Lecture 6: Moderately Large Deflection Theory of Beams

6.1 General Formulation

Compare to the classical theory of beams with infinitesimal deformation, the moderately large deflection theory introduces changes into the strain-displacement relation and vertical equilibrium, but leaves the constitutive equation and horizontal equilibrium unchanged. The kinematical relation, Eq. (2.68) acquires now a new term due to finite rotations of beam element.

$$\epsilon^\circ = \frac{\mathrm{d}u}{\mathrm{d}x} + \boxed{\frac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2} - \text{new term} \tag{6.1}$$

The definition of curvature has also a nonlinear rotation term

$$\kappa = -\frac{\dfrac{\mathrm{d}^2w}{\mathrm{d}x^2}}{\left[1+\left(\dfrac{\mathrm{d}w}{\mathrm{d}x}\right)^2\right]^{3/2}} \tag{6.2}$$

The square of the slope can be large, as compared with the term $\frac{\mathrm{d}u}{\mathrm{d}x}$ and must be retained in Eq. (6.1). At the same time the square of the slope (beam rotation) are small compared to unity. Why? This is explained in Fig. (6.1), where the square of the slope is plotted against the slope.

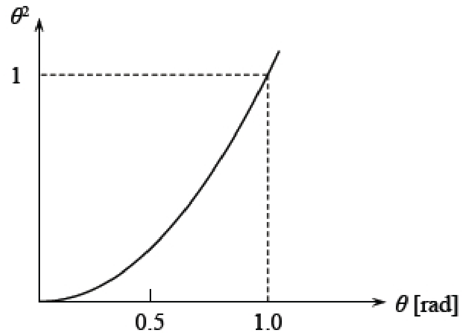

Figure 6.1: The significance of the square of the slope term.

At $\theta = 1$ rad $= 57$ degrees the two terms in the denominator of Eq. (6.2) are equal. However, the theory of moderately large deflections are valid up to $\theta = 10^\circ \approx 0.175$ rad. The term θ^2 amounts to 0.03, which is negligible compared to unity. Therefore the curvature is defined in the same way as in the theory of small deflections

$$\kappa = -\frac{\mathrm{d}^2w}{\mathrm{d}x^2} \tag{6.3}$$

It was shown in Lecture 3 that the equation of equilibrium in the horizontal direction is not affected by the finite rotation. Therefore, we infer from Eq. (3.77) that the axial force is either constant or zero

$$N = \text{constant} \tag{6.4}$$

The vertical equilibrium, given by Eq. (3.79) has a new nonlinear term

$$\frac{d^2M}{dx^2} + \boxed{N\frac{d^2w}{dx^2}} \overset{-\text{new term}}{+q = 0} \tag{6.5}$$

Finally, the elasticity law is unaffected by finite rotation

$$N = EA\epsilon^\circ \tag{6.6a}$$

$$M = EI\kappa \tag{6.6b}$$

The solution to the coupled problem depends on the boundary conditions in the horizontal direction. Referring to Fig. 5.1, two cases must be considered:

- Case 1, beam free to slide, $N = 0$, $u \neq 0$.
- Case 2, beam fixed, $u = 0$, $N \neq 0$.

6.2 Solution for a Beam on Roller Support

Consider first case 1. From the constitutive equation, zero axial force beams that there is no extension of the beam axis, $\epsilon^\circ = 0$. Then, from Eq. (6.1)

$$\frac{du}{dx} = -\frac{1}{2}\left(\frac{dw}{dx}\right)^2 \tag{6.7}$$

At the same time, the nonlinear term in the vertical equilibrium vanishes and the beam response is governed by the linear differential equation

$$EI\frac{d^4w}{dx^4} = q(x) \tag{6.8}$$

which is identical to the one derived for the infinitesimal deflections. As an example, consider the pin-pin supported beam under mid-span point load. From Eqs. (5.54) and (5.55), the deflection profile is

$$w(x) = w_o\left[3\frac{x}{l} - 4\left(\frac{x}{l}\right)^3\right] \tag{6.9}$$

and the slope is

$$\frac{dw}{dx} = \frac{w_o}{l}\left[3 - 12\left(\frac{x}{l}\right)^2\right] \tag{6.10}$$

where w_o is the central deflection of the beam. Now, Eq. (6.7) can be used to calculate relative horizontal displacement Δu. Integrating Eq. (6.7) in the limits $(0, l)$ gives

$$\int_0^l \frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x = u\Big|_0^l = u(l) - u(0) = \Delta u = -\int_0^l \frac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 \mathrm{d}x \tag{6.11}$$

