Quiz 3 Solutions
December 11, 2013

Problem 1:
Use your knowledge of the plastic material membrane and the material testing lab in the following problem:

a) Re-derive the solution for the load displacement relation of a circular thin clamped plastic membrane loaded by a concentrated force at the middle using the principle of virtual velocity and assume the material to be rigid-plastic. Make clear assumptions about out-of-plane displacement w(r) and in-plane displacement u(r)

b) Calculate all components of the in-plane strain tensor

c) Define the component of the through-thickness strain, ϵ_{zz}, using the engineering definition of strain, and determine the current thickness from the condition of plastic incompressibility.

d) Compare the solution for the load displacement for constant thickness and thickness varying with strain. Find the correction factor when the variable thickness is included in the solution.

Extra Credit:
Calculate the coordinates of the point of instability (maximum force) for both linear and logarithmic definition of strain

Problem 1 Solutions:

a) Membrane Solution:

$$u = 0$$

$$\dot{w}(r) = \dot{w_0}\left(1 - \frac{r}{R}\right)$$

$$\int (N_r\dot{\varepsilon_r} + N_\theta\dot{\varepsilon_\theta})2\pi r dr = P\dot{w}_0$$

$$\varepsilon_\theta = \frac{u}{r} = 0$$

$$\varepsilon_r = \frac{1}{2}\left(\frac{dw}{dr}\right)^2 + \frac{du}{dr}$$

$$\dot{\varepsilon}_r = \frac{dw}{dr}\frac{d\dot{w}}{dr}$$

$$\frac{dw}{dr} = \frac{w_0}{R}$$

$$\dot{\varepsilon}_r = \frac{w_0 \dot{w}_0}{R^2}$$

$$\frac{2\pi N_r w_0 \dot{w}_0}{R^2} \int_0^R r dr = P\dot{w}_0$$

$$\frac{2\pi N_r w_0 \dot{w}_0 R^2}{2R^2} = P\dot{w}_0$$

$$P = \pi N_R w_0$$

$$N_r = \sigma_y t$$

$$P = \pi \sigma_y t w_0$$

b) In-plane strain tensor

$$\begin{bmatrix} \varepsilon_{rr} & \varepsilon_{r\theta} \\ \varepsilon_{\theta r} & \varepsilon_{\theta\theta} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(\frac{dw}{dr}\right)^2 & 0 \\ 0 & 0 \end{bmatrix}$$

c) Engineering definition of ϵ_{zz}

$$\epsilon_{zz} = \frac{t - t_0}{t_0}$$

Incompressibility: $\epsilon_{rr} + \epsilon_{\theta\theta} + \epsilon_{zz} = 0$

Therefore, since $\epsilon_{\theta\theta} = 0$,

$$\epsilon_{zz} = -\epsilon_{rr} = -\frac{1}{2}\left(\frac{dw}{dr}\right)^2$$

The expression for the thickness and correction factor:

$$t = t_0\left(1 - \frac{1}{2}\left(\frac{dw}{dr}\right)^2\right)$$

$$\frac{dw}{dr} = \frac{w_0}{R}$$

$$t = t_0\left(1 - \frac{1}{2}\left(\frac{w_0}{R}\right)^2\right)$$

d) Compare solution with uniform thickness and variable thickness:

$$P = \pi \sigma_y t_0 w_0$$

$$P = \pi \sigma_y t_0 w_0 \left(1 - \frac{1}{2}\left(\frac{w_0}{R}\right)^2\right)$$

Decreases by a factor of $\left(1-\frac{1}{2}\left(\frac{w_0}{R}\right)^2\right)$

Extra Credit Solution:

At instability or maximum force, the slope of the force-displacement curve is 0

$$\frac{\partial}{\partial w_0}\left(w_0-\frac{w_0{}^3}{2R^2}\right)=0=\left(1-\frac{1}{2}\left(\frac{w_0}{R}\right)^2\right)$$

$$\frac{\partial P}{\partial w_0}=0=\frac{\partial}{\partial w_0}\left(w_0-\frac{w_0{}^3}{2R^2}\right)$$

$$\frac{\partial P}{\partial w_0}=1-\frac{3w_0{}^2}{2R^2}$$

$$w_0=\sqrt{\frac{2}{3}}R$$

Substitute back to determine the corresponding force:

$$P=\sqrt{\frac{2}{3}}\pi\sigma_y t_0 R\left(1-\frac{1}{2}\left(\frac{\sqrt{\frac{2}{3}}R}{R}\right)^2\right)$$

$$P=\frac{2}{3}\sqrt{\frac{2}{3}}\pi\sigma_y t_0 R$$

Instability will occur at $w_0=\sqrt{\frac{2}{3}}R$ and $P=\frac{2}{3}\sqrt{\frac{2}{3}}\pi\sigma_y t_0 R$

