Lecture 9: Stability of Elastic Structures

In lecture 8 we have formulated the condition of static equilibrium of bodies and structures by studying a small change (variation) of the total potential energy. The system was said to be in equilibrium if the first variation of the total potential energy vanishes. The analysis did not say anything about the stability of equilibrium. The present lecture will give an answer to that question by looking more carefully what is happening in the vicinity of the equilibrium state.

To illustrate the concept, consider a rigid body (a ball) siting in an axisymmetric paraboloid. shown in Fig. (9.1).

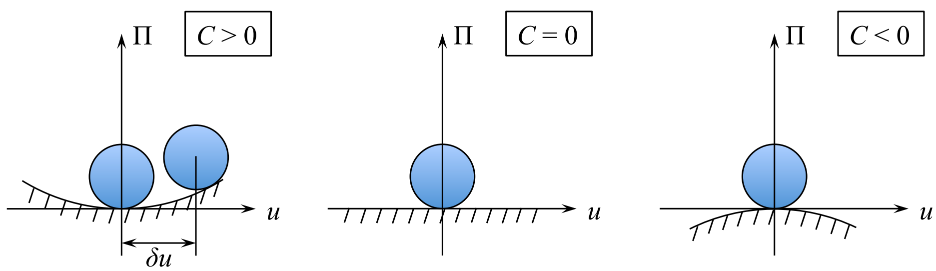

Figure 9.1: Illustration of stable, neutral and unstable equilibrium.

In the case of a rigid body the total potential energy is just the potential energy

$$\Pi = mgh = Cu^2 \tag{9.1}$$

where u is the horizontal displacement of the ball from the resting position. Let's calculate the first and second variation of the function $\Pi(u)$

$$\delta\Pi = \frac{\partial \Pi}{\partial u}\delta u = 2Cu\delta u \tag{9.2}$$

$$\delta^2\Pi = \delta(\delta\Pi) = 2C\delta u\delta u \tag{9.3}$$

At the origin of the coordinate system $u = 0$, so the first variation of Π is zero no matter what the sign of the coefficient C is. In the expression for the second variation, the product $\delta u\delta u = (\delta u)^2$ is always non-negative. Therefore, the sign of the second variation depends on the sign of the coefficient C. From Fig. 9.1 we infer that $C > 0$ corresponds to a *stable configuration*. The ball displaced by a small amount δu will return to the original position. By contrast, for $C < 0$, the ball, when displaced by a tiny amount δu, will roll down and disappear. We call this an *unstable behavior*. The case $C = 0$ corresponds to the *neutral equilibrium*.

One can formalize the above consideration to the elastic body (structure), where the total potential energy is a function of a scalar parameter, such as a displacement amplitude u. The function $\Pi(u)$ can be expanded in Taylor series around the reference point u_o

$$\Pi(u) = \Pi(u_o) + \left.\frac{\mathrm{d}\Pi}{\mathrm{d}u}\right|_{u=u_o}(u-u_o) + \frac{1}{2}\left.\frac{\mathrm{d}^2\Pi}{\mathrm{d}u^2}\right|_{u=u_o}(u-u_o)^2 + \cdots \tag{9.4}$$

The incremental change of the potential energy $\Delta\Pi = \Pi(u) - \Pi(u_o)$ upon small variation of the argument $\delta u = u - u_o$ is

$$\Delta\Pi = \frac{\mathrm{d}\Pi}{\mathrm{d}u}\delta u + \frac{1}{2}\frac{\mathrm{d}^2 u}{\mathrm{d}u^2}(\delta u)^2 = \delta\Pi + \delta^2\Pi + \cdots \tag{9.5}$$

For the system in equilibrium the first variation must be zero. Therefore, to the second term expansion, the sign of the increment of Π depends on the sign of the second variation of the potential energy. We can now distinguish three cases

$$\delta^2\Pi \begin{cases} > 0, & \text{Positive (stable equilibrium)} \\ = 0, & \text{Zero (neutral equilibrium)} \\ < 0, & \text{Negative (unstable equilibrium)} \end{cases}$$

9.1 Trefftz Condition for Stability

In 1933 the German scientist Erich Trefftz proposed the energy criterion for the determination of the stability of elastic structures. We shall explain this criterion on a simple example of a one-degree-of-freedom structure. Consider a rigid column free at one end and hinged at the other. There is a torsional spring mounted at the hinge. Upon rotation by an angle θ, a bending moment develops at the hinge, resisting the motion

$$M = K\theta \tag{9.6}$$

where K is the constant of the rotational spring. The column is initially in the vertical position and is loaded by a compressive load P, Fig. (9.2). In the deformed configuration, the force P exerts a work on the displacement u

$$u = l(1 - \cos\theta) \cong l\frac{\theta^2}{2} \tag{9.7}$$

The total potential energy of the system is

$$\Pi = \frac{1}{2}M\theta - Pu = \frac{1}{2}K\theta^2 - \frac{1}{2}Pl\theta^2 \tag{9.8}$$

Upon load application the column is of course rigid and remains straight up to the critical point $P = P_c$. The path $\theta = 0$ is called the *primary equilibrium path*. If the column were elastic rather than rigid, there would be only axial compression along that path. This stage is often referred to as a *pre-buckling configuration*. At the critical load P_c the structure has two choices. It can continue resisting the force $P > P_c$ and remain straight. Or it *can bifurcate to* the neighboring configuration and continue to rotate at a constant force. The bifurcation point is the *buckling point*. The structure is said to buckle from the purely compressive stage to the stage of a combined compression and bending.

The above analysis have shown that consideration of the equilibrium with nonlinear geometrical terms, Eq. (9.7) predicts two distinct equilibrium paths and a bifurcation (buckling) point. Let's now explore a bit further the notion of stability and calculate the second variation of the total potential energy

$$\delta^2\Pi = (K - Pl)\delta\theta\delta\theta \tag{9.9}$$

The plot of the normalized second variation $\delta^2\Pi/\delta\theta\delta\theta$ is shown in Fig. (9.2).