The result of the integration is

$$\Delta u \approx 7\frac{w_o^2}{l} \tag{6.12}$$

In order to get a physical sense of the above result, the vertical and horizontal displacements are normalized by the thickness h of the beam

$$\frac{\Delta u}{h} = \frac{7}{l/h}\left(\frac{w_o}{h}\right)^2 \tag{6.13}$$

For a beam with $\dfrac{l}{h} = 21$, the result

$$\frac{\Delta u}{h} = \frac{1}{3}\left(\frac{w_o}{h}\right)^2 \tag{6.14}$$

is ploted in Fig. (6.2).

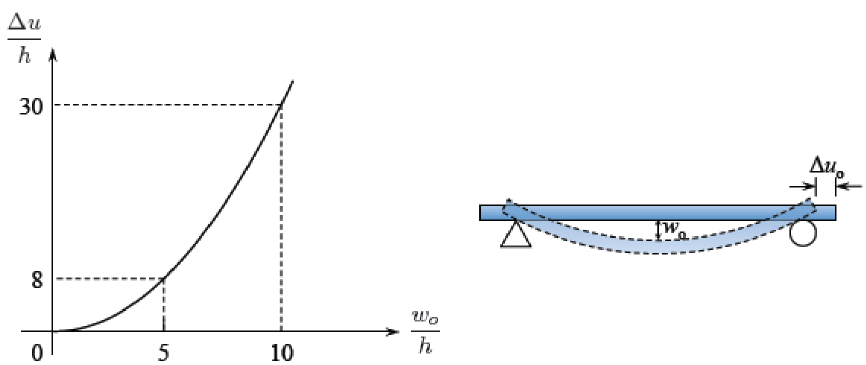

Figure 6.2: Sliding of a beam from the roller support.

It is seen that the amount of sliding in the horizontal direction can be very large compared to the thickness.

To summarize the results, the roller supported beam can be treated as a classical beam even though the displacements and rotations are large (moderate). The solution of the linear differential equation can then be used *a posteriori* to determine the magnitude of sliding. The analysis of fully restrained beam is much more interesting and difficult. This is the subject of the next section.

6.3 Solution for a Beam with Fixed Axial Displacements

The problem is solved under the assumption of simply-supported end condition, and the line load is distributed accordingly to the cosine function.The beam is restrained in the axial direction. There is a considerable strengthening effect of the beam response due to finite rotations of beam elements. The axial force N (non-zero this time) is calculated from Eq. (6.6) with Eq. (6.1)

$$N = EA\left[\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2\right] \tag{6.15}$$

From Eq. (6.4) we know that N is constant but unknown. In order to make use of the kinematic boundary conditions, let us integrate both sides of Eq. (6.15) with respect to x

$$\frac{Nl}{EA} = u(l) - u(0) + \int_0^l \frac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 \mathrm{d}x \tag{6.16}$$

Using the boundary conditions for u, the axial force is related to the square of the slope by

$$\frac{Nl}{EA} = \frac{1}{2}\int_0^l \left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 \mathrm{d}x \tag{6.17}$$

In order to determine the deflected shape of the beam, consider the equilibrium in the vertical direction given by Eq. (6.5)

$$-EI\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} + N\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + q = 0 \tag{6.18}$$

Dividing both sides by $(-EI)$ one gets

$$\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} - \lambda^2\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = \frac{q_o}{EI}\cos\frac{\pi x}{l} \tag{6.19}$$

where

$$\lambda^2 = \frac{N}{EI} \tag{6.20}$$

The roots of the characteristic equations are 0, 0, $\pm\lambda$. Therefore the general solution of the homogeneous equation is

$$w_\mathrm{g} = C_o + C_1 x + C_2\cosh\lambda x + C_3\sinh\lambda x \tag{6.21}$$

