Recitation 1: Vector/Tensor Analysis and Review of Static Equilibrium

## 1.1 Scalars, Vectors, and Tensors

### 1.1.1 Scalars

Physical quantities that are described by a single real number are called scalars.

Examples: density, energy, temperature, pressure

### 1.1.2 Vectors

Vectors are physical quantities that are completely characterized by a magnitude and direction.

Examples: force, velocity, displacement

### 1.1.3 Tensors

A tensor can be thought of as a linear operator that acts on one vector to generate a new vector.

Example: Cauchy's stress theorem

$$\boldsymbol{t} = \sigma \boldsymbol{n} \tag{1.1}$$

where $\boldsymbol{t}$ is the teaction vector, $\boldsymbol{n}$ is the normal vector, and $\sigma$ is the stress tensor.

### 1.1.4 Indicial Notation

**Range Convention**

Wherever a subscript appears only once in a term (called a *free* or *live* index), the subscript takes on all the values of the coordinate space (i.e., 1,2,3 for a 3D space).

**Examples:**

$$A_i = (A_1, A_2, A_3) \quad \text{(3D vector)} \tag{1.2}$$

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \quad \text{(3X3 tensor)} \tag{1.3}$$

Typically, indices $i$, $j$ represent 3D space. Indices $\alpha$, $\beta$ represent 2D space (e.g., plane strain or plane stress). A scalar quantity has 0 free indices, a vector has 1 free index, and a tensor has 2 (or more) free indices.

**Summation Convention (Einstein Notation)**

If an index appears twice in a term (called a *dummy* index), summation over the range of the index is implied.

**Examples:**

$$a_{ii} = a_{11} + a_{22} + a_{33} \tag{1.4}$$

$$a_i a_i = a_1^2 + a_2^2 + a_3^2 \tag{1.5}$$

**Comma convention**

A subscript comma followed by an index $i$ indicates partial differentiation with respect to each coordinate $x_i$. The summation and range conventions apply to indices following the comma as well.

**Examples:**

$$u_{i,i} = \frac{\partial u_1}{\partial z_1} + \frac{\partial u_2}{\partial z_2} + \frac{\partial u_3}{\partial z_3} \tag{1.6}$$

**Kronecker Delta, $\delta_{ij}$**

$$\delta_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j \equiv \begin{cases} 1, \text{ if } i = j \\ 0, \text{ if } i \neq j \end{cases} \tag{1.7}$$

**Examples:**

$$u_i \delta_{ij} = u_j \tag{1.8}$$

$$\delta_{ij} A_{jk} = A_{ik} \tag{1.9}$$

**Permutation Symbol, $\varepsilon_{ijk}$**

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1 \tag{1.10}$$

$$\varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} = -1 \tag{1.11}$$

$$\varepsilon_{ink} = 0 \text{ if there is a repeated index} \tag{1.12}$$

**Scalar Products (dot products)**

The scalar (dot) product of two vectors is defined by:

$$\boldsymbol{u} \cdot \boldsymbol{v} = |\boldsymbol{u}||\boldsymbol{v}| \cos(\boldsymbol{u}, \boldsymbol{v}) \tag{1.13}$$

where $(\boldsymbol{u}, \boldsymbol{v})$ is the angle between the two vectors.

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = u_i v_i \tag{1.14}$$

**Examples:**

$$\text{Work, } W = \boldsymbol{F} \cdot \boldsymbol{u} \tag{1.15}$$

**Vector Products (cross products)**

The vector (cross) product of two vectors is defined by:

$$\boldsymbol{w} = \boldsymbol{u} \times \boldsymbol{v} \tag{1.16}$$

where $\boldsymbol{w}$ is a new vector and orthogonal to $\boldsymbol{u}$ and $\boldsymbol{v}$. Its magnitude is given by

$$|\boldsymbol{w}| = |\boldsymbol{u}||\boldsymbol{v}| \sin(\boldsymbol{u}, \boldsymbol{v}) \tag{1.17}$$

**Examples:**

$$\text{Moment, } \boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F} \tag{1.18}$$

The cross product can be calculated as

$$\boldsymbol{u} \times \boldsymbol{v} = \det \begin{bmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} \tag{1.19}$$

In terms of the permutation symbol, the cross product can be written as

$$\boldsymbol{u} \times \boldsymbol{v} = \varepsilon_{ijk} u_j v_k \tag{1.20}$$

**Example putting it all together:**

1. Gradient of a scalar:

$$\boldsymbol{\nabla}\phi = \frac{\partial \phi}{\partial x}\boldsymbol{i} + \frac{\partial \phi}{\partial y}\boldsymbol{j} + \frac{\partial \phi}{\partial z}\boldsymbol{k} = \phi_{,i} \tag{1.21}$$

2. Divergence of a vector:

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = v_{i,i} \tag{1.22}$$

3. Hooke's Law:

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} \tag{1.23}$$

How many equations does this represent?

## 1.2 Static Equilibrium (of rigid structures)

### 1.2.1 Framework for Structural Analysis

Structural actions (i.e., loads):

- Tension
- Compression
- Bending (combined tension + compression)
- Shear
- Torsion

Structural elements:

- Rods
- Beams
- Columns
- Plates
- Shells

Structural systems:

- Buildings
- Bridges
- Ships, etc.

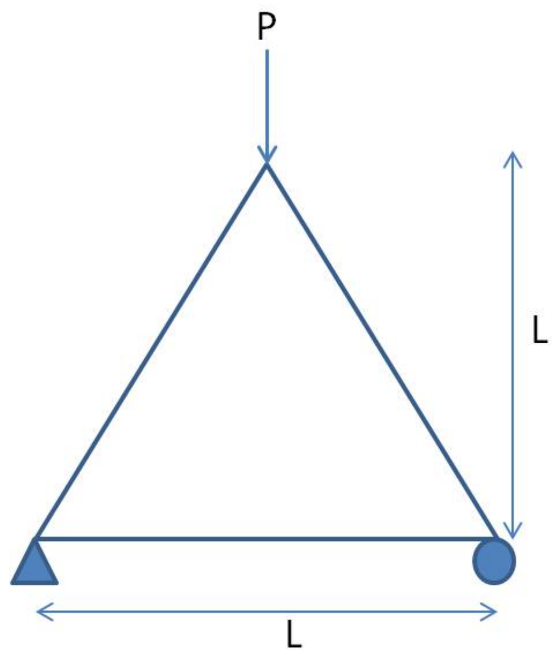
What influences how a structural behaves?

### 1.2.2 Static Equilibrium

$$\Sigma \boldsymbol{F} = 0 \tag{1.24}$$

$$\Sigma \boldsymbol{M} = 0 \tag{1.25}$$

**Example:** Find the minimum required cross-sectional area of each member to avoid yield.



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