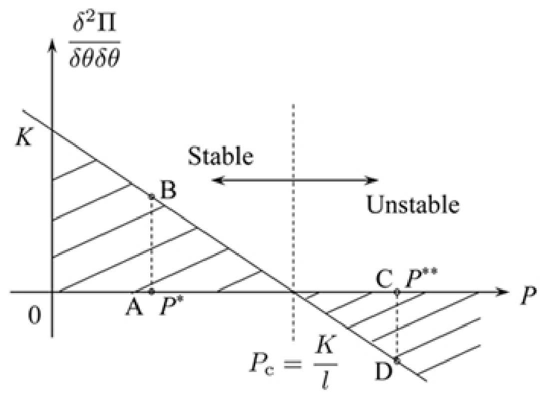

Figure 9.2: Stable and unstable range in column buckling.

It is seen that in the range $0 < P < P_c$, the second variation of the total potential energy is positive. In the range $P > P_c$, that function is negative. A transition from the stable to unstable behavior occurs at $\delta^2\Pi = 0$. Therefore, vanishing of the second variation of the total potential energy identifies the point of structural instability or buckling.

Physically, the test for stability looks like this. We bring the compressive force to the value P^*, still below the critical load. We then apply a small rotation $\pm\delta\theta$ in either direction of the buckling plane. The product $\delta\theta\delta\theta$ is always non-negative.

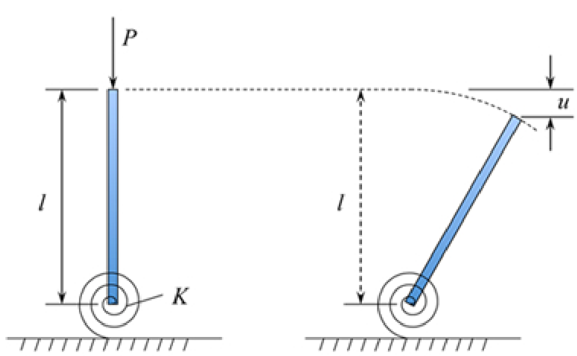

Figure 9.3: A discrete Euler column in the undeformed and deformed configuration.

For equilibrium the first variation of the total potential energy should banish, $\delta\Pi = 0$, which gives

$$(K - Pl)\theta\delta\theta = 0 \tag{9.10}$$

There are two solutions of the above equation, which corresponds to two distinct equilibrium paths:

- $\theta = 0$ – *primary equilibrium path*
- $P = P_c = \dfrac{K}{l}$ – *secondary equilibrium path*

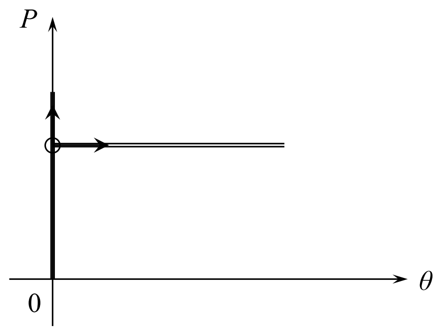

Figure 9.4: Two equilibrium paths intersects at the bifurcation point.

And so is the second variation of the total potential energy (length AB in Fig. (9.2)). When the lateral load needed to displace the column by $\delta\theta$ is released, the spring system will return to the undeformed, straight position.

We repeat the same test under the compressive force $p^{**} > P_c$. The application of the infinitesimal rotation $\delta\theta$ will make the function $\delta^2/\delta\theta\delta\theta$ negative. This is a range of unstable behavior. Upon releasing of the transverse force, the column will not returned to the vertical position, but it will stay in the deformed configuration. It should be pointed up that the foregoing analysis pertains to the problem of *stability of the primary equilibrium path*. The secondary equilibrium path is stable, as will be shown below.

To expression for total potential, it can be constructed with the exact equation for the displacement u rather than the first two-term expansion, Eq. (9.7)

$$\Pi = \frac{1}{2}K\theta^2 - lP(1 - \cos\theta) \tag{9.11}$$

The secondary equilibrium path obtained from $\delta\Pi = 0$ is

$$\frac{P}{P_c} = \frac{\theta}{\sin\theta} \tag{9.12}$$

The plot of the above function is shown is Fig. (9.5).

For small values of the column rotation, the force P is almost constant, as predicted by the two-term expansion of the cosine function. For larger rotations, the column resistance increases with the angle θ. Such a behavior is inherently stable. The force is monotonically increasing and reach infinity at $\theta \to \pi$.

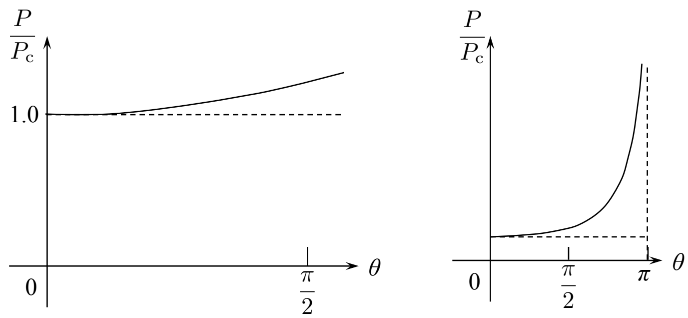

Figure 9.5: Plot of the secondary equilibrium path.

9.2 Stability of Elastic Column Using the Energy Method

The Trefftz condition for stability can now be extended to the elastic column subjected to combined bending and compression. The elastic strain energy stored in the column is a sum of the bending and axial force contribution

$$U = \int_0^l \left(\frac{1}{2} M\kappa + \frac{1}{2} N\epsilon_o \right) \mathrm{d}x \tag{9.13}$$

It is assumed that the column is fixed at one end against axial motion and allow to move in the direction of the axial force.

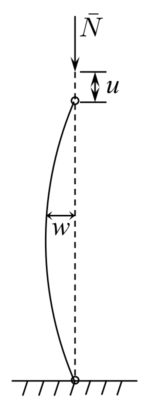

Figure 9.6: Initial and deformed elastic column.