As the particular solution of the inhomogeneous equation we can try

$$w_\mathrm{p}(x) = C\cos\frac{\pi x}{l} \tag{6.22a}$$

$$\frac{\mathrm{d}^2 w_\mathrm{p}}{\mathrm{d}x^2} = -\frac{\pi^2}{l^2}C\cos\frac{\pi x}{l} \tag{6.22b}$$

$$\frac{\mathrm{d}^4 w_\mathrm{p}}{\mathrm{d}x^4} = \frac{\pi^4}{l^4}C\cos\frac{\pi x}{l} \tag{6.22c}$$

Substituting the above solution to the governing equation (6.18) one gets

$$\left[\frac{\pi^4}{l^4}C - \lambda^2\frac{\pi^2}{l^2}C - \frac{P_o}{EI}\right]\cos\frac{\pi x}{l} = 0 \tag{6.23}$$

The above solution satisfy the differentia equation if the amplitude C is related to input parameters and the unknown tension N

$$C = \frac{\frac{q_o}{EI}}{\frac{\pi^2}{l^2}\left(\lambda^2 + \frac{\pi^2}{l^2}\right)} = \frac{q_o}{EI\left(\frac{\pi}{l}\right)^4 + N\left(\frac{\pi}{l}\right)^2} \tag{6.24}$$

The general solution of Eq. (6.18) is a sum of the particular solution of the inhomogeneous equation w_{p} and general solution of the homogeneous equation, w_{g}

$$w(x) = w_{\mathrm{g}} + w_{\mathrm{p}} \tag{6.25}$$

There are five unknowns, C_o, C_1, C_2, C_3 and N and five equations. Four boundary conditions for the transverse deflections

$$w = 0, \quad \frac{\mathrm{d}^2w}{\mathrm{d}x^2} = 0 \quad \text{at} \quad x = \pm\frac{l}{2} \tag{6.26}$$

and equation (6.17) relating the horizontal and vertical response. The determination of the integration constants is straightforward. Note that the problem is symmetric. Therefore the solution should be an even function[1] of x. The terms C_1x and $C_3\sinh\lambda x$ are odd functions. Therefore the respective coefficients should vanish

$$C_1 = C_3 = 0 \tag{6.27}$$

$$w(x) = C_o + C_2\cosh\lambda x + C\cos\frac{\pi x}{l} \tag{6.28}$$

The remaining two coefficients are determined only from the boundary conditions at one side of the beam

$$\begin{aligned} w(x = \tfrac{l}{2}) = 0 \quad &\rightarrow \\ \left.\frac{\mathrm{d}^2w}{\mathrm{d}x^2}\right|_{x=\frac{l}{2}} = 0 \quad &\rightarrow \end{aligned} \begin{cases} C_o + C_2\cosh\dfrac{\lambda l}{2} = 0 \\ C_2\lambda^2\cosh\dfrac{\lambda l}{2} = 0 \end{cases} \tag{6.29}$$

The solution of the above system is

$$C_o = C_2 = 0 \tag{6.30}$$

The slope of the deflection curve, calculated from Eq. (6.28) is

$$\frac{\mathrm{d}w}{\mathrm{d}x} = -C\frac{\pi}{l}\sin\frac{\pi x}{l} \tag{6.31}$$

[1] The function is even when $f(-A) = f(A)$. The function is odd when $f(-A) = -f(A)$.