Consider the logarithmic definition of strain:

$$\epsilon_{zz}=\ln\left(\frac{t}{t_0}\right)=-\frac{1}{2}\left(\frac{dw}{dr}\right)^2$$

$$t=t_0 e^{-\frac{1}{2}\left(\frac{w_0}{R}\right)^2}$$

$$P=\pi\sigma_y w_0 t_0 e^{-\frac{1}{2}\left(\frac{w_0}{R}\right)^2}$$

$$\frac{\partial P}{\partial w_0}=0=\frac{\partial}{\partial w_0}\left(e^{-\frac{1}{2}\left(\frac{w_0}{R}\right)^2}+e^{-\frac{1}{2}\left(\frac{w_0}{R}\right)^2}\left(\frac{-w_0{}^2}{R^2}\right)\right)$$

$$\frac{w_0^{\ 2}}{R^2} = 1$$

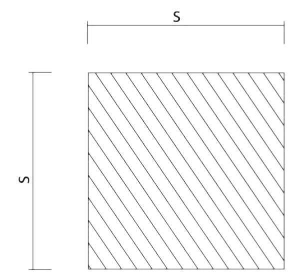
$$\boxed{w_0 = R}$$

Plug back into expression for P

$$\boxed{P = \pi \sigma_y w_0 t_0 e^{-\frac{1}{2}}}$$

Problem 2:

Consider an elastic-perfectly plastic material of a square cross-section of length s. What is the moment capacity of the cross section when the curvature is twice the maximum elastic curvature?



$$M = M_p\left(1 - \left(\frac{\kappa}{\kappa_e}\right)^2\right)$$

Where

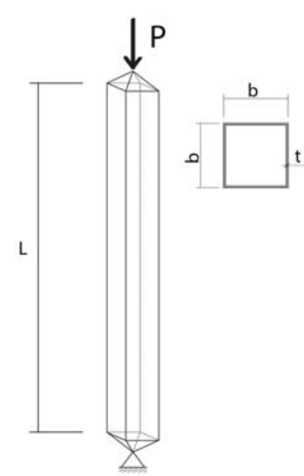
$$\kappa = 2\kappa_e$$

$$M_p = \frac{\sigma_y s^3}{4}$$

$$M = \frac{11}{12}M_p = \frac{11\sigma_y s^3}{48}$$

Problem 3:

Consider a geometrically perfect square thin-walled elastic box column of the length L, width b, thickness t, and material properties: σ_y, E, υ. The column is simply supported at the ends and subjected to a compressive load P. What is the expression for the thickness, "t", in terms of the input parameters such that the global Euler buckling load is equal to the ultimate strength of the buckled plates (effective width theory).



Extra Credit:

Given the geometry that satisfies Problem 3, which failure mode will occur first if the column is restrained at the mid-point (shown below)

Problem 3 Solutions:

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

$$P_{ult} = \sigma_y * b_{eff} * A = \sigma_y * \frac{1.9t}{b}\sqrt{\frac{E}{\sigma_y}} * 4bt \ (Eq. 11.56)$$

Equate both

$$\frac{\pi^2 EI}{L^2} = 7.6t^2\sqrt{E\sigma_y}$$

Find for moment of inertia

$$I = \frac{b^4}{12} - \frac{(b-2t)^2}{12} = \frac{2}{3}b^3 t$$

Substitute

$$\frac{\pi^2 E \frac{2}{3} b^3 t}{L^2} = 7.6t^2\sqrt{E\sigma_y}$$

Solve for t:

$$t = \frac{\pi^2 b^3}{11.4L^2}\sqrt{\frac{E}{\sigma_y}}$$

Extra Credit Solution:

The plate-buckling ultimate load will occur first because the Euler buckling load will increase by 4.

2.080J / 1.573J Structural Mechanics
Fall 2013