To maintain generality, no static or kinematic boundary conditions are introduced in the present derivation. The work of external forces is

$$W = \bar{N} u_o \tag{9.14}$$

The first variation of the total potential energy is

$$\delta\Pi = \delta(U - W) = \frac{1}{2} \int_0^l (\delta M\kappa + M\delta\kappa + \delta N\epsilon + N\delta\epsilon) \mathrm{d}x - N\delta u_o \tag{9.15}$$

For linear elastic material

$$M = EI\kappa, \quad \delta M = EI\delta\kappa \tag{9.16a}$$

$$N = EA\epsilon, \quad \delta N = EA\delta\epsilon \tag{9.16b}$$

By eliminating the bending and axial rigidity between the above equation, the following identities hold

$$M\delta\kappa = \delta M\kappa \tag{9.17a}$$

$$N\delta\epsilon = \delta N\epsilon \tag{9.17b}$$

Therefore, the expression (9.15) is reduced to

$$\delta\Pi = \int_0^l (M\delta\kappa + N\delta\epsilon)\,\mathrm{d}x - \bar{N}\delta u_o \tag{9.18}$$

Now, recall the strain-displacement relations in the theory of moderately large deflection of beams

$$\kappa = -w'', \quad \delta\kappa = -(\delta w)'' \tag{9.19a}$$

$$\epsilon = u' + \frac{1}{2}(w')^2, \quad \delta\epsilon = (\delta u)' - w'\delta w' \tag{9.19b}$$

Substituting the increments $\delta\kappa$ and $\delta\epsilon$ into Eq. (9.18) yields

$$\delta\Pi = -\int_0^l M\delta w''\,\mathrm{d}x + N\int_0^l w'\delta w'\,\mathrm{d}x + N\int_0^l \delta u'\,\mathrm{d}x - N\delta u_o \tag{9.20}$$

In the above expression the axial load is unknown but constant over the length of the column. Therefore the load N could be brought outside the integrals. Consider now the last two terms in Eq. (9.20)

$$N\int_0^l \delta u'\,\mathrm{d}x - N\delta u_o = N\delta u\Big|_{x=0}^{x=l} - N\delta u = N\delta u_o - N\delta u_o = 0 \tag{9.21}$$

With the above simplification we calculate now the second variation of the total potential energy

$$\delta^2\Pi = \delta(\delta\Pi) = -\int_0^l \delta M\delta w''\,\mathrm{d}x + N\int_0^l \delta w'\delta w'\,\mathrm{d}x \tag{9.22}$$

According to the Trefftz stability criterion $\delta^2\Pi = 0$,

$$-\int_0^l EI\delta w''\delta w''\,\mathrm{d}x + N\int_0^l \delta w'\delta w'\,\mathrm{d}x = 0 \tag{9.23}$$

The buckling load $N = N_c$ is then

$$N = EI\frac{-\int_0^l \delta w''\delta w''\,\mathrm{d}x}{\int_0^l \delta w'\delta w'\,\mathrm{d}x} \tag{9.24}$$

Let's express the out-of-plane deflection of the column as a product of the amplitude A and the normalized shape function $\phi(x)$. The shape function should satisfy kinematic boundary condition of a problem

$$w(x) = A\phi(x) \tag{9.25}$$

We can calculate now the first and second derivatives of the function $w(x)$ and their variations

$$w' = A\phi', \quad \delta w' = \delta A\phi' \tag{9.26a}$$

$$w'' = A\phi'', \quad \delta w'' = \delta A\phi'' \tag{9.26b}$$

Substituting the above expression into Eq. (9.24), we get

$$N_\mathrm{c} = EI\frac{\int_0^l \delta A\phi''\delta A\phi''\,\mathrm{d}x}{\int_0^l \delta A\phi'\delta A\phi'\,\mathrm{d}x} = EI\frac{\int_0^l \phi''\phi''\,\mathrm{d}x}{\int_0^l \phi'\phi'\,\mathrm{d}x} \tag{9.27}$$

where $N_\mathrm{c} = -N$ is he compressive buckling load.

The above equation for the critical buckling load of a column is called the *Raleigh-Ritz quotient*. The Trefftz criterion does not provide the shape function but for a given shape calculates the approximate value of the buckling load. This is always an upper bound. Should the shape function coincide with the exact buckling shape, the Raleigh-Ritz quotient will give the absolute minimum value.

As an illustration, consider the pin-pin supported column and assume the following buckling shape

$$\phi(x) = x(l - x) \tag{9.28}$$

which satisfies identically kinematic boundary conditions $\phi(x = 0) = \phi(x = l) = 0$. The first and second derivatives of the shape function are

$$\phi'(x) = 2x - l \tag{9.29a}$$

$$\phi''(x) = 2 \tag{9.29b}$$

After straightforward integration, the calculated buckling load is

$$N_\mathrm{c} = \frac{12EI}{l^2} \tag{9.30}$$

Can the accuracy of the above solution be improved? Let's try and assume as a shape function the solution for the pin-pin beam under the uniform line load

$$\phi(x) = x^4 - 2lx^3 + l^3x \tag{9.31}$$

The above function satisfies the simple support boundary condition at both ends. The slope and the curvature of the deflected shape are

$$\phi'(x) = 4x^3 - 6lx^2 + l^3 \tag{9.32a}$$

$$\phi''(x) = 12x^2 - 12lx \tag{9.32b}$$

Because the curvature at both ends vanish, so does the bending moment. Also the slope at mid-span is zero. This means that the static (zero bending moments) boundary conditions are also satisfied. The previously considered shape function, Eq. (9.28) led to the constraint curvature, meaning that the static boundary conditions were violated. After straightforward calculation, the numerical coefficient become $\frac{1680}{170} = 9.88$. There was over 20% improvements in the accuracy of the solution

$$N_{\rm c} = 9.88\frac{EI}{l^2} \tag{9.33}$$

Can the solution be further improved (lowered)? yes, but not by much. Assume a sinusoidal shape of the buckling shape

$$\phi = \sin\frac{\pi x}{l} \tag{9.34a}$$

$$\phi' = \frac{\pi}{l}\cos\frac{\pi x}{l} \tag{9.34b}$$

$$\phi'' = -\left(\frac{\pi}{l}\right)^2 \sin\frac{\pi x}{l} \tag{9.34c}$$

The expression for the critical buckling load becomes

$$N_{\rm c} = \frac{EI(\frac{\pi}{l})^4}{(\frac{\pi}{l})^2}\frac{\int_0^l \sin^2\frac{\pi x}{l}\,{\rm d}x}{\int_0^l \cos^2\frac{\pi x}{l}\,{\rm d}x} \tag{9.35}$$

Both integrals are identical and the solution becomes

$$N_{\rm c} = \frac{\pi^2 EI}{l^2} \tag{9.36}$$

Because $\pi^2 = 9.86$, the sinusoidal solution is slightly lower than the previous polynomial solution. This is the lowest possible coefficient meaning that it must be an exact solution to the buckling problem. To prove it, it is sufficient to check if the local equilibrium equation is satisfied

$$EIw^{\rm IV} + Nw'' = 0 \tag{9.37}$$

Indeed, substituting Eqs. (9.34) and (9.36) into the equilibrium equation brings the left hand side of this equation identically to zero.