Expressing N in terms of x in Eq. (6.17) gives

$$\lambda^2 \left(\frac{Il}{A}\right) = \frac{1}{2}\int_0^l \left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 \mathrm{d}x \tag{6.32}$$

Combining the above two equations one gets

$$\lambda^2 \left(\frac{Il}{A}\right) = \frac{1}{2}\int_0^l \left(-\frac{C\pi}{l}\sin\frac{\pi x}{l}\right)^2 \mathrm{d}x \tag{6.33}$$

or after integration

$$\frac{\lambda^2 Il}{A} = \frac{1}{4}C^2\left(\frac{\pi}{l}\right)^2\frac{l}{2} \tag{6.34}$$

Recalling the definition of λ, the membrane force N becomes a quadratic function of the deflection amplitude

$$N = \frac{EA}{4}C^2\left(\frac{\pi}{l}\right)^2 \tag{6.35}$$

The membrane force can be eliminated between Eqs. (6.24) and (6.35) to give the cubic equation for the deflection amplitude C

$$C + C^3\frac{A}{4I} = \frac{q_o}{EI}\left(\frac{l}{\pi}\right)^4 \tag{6.36}$$

To get a better sense of the contribution of various terms, consider a beam of the square cross-section $h \times h$, for which

$$I = \frac{h}{12},\ A = h^2,\ \frac{A}{I} = \frac{12}{h^2} \tag{6.37}$$

Also, the ventral deflection is dimensionalized with respect to the beam thickness $\bar{w_o} = \dfrac{C}{h}$

$$\bar{w_o} + 3w_o^3 = \left(\frac{q_o}{Eh}\right)\left(\frac{l}{\pi h}\right)^4 \tag{6.38}$$

The present solution is exact and involves all information about the material (E), load intensity (q_o), length (l) and cross-sectional dimension. The distribution of line load and boundary conditions are reflected in the specific numerical coefficients in the respective terms.

In order to get a physical insight about the contributions of all terms in the above solution, consider two limiting cases:

(i) Pure bending solution for which $N\dfrac{\mathrm{d}w}{\mathrm{d}x} = 0$.

(ii) Pure membrane (string, cable) solution with zero flexural resistance (bending rigidity, $EI \rightarrow 0$).

(i) The bending solution is obtained by dropping the cubic term in Eqs. (6.36) or (6.38)

$$C = \frac{q_o l^4}{EI}\frac{1}{\pi^4} \tag{6.39}$$

where the coefficient $\pi^4 = 97.4$. this result for the sinusoidal distribution of the line load should be compared with the uniform line load (coefficient 77) and point load (coefficient 48).

(ii) The membrane solution is recovered by neglecting the first linear term

$$C^3 = \frac{q_o l^4}{EI}\frac{4}{\pi^4} \tag{6.40}$$

The plot of the full bending/membrane solution and two limiting cases is shown in Fig. (6.3).

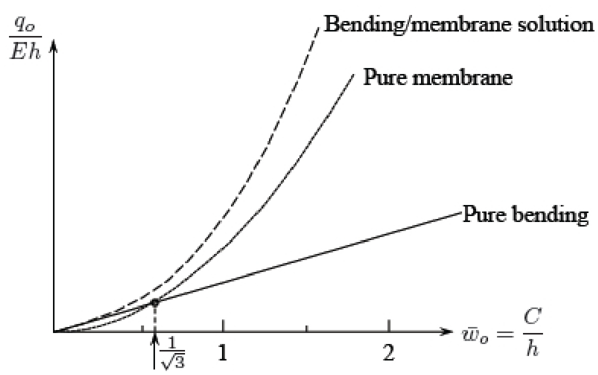

Figure 6.3: Comparison of a bending and membrane solution for a beam.

The question is at which magnitude of the central deflection relative to the beam thickness the non-dimensional load $\frac{q_o}{Eh}$ is the same. This is the intersection of the straight line with the third order parabola. By eliminating the load parameter between Eqs. (6.39) and (6.40) one gets

$$C^2 = \frac{4I}{A} = \frac{4\rho^2 A}{A} = 4\rho^2 \tag{6.41}$$

where ρ is the radius of gyration of the cross-section. For a square cross-section

$$C = 2\rho = 2\sqrt{\frac{I}{\rho}} = 2\sqrt{\frac{h^4}{12h^2}} = \frac{h}{\sqrt{3}} \cong 0.58h \tag{6.42}$$

It is concluded that the transition from bending to membrane action occurs quite early in the beam response. As a rule of thumb, the bending solution in the beam restrained from axial motion is restricted to deflections equal to half of the beam thickness. If deflections are larger, the membrane response dominate. For example, if beam deflection reaches three thicknesses, the contribution of bending and membrane action is 3:81. In the upper limit of the applicability of the theory of moderately large deflection of beams $C \cong 10h$, the contribution of bending resistance is only 0.3% of the membrane strength.

The rapid transition from bending to membrane action is only present for axially restrained beams. If the beam is free to slide in the axial direction, no membrane resistance is developed and load is always linearly related to deflections.

The above elegant closed-form solution was obtained for the particular sinusoidal distribution of line load, which coincide with the deflected shape of the beam. For an arbitrary loading, only approximate solutions could be derived. One of such solution method, applicable to the broad class of non linear problems for plates and shells is called the *Galerkin method.*

6.4 Galerkin Method of Solving Non-linear Differential Equation

Beris Galerkin, a Russian scientist, mathematician and engineer was active in the first forty ears of the 20^{th} century. He is an example of a university professor who applied methods of structural mechanics to solve engineering problems. At that time (World War I), the unsolved problem was moderately large deflections of plates. In 1915, he developed an approximate method of solving the above problem and by doing it made an important and everlasting contribution to mechanics.

The theoretical foundation of the Galerkin method goes back to the Principle of Virtual Work. We will illustrate his idea on the example of the moderately large theory of beams. If we go back to Lecture 3 and follow the derivation of the equations of equilibrium from the variational principle, the so called "weak" form of the equilibrium is given by Eq. (3.54). Adding the non-linear term representing the contribution of finite rotations, this equation can be re-written as

$$\int_0^l (M'' + Nw'' + q)\delta w \mathrm{d}x + \int_0^l N'\delta u \mathrm{d}x + \text{Boundary terms} \tag{6.43}$$

where

$$M = -EIw'' \tag{6.44a}$$

$$N = EA[u' + \frac{1}{2}(w')^2] \tag{6.44b}$$

From the weak (global) equilibrium one can derive the strong (local) equilibrium by considering an infinite class of variations. But, what happens if, instead of a "class", we consider only one specific variation (shape) that satisfies kinematic boundary conditions? The equilibrium will be violated locally, but can be satisfied globally in average

$$\int_0^l \left[-EIw^{\mathrm{IV}} + EA\left(u' + \frac{1}{2}(w')^2 w'' + q\right)\right] \delta w \, \mathrm{d}x = 0 \tag{6.45}$$

Consider the example of a simply supported beam, restrained from axial motion. The exact solution of this problem fro the sinusoidal distribution of load was given in the previous section. Assume now that the same beam is loaded by a uniform line load $q(x) = q$. No exact solution of this problem exists.

Let's solve this problem approximately by means of the Galerkin method. As a trial approximate deflected shape, we take the same shape that was found as a particular solution of the full equation

$$w(x) = C \sin \frac{\pi x}{l} \tag{6.46a}$$

$$\delta w(x) = \delta C \sin \frac{\pi x}{l} \tag{6.46b}$$

With the condition of ends fixity in the axial direction, $u = u' = 0$, and Eq. (6.45) yields

$$\delta C \int_0^l \left[-EIw^{\mathrm{IV}} + \frac{EA}{2}(w')^2 w'' + q \right] \sin \frac{\pi x}{l} \mathrm{d}x = 0 \tag{6.47}$$

Evaluating the derivatives and integrating, the following expression is obtained

$$\frac{l}{2}C + \frac{l}{8}\frac{C^3 A}{2I} - \frac{q_1}{EI(\frac{\pi}{l})^4}\frac{2l}{\pi} = 0 \tag{6.48}$$

After re-arranging, the dimensionless deflection amplitude $\dfrac{C}{h} = \dfrac{w_o}{h}$ is related to the remaining

$$\frac{w_o}{h} + \frac{3}{2}\left(\frac{w_o}{h}\right)^3 = \left(\frac{q_1}{Eh}\right)\frac{48}{\pi^5}\left(\frac{l}{h}\right)^4 \tag{6.49}$$

The above cubic equation has a simple solution.