9.3 Effect of Structural Imperfections

Consider the same discrete strutter as in Section 9.1. This time the rigid rod is not straight but is rotated by the angle θ_o before the vertical load is applied. Upon load application the column is subjected to additional rotation θ, measured from the theoretical vertical direction, Fig. (9.7).

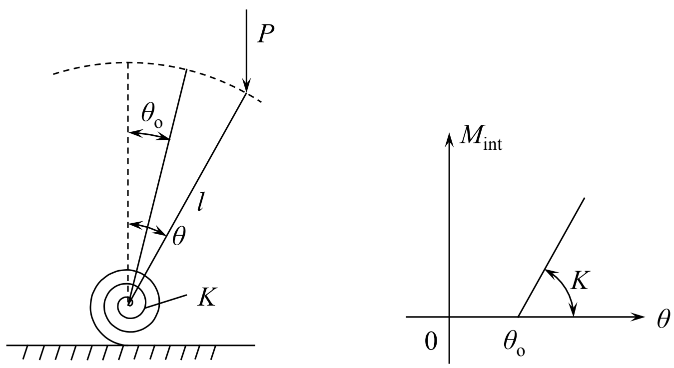

Figure 9.7: The initial inclination angle θ_o is a measure of structural imperfection.

The problem will be solved by means of local equilibrium. The external bending moment at the base is

$$M_{\text{ext}} = Pl \sin\theta, \text{ for } \theta \geqslant \theta_o \tag{9.38}$$

where $l \sin\theta$ is the arm of the force P. In the case of small angle approximation $M_{\text{ext}} = Pl\theta$. The internal resisting bending moment is

$$M_{\text{int}} = K(\theta - \theta_o) \tag{9.39}$$

Equating the external and internal bending moments

$$Pl\theta = K(\theta - \theta_o) \tag{9.40}$$

For a geometrically perfect column $\theta_o = 0$ and from Eq. (9.40)

$$P = P_c = \frac{K}{l} \tag{9.41}$$

Equation (9.40) can be re-written in terms of the normalized compressive force P/P_c

$$\frac{P}{P_c}\theta = \theta - \theta_o \tag{9.42}$$

Solving this equation for θ yields

$$\theta = \theta_o \frac{1}{1 - \dfrac{P}{P_c}} \tag{9.43}$$

The plot of the above function is shown in Fig. (9.8). The term $1/(1-\dfrac{P}{P_c})$ is called the magnification factor. It predicts how much the initial imperfections are magnified for a given magnitude of load. When structural imperfections are present, there are no primary and secondary equilibrium paths. There is only one smooth load-deflection curve called the equilibrium path.

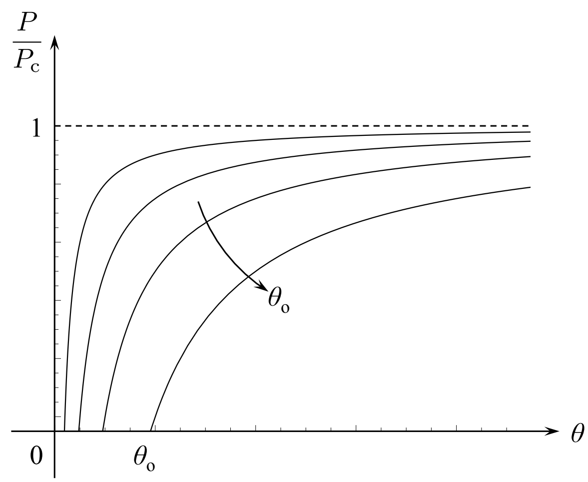

Figure 9.8: A family of equilibrium paths for different values of imperfections.

It is interesting to note that with smaller and smaller initial imperfections, the equilibrium paths are approaching the bifurcation point but never reach it. This type of behavior is common to all imperfect structures.

As another example of an imperfect structure consider a pin-pin elastic column. The following notation is introduced:

- $\bar{w}(x)$ – shape of initial imperfection
- $\bar{w}_o$ – amplitude of initial imperfection
- $w(x)$ – actual buckled shape measured from the vertical (perfect) position
- w_o – central amplitude of the actual deflection

The internal bending moment is

$$M_{\text{int}} = EI\Delta\kappa = -EI(w'' - \bar{w}'') \tag{9.44}$$

where $\Delta\kappa$ is the change of curvature from the initial curved (imperfect) column. For a simply supported column, the end (reaction) moments are zero so the external bending moment is

$$M_{\text{ext}} = Pw \tag{9.45}$$

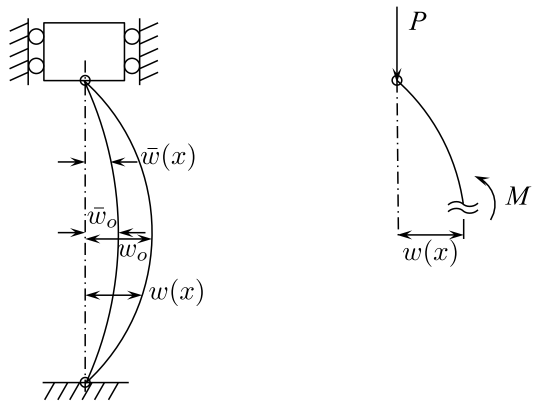

Figure 9.9: A continuous imperfect column and the free body diagram.

Equating the internal and external bending moments one gets

$$EIw'' + Pw = EI\bar{w}(x) \tag{9.46}$$

This is a second order, linear inhomogeneous differential equation, where the right hand side is a known shape of initial imperfection. The solution to this equation exists in terms of quadratures, but the integrals are difficult to evaluate for complex shapes of imperfections.

Let's consider the simplest case of a sinusoidal shape of imperfections. It can be shown that the solution $w(x)$ is also of the sinusoidal shape

$$w(x) = w_o \sin \lambda x \tag{9.47a}$$

$$\bar{w}(x) = \bar{w}_o \sin \lambda x \tag{9.47b}$$

The kinematic boundary conditions are

$$w(0) = w(l) = 0$$

which implies that

$$\sin \lambda l = 0 \quad \rightarrow \quad \lambda l = n\pi \tag{9.48}$$

Substituting Eq. (9.47) into the governing equation (9.46)

$$-EI\lambda^2(w_o - \bar{w}_o)\sin \lambda x - Pw_o \sin \lambda x = 0 \tag{9.49}$$

which is satisfied if

$$Pw_o = EI(w_o - \bar{w}_o)\lambda^2 \tag{9.50}$$

For a perfect column $\bar{w}_o = 0$, and Eq. (9.50) yields

$$(P_c - EI\lambda^2)w_o = 0$$

$$\text{or } P_c = EI\lambda^2 = \frac{n^2\pi^2 EI}{l^2} \tag{9.51}$$

For an imperfect column

$$P w_o = P_c (w_o - \bar{w}_o) \tag{9.52}$$

or solving for w_o

$$w_o = \bar{w}_o \frac{1}{1 - \frac{P}{P_c}} \tag{9.53}$$

The form of the magnification factor is identical to the one derived for the district column. The only difference is that a continuous column has infinity buckling mode where $n = 1$ corresponds to the lowest buckling load. The buckling load corresponding to the second buckling mode is four times larger and so on.

9.4 Stability in Tension

For some materials instability in tension manifest itself by a development of a local neck, Fig. (9.10).

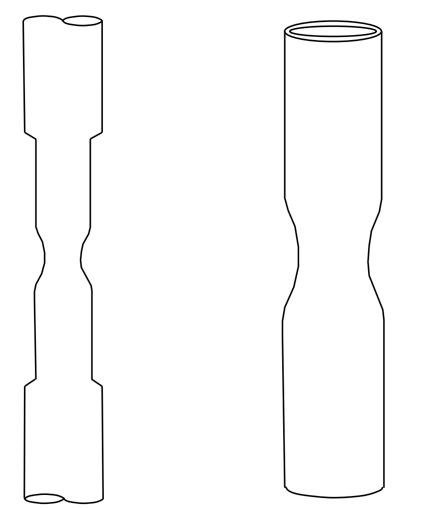

Figure 9.10: Necking in a solid section bar and thin-walled tube under tension.

Consider a round bar of the initial cross-sectional area A_o subjected to a tensile force P. The bar becomes longer and because the Poisson effect its cross-section shrinks to a current value A. The present analysis is valid for materials that are incompressible, that is do not change volume but only shape. Certain polymers, rubber and metals (in the plastic range) are incompressible.

The volume of an infinitesimal length l is

$$V = lA \tag{9.54}$$

The increment of volume for the incompressible material must be equal to zero

$$\delta V = \delta(lA) = \delta l A + l \delta A = 0 \tag{9.55}$$

Take the logarithmic definition of the axial strain

$$\epsilon = \ln \frac{l}{l_o}; \quad \delta\epsilon = \frac{\delta l}{l} \tag{9.56}$$

From the above two equations

$$\delta\epsilon = \frac{\delta l}{l} = -\frac{\delta A}{A} \tag{9.57}$$

Integrating both sides

$$\epsilon = -\ln A + C$$

At $A = A_o$, $\epsilon = 0$ so $C = \ln A_o$.

Therefore, the expression for the axial strain becomes

$$\epsilon = \ln \frac{A_o}{A} = \ln \frac{l}{l_o} \tag{9.58}$$

We conclude that axial strain can be determined by either measuring the change in length or the change in cross-sectional area. The true (Cauchy) stress is defined as the load divided by the current cross-section A

$$\sigma = \frac{P}{A} \tag{9.59}$$

Let's construct the total potential energy and its first variation

$$\delta\Pi = \int_V \sigma\delta\epsilon \, \mathrm{d}v - P\delta u \tag{9.60}$$

Before instability occurs, the deformation and stress (uniaxial tension) is uniform across the section of the bar of the length l

$$u = l\epsilon = l \ln \frac{A_o}{A} \tag{9.61a}$$

$$\delta u = -l\frac{\delta A}{A} \tag{9.61b}$$

Thus, from Eqs. (9.60) and (9.61)

$$\delta\Pi = \int_V \sigma\delta\epsilon \, \mathrm{d}v + Pl\frac{\delta A}{A} \tag{9.62}$$

The second variation of the total potential energy is

$$\delta^2\Pi = \int_V \delta\sigma\delta\epsilon \, \mathrm{d}v - Pl\frac{\delta A \delta A}{A^2} \tag{9.63}$$

Applying the Trefftz stability condition $\delta^2\Pi = 0$ one gets

$$lA\delta\sigma\delta\epsilon = Pl\delta\epsilon\delta\epsilon \tag{9.64}$$

or

$$\delta\sigma = \frac{P}{A}\delta\epsilon = \sigma\delta\epsilon \tag{9.65}$$

and finally

$$\frac{\delta\sigma}{\delta\epsilon} = \sigma \tag{9.66}$$

The incompressible bar is losing stability in tension when the local tangent to the stress-strain curve becomes equal to the value of stress at that point. A graphical interpretation is shown in Fig. (9.11).

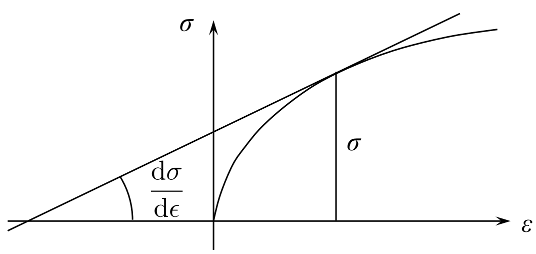

Figure 9.11: The construction of Considere' who was the first to derive Eq. (9.66).