Let's discuss the two limiting cases. Without the non-linear term, Eq. (6.49) predicts the following deflection of the beam under pure bending action for the square section

$$\frac{w_o}{h} = \left(\frac{q_1}{Eh}\right)\frac{48}{\pi^5}\left(\frac{l}{h}\right)^4 \tag{6.50}$$

In the exact solution of the same problem, the numerical coefficient is $\dfrac{60}{384} = \dfrac{1}{6.4}$, which is only 1.5% smaller than the present approximate solution $\dfrac{48}{\pi^5} = \dfrac{1}{6.3}$. If on the other hand the flexural resistance is small, $EI \to 0$, the first term in Eq. (6.49) vanishes giving a cubic load-deflection relation

$$\left(\frac{w_o}{h}\right)^3 = \frac{32}{\pi^5}\left(\frac{q_1}{Eh}\right)\left(\frac{l}{h}\right)^4 \tag{6.51}$$

There is no closed-form solution for the pure membrane response of the beam under uniform pressure. However, the present prediction compares favorably with the Eq. (6.40) for the moderately large deflection, if the total load under the uniform and sinusoidal pressure is the same

$$P = q_1 l = q_o \int_0^l \sin \frac{\pi x}{l} \mathrm{d}x = q_o \frac{2l}{\pi} \tag{6.52}$$

Replacing q_1 by $\frac{2}{\pi}q_o$, the pure membrane solution takes the final form

$$\left(\frac{w_o}{h}\right)^3 = \frac{6.4}{\pi^4}\left(\frac{q_o}{Eh}\right)\left(\frac{l}{h}\right)^4 \tag{6.53}$$

One can see that not only the dimensionless form of the exact and approximate solutions are identical, but also the coefficient 6.4 in Eq. (6.53) is of the same order as the coefficient 4 in Eq. (6.41).

6.5 Generalization to Arbitrary Non-linear Problems in Plates and Shells

The previous section felt with the application of the Galerkin method to solve the non-linear ordinary differential equation for the bending/membrane response of beams. Galerkin name is forever attached to the analytical or numerical solution of partial differential equation, such as describing response of plates and shells. In the literature you will often encounter such expression as Galerkin-Bubnov method, Petrov-Galerkin method, the discontinuous Galerkin method or the weighted residual method. The essence of this method is sketched below.

Denote by $F(w, \boldsymbol{x})$ the non-linear operator (the left hand side of the partial differential equation) is defined over a certain fixed domain in the 2-D space S. Now, a distinction is made between the exact solution $w^*(\boldsymbol{x})$ and the approximate solution $w(\boldsymbol{x})$. The approximate solution is often referred to as a *trial function*. The exact solution makes the operator F to vanish

$$F(w^*, \boldsymbol{x}) = 0 \tag{6.54}$$

The approximate solution does not satisfy exactly the governing equation, so instead of zero, there is a *residue* on the right hand of the Eq. (6.55)

$$F(w, \boldsymbol{x}) = R(\boldsymbol{x}) \tag{6.55}$$

The residue can be positive over part of S and negative elsewhere. If so, we can impose a weaker condition that the residue will become zero "in average" over S, when multiplied by a weighting function $w(\boldsymbol{x})$

$$\int_S R(\boldsymbol{x}) w(\boldsymbol{x})\, \mathrm{d}S = 0 \tag{6.56}$$

Mathematically we say that these two functions are *orthogonal*. In general, there are also boundary terms in the Galerkin formulation. For example, in the theory of moderately large deflection of plates, Eq. (6.56) takes the form

$$\int_S (D\nabla^4 w - N_{\alpha\beta} w_{,\alpha\beta}) w\, \mathrm{d}S = 0 \tag{6.57}$$

The counterpart of Eq. (6.57) in the theory of moderately large deflection of beams is Eq. (6.47) which was solved in the previous section of the notes. The solution of partial differential equations for both linear and non-linear problems is extensively covered in textbooks on the finite element method and therefore will not be covered here.

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