At what strain an instability develops for an elastic material? In uni-axial stress

$$\sigma = E\epsilon \tag{9.67a}$$

$$\frac{d\sigma}{d\epsilon} = E \tag{9.67b}$$

Equation (9.66) is satisfied if $\epsilon = 1$. For metals such strain is not attainable in the elastic range because yield will be reached at the strain $\epsilon_y = \frac{\sigma_y}{E} \cong 10^{-3}$. However, for rubber and similar polymeric materials the Young's modulus is four orders of magnitude smaller, so necking is of common occurrence. The derivation of the instability condition (9.66) was done without any assumption on the stress-strain relation of the material. Therefore this condition is valid for an elastic as well as plastic material. This brings us to the next topic which is *plastic buckling of columns.*

9.5 Plastic Buckling of Columns

Let's consider the pin-pin column for which the critical buckling load is

$$P_c = \frac{\pi^2 EI}{l^2} \tag{9.68}$$

The corresponding critical buckling stress σ_c is

$$\sigma_c = \frac{P_c}{A} = \frac{\pi^2 E}{l^2}\frac{I}{A} \tag{9.69}$$

where A is the cross-sectional area. Note that the stress is calculated over the pre-buckling, primary equilibrium path, for which there is no bending. Denoting by ρ the radius of gyration, $A\rho^2 \equiv I$, Eq. (9.69) can be re-written in terms of the slenderness ratio $\beta = l/\rho$

$$\sigma_c = \pi^2 E \frac{1}{\beta^2} \tag{9.70}$$

The buckling stress is small for long, slender column and is rapidly increasing for short columns. At some critical column length, the yield stress of the material σ_y will be reached, Fig. (9.12).

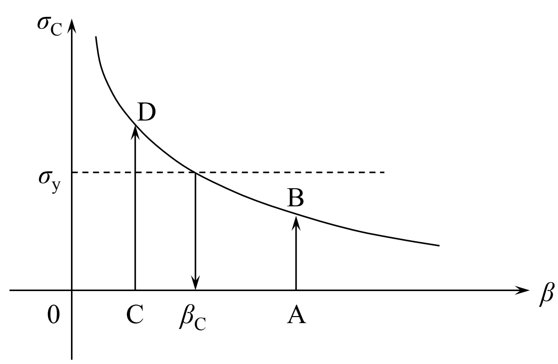

Figure 9.12: A hyperbolic dependence of the buckling stress on the slenderness ratio.

The critical slenderness ratio at which the buckling stress reaches the yield stress of the material is obtained from Eq. (9.70) by setting $\sigma_c = \sigma_y$

$$\beta_c = \frac{l_c}{\rho} = \pi\sqrt{\frac{E}{\sigma_y}} \tag{9.71}$$

To give you the feel of the critical slenderness, consider a mild steel column with $E = 210$ GPa, $\sigma_y = 250$ MPa and square cross-section $A = h^2$, for which the radius of gyration is $\rho^2 = h^2/12$

$$\left(\frac{l}{h}\right)_c = \pi\sqrt{\frac{E}{\sigma_y}} \cdot \frac{1}{2\sqrt{3}} \approx 30 \tag{9.72}$$

Columns more slender than the critical will buckle elastically before yielding (path AB). Shorter column or stocky column will yield before buckling. What will happen with such columns? They will deform plastically in axial compression and eventually buckle in the plastic range.

Gerrard (1948) extended the predictive capability of Eq. (9.70) into the plastic range by replacing the elastic modulus by the tangent modulus $E_t = \dfrac{d\sigma}{d\epsilon}$

$$(\sigma_c)_{plastic} = \pi^2 E_t \frac{1}{\beta^2}, \text{ for } \beta < \beta_c \tag{9.73}$$

For example, for a plastic material obeying the power hardening rule,

$$\sigma = B \cdot \epsilon^n \tag{9.74a}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\epsilon} = nB\epsilon^{n-1} \tag{9.74b}$$

Substituting Eq. (9.74) into Eq. (9.73), the critical buckling strain ϵ_c is

$$\epsilon_\mathrm{c} = \frac{\pi^2 n}{\beta^2} \tag{9.75}$$

Using the hardening rule, the buckling stress is

$$\sigma_\mathrm{c} = B\left[\frac{\pi^2 n}{\beta^2}\right]^n \tag{9.76}$$

In the above equation B is the amplitude of the hardening law and n is the hardening exponent. For most structural steels $n \approx$ 0.1-0.2.

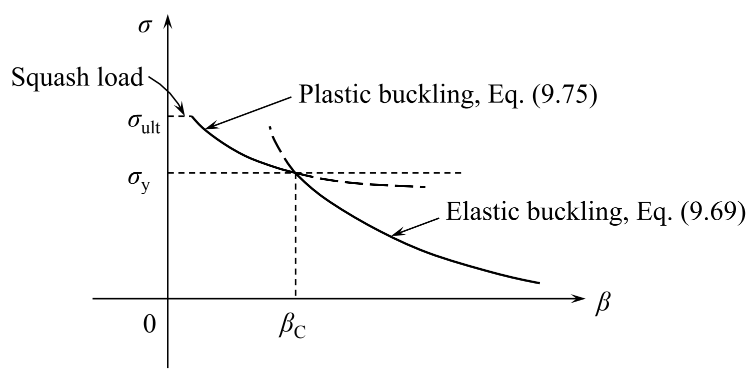

Figure 9.13: Range of elastic and plastic buckling.

Very short columns are beyond the scope of the elementary theory of thin and slender beams. They will never buckle but flatten as a pancake.

ADVANCED TOPIC

9.6 Mode Transition

Moment Equilibrium Equation

For a pin-pin supported column, the shape of the imperfection $\bar{w}(x)$ and the deformation $w(x)$ satisfy the moment equilibrium equation

$$EIw'' + Pw = EI\bar{w}'' \tag{9.77}$$

The solutions should satisfy the boundary conditions

$$w(0) = 0 \tag{9.78a}$$
$$w''(0) = 0 \tag{9.78b}$$
$$w(l) = 0 \tag{9.78c}$$
$$w''(l) = 0 \tag{9.78d}$$

Of course, the solutions should also satisfy the continuous conditions: $w(x)$ and $w'(x)$ are continuous along the entire length of the column, namely, no step or kink occurs in the solutions.

We can expand the imperfection $\bar{w}(x)$ in Fourier series as

$$\bar{w}(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l} \tag{9.79}$$

The coefficients A_n can be determined by Fourier transformation of $\bar{w}$:

$$A_n = \frac{2}{l}\int_0^l \bar{w}(x)\sin\frac{n\pi x}{l}dx \tag{9.80}$$

The deformation $w(x)$ under a load P can be written as a summation of a complete set of Fourier series

$$w(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \tag{9.81}$$

where B_n can be determined by Eq. (9.77).

Eq. (9.77) now becomes

$$-\sum_{n=1}^{\infty} B_n\left(\frac{n\pi}{l}\right)^2\sin\frac{n\pi x}{l} + \frac{P}{EI}\sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{l} = -\sum_{n=1}^{\infty} A_n\left(\frac{n\pi}{l}\right)^2\sin\frac{n\pi x}{l} \tag{9.82}$$

To make the equation hold, the coefficients should satisfy

$$-B_n\frac{n^2\pi^2}{l^2} + \frac{P}{EI}B_n = -A_n\frac{n^2\pi^2}{l^2} \tag{9.83}$$

Solve for B_n, we get

$$B_n = A_n \frac{1}{1 - \bar{P}/n^2} \tag{9.84}$$

here, we defined $\bar{P} = P/P_c$ and $P_c = \dfrac{\pi^2 EI}{l^2}$. So, the deformation $w(x)$ is

$$w(x) = \sum_{n=1}^{\infty} A_n \frac{1}{1 - \bar{P}/n^2} \sin\frac{n\pi x}{l} \tag{9.85}$$

The solution tells us what is the shape of the deformation, but it does not tell us any thing about the stability of the equilibrium shape. If we want to study the stability, we have to use potential energy method.

Potential Energy Method

Under a load P, the total potential energy of the column system is (due to Eq. (10.26) in Lecture 10):

$$\Pi = \frac{EI}{2}\int_0^l (w'' - \bar{w}'')^2 \, \mathrm{d}x - \frac{P}{2}\int_0^l (w'^2 - \bar{w}'^2) \, \mathrm{d}x \tag{9.86}$$

Substitute Eqs. (9.79) and (9.81) into it, we have

$$\begin{aligned}
\Pi &= \frac{EI}{2}\int_0^l (w'' - \bar{w}'')^2 \, \mathrm{d}x - \frac{P}{2}\int_0^l (w'^2 - \bar{w}'^2) \, \mathrm{d}x \\
&= \frac{EI}{2}\int_0^l \left[-\sum_{n=1}^{\infty} B_n \left(\frac{n\pi}{l}\right)^2 \sin\frac{n\pi x}{l} + \sum_{n=1}^{\infty} A_n \left(\frac{n\pi}{l}\right)^2 \sin\frac{n\pi x}{l} \right]^2 \mathrm{d}x \\
&\quad - \frac{P}{2}\int_0^l \left\{ \left[\sum_{n=1}^{\infty} B_n \left(\frac{n\pi}{l}\right) \cos\frac{n\pi x}{l}\right]^2 - \left[\sum_{n=1}^{\infty} A_n \left(\frac{n\pi}{l}\right) \cos\frac{n\pi x}{l}\right]^2 \right\} \mathrm{d}x \\
&= \frac{\pi^4 EI}{4l^3} \sum_{n=1}^{\infty} (B_n - A_n)^2 n^4 - \frac{\pi^2 P}{4l}\sum_{n=1}^{\infty} B_n^2 n^2 + \frac{\pi^2 P}{4l}\sum_{n=1}^{\infty} A_n^2 n^2 \\
&= \frac{\pi^2 P_c}{4l}\left\{ \sum_{n=1}^{\infty} \left[(B_n - A_n)^2 n^4 - \bar{P}(b_n^2 - A_n^2) n^2 \right] \right\}
\end{aligned} \tag{9.87}$$

The orthogonality of Fourier series is used to simplify the integration.

In order to obtain the equilibrium solution, we need the first derivative of potential energy

$$\frac{\partial \Pi}{\partial B_n} = 0 \quad \rightarrow \quad B_n = A_n \frac{1}{1 - \bar{P}/n^2} \tag{9.88}$$

which is exactly the same as the solution given by solving the equilibrium equation.

To see the stability of the solution, we need the second derivative of potential energy

$$\frac{\partial^2 \Pi}{\partial B_n^2} > 0 \quad \rightarrow \quad \bar{P} < n^2 \tag{9.89}$$

We can see the following points directly from Eqs. (9.88) and (9.89):

- The critical buckling load for the n^{th} mode is $P_c = \dfrac{n^2\pi^2 EI}{l^2}$.
- The modes that satisfy $n^2 > \bar{P}$ are in stable equilibrium.
- For the modes that satisfy $n^2 < \bar{P}$, we can still solve for a value of B_n, but those modes are unstable and will snap into either plus or minus infinity.

Examples 1

The imperfection $\bar{w}(x)$ consists of only the first two modes, namely

$$\bar{w}(x) = A_1 \sin\frac{\pi x}{l} + \sin\frac{2\pi x}{l} \tag{9.90}$$

If $A_1 = 0$, the zero point is at the center of the column. If $A_1 \neq 0$, the zero point is displaced by a distance u. u is given by

$$u = \frac{\sin^{-1}\frac{A_1}{2}}{\pi} l \ , \qquad 0 \leq A_1 \leq 2 \tag{9.91}$$

The deformation amplitudes vs. load curves are plotted in Fig. (9.14) for the case $A_1 = 0.5$, Fig. (9.15) for $A_1 = 1$, and Fig. (9.16) for $A_1 = 1.5$.

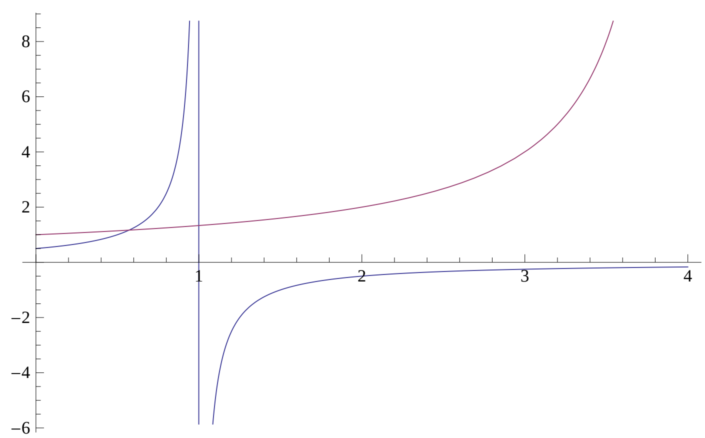

Figure 9.14: The deformation amplitudes B_1, B_2 vs. load $\bar{P}$ curves, for the case $A_1 = 0.5$.

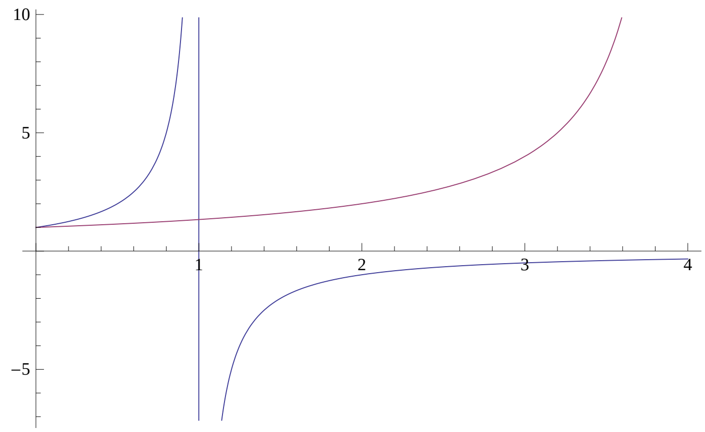

Figure 9.15: The deformation amplitudes B_1, B_2 vs. load $\bar{P}$ curves, for the case $A_1 = 1$.

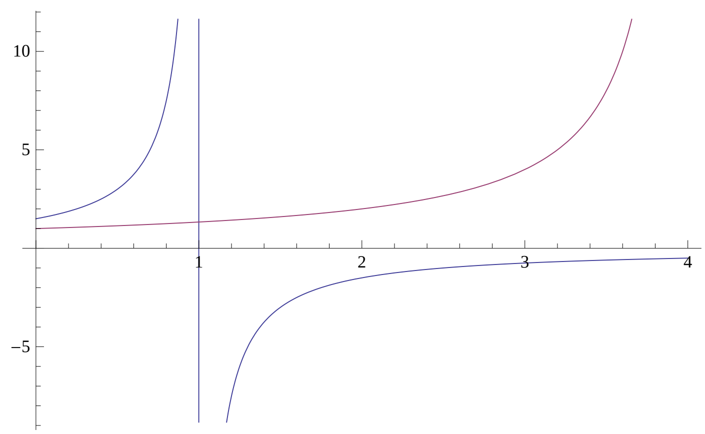

Figure 9.16: The deformation amplitudes B_1, B_2 vs. load $\bar{P}$ curves, for the case $A_1 = 1.5$.

Examples 2

The imperfection is in such a shape that the zero point is displaced while both sections are self-symmetric. Such a shape can be described as

$$\bar{w}(x) = \begin{cases} \sin \dfrac{\pi x}{\eta l} & 0 < x < \eta l \\ -\dfrac{1-\eta}{\eta} \sin \dfrac{\pi(x - \eta l)}{(1-\eta) l} & \eta l < x < l \end{cases} \tag{9.92}$$

where $0.5 \leq \eta \leq 1$. When $\eta = 0.7$, $\bar{w}(x)$ can be expanded in Fourier series as:

$$\bar{w}(x) = 0.634 \sin \frac{\pi x}{l} + 0.563 \sin \frac{2\pi x}{l} - 0.174 \sin \frac{3\pi x}{l} + 0.071 \sin \frac{4\pi x}{l} + \cdots \tag{9.93}$$

As expected, in this case, the first two modes dominate, but there are still higher modes in the expansion.

We plot the mode amplitudes vs. load in Fig. (9.17). If the load $\bar{P} > 4$, for example, $\bar{P} = 7.5$, the amplitude of mode III becomes largest. So, the solved deformation shape looks more like mode III, although the initial imperfection seems having nothing to do with mode III. Nevertheless, this shape is very unstable; since $\bar{P} > 2^2$, both mode I and mode II are in unstable equilibrium. Under such a load, mode I and mode II will amplify exponentially.

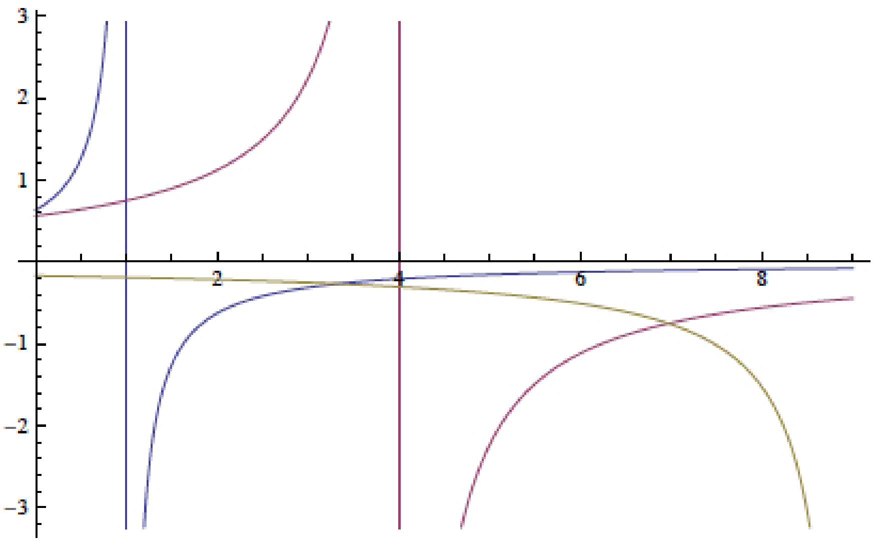

Figure 9.17: The deformation amplitudes of the first three modes vs. load $\bar{P}$ curves, for the imperfection shape described by Eqn. 9.92.

END OF ADVANCED TOPIC

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