

Lecture 4: Development of Constitutive Equations for Continuum, Beams and Plates

This lecture deals with the determination of relations between stresses and strains, called the *constitutive equations*. For an elastic material the term *elasticity law* or the *Hooke's law* are often used. In one dimension we would write

$$\sigma = E\epsilon \quad (4.1)$$

where E is the Young's (elasticity) modulus. All types of steels, independent on the yield stress have approximately the same Young modulus $E = 2. \text{ GPa}$. The corresponding value for aluminum alloys is $E = 0.80 \text{ GPa}$. What actually is σ and ϵ in the above equation? We are saying the "uni-axial" state but such a state does not exist simultaneously for stresses and strains. One dimensional stress state produces three-dimensional strain state and vice versa.

4.1 Elasticity Law in 3-D Continuum

The second question is how to extend Eq.(4.1) to the general 3-D state. Both stress and strain are tensors so one should seek the relation between them as a linear transformation in the form

$$\sigma_{ij} = C_{ij,kl}\epsilon_{kl} \quad (4.2)$$

where $C_{ij,kl}$ is the matrix with $9 \times 9 = 81$ coefficients. Using symmetry properties of the stress and strain tensor and assumption of material isotropy, the number of independent constants are reduced from 81 to just two. These constants, called the Lamé' constants, are denoted by (χ, μ) . The general stress strain relation for a linear elastic material is

$$\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda\epsilon_{kk}\delta_{ij} \quad (4.3)$$

where δ_{ij} is the identity matrix, or Kronecker "delta", defined by

$$\delta_{ij} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \quad \text{or} \quad \begin{cases} \delta_{ij} = 1 & \text{if } i = j \\ \delta_{ij} = 0 & \text{if } i \neq j \end{cases} \quad (4.4)$$

and ϵ_{kk} is, according to the summation convention,

$$\epsilon_{kk} = \epsilon_{11} + \epsilon_{22} + \epsilon_{33} = \frac{dV}{V} \quad (4.5)$$

In the expanded form, Eq. (4.3) reads

$$\sigma_{11} = 2\mu\epsilon_{11} + \lambda(\epsilon_{11} + \epsilon_{22} + \epsilon_{33}), \quad \delta_{11} = 1 \quad (4.6a)$$

$$\sigma_{22} = 2\mu\epsilon_{22} + \lambda(\epsilon_{11} + \epsilon_{22} + \epsilon_{33}), \quad \delta_{22} = 1 \quad (4.6b)$$

$$\sigma_{33} = 2\mu\epsilon_{33} + \lambda(\epsilon_{11} + \epsilon_{22} + \epsilon_{33}), \quad \delta_{33} = 1 \quad (4.6c)$$

$$\sigma_{12} = 2\mu\epsilon_{12} \quad \delta_{12} = 0 \quad (4.6d)$$

$$\sigma_{23} = 2\mu\epsilon_{23} \quad \delta_{23} = 0 \quad (4.6e)$$

$$\sigma_{31} = 2\mu\epsilon_{31} \quad \delta_{31} = 0 \quad (4.6f)$$

Our task is to express the Lamé constants by a pair of engineering constants ($E(\nu)$, where ν is the *Poisson ratio*). For that purpose, we use the virtual experiment of tension of a rectangular bar

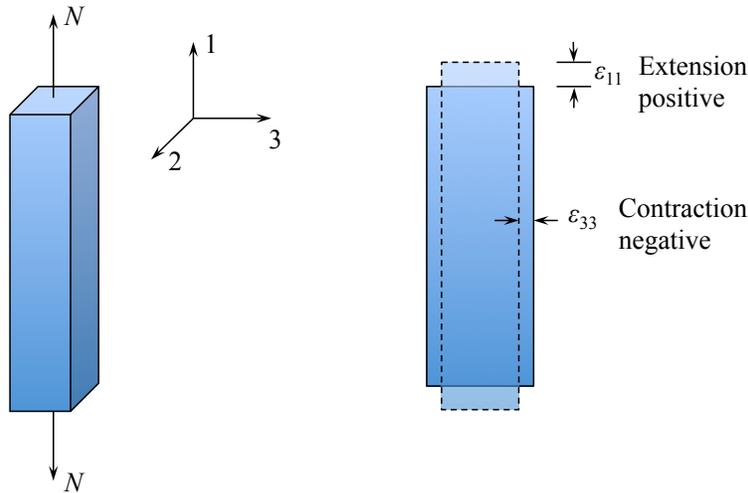


Figure 4.1: Uniaxial tension of a bar.

In the conceptual test, measured are the force, displacement and change in the cross-sectional dimension. The experimental observations can be summarized as follows:

- σ_{11} is proportional to ϵ_{11} , $\sigma_{11} = E\epsilon_{11}$
- ϵ_{22} is proportional to ϵ_{11} , $\epsilon_{22} = -\nu\epsilon_{11}$
- ϵ_{33} is proportional to ϵ_{11} , $\epsilon_{33} = -\nu\epsilon_{11}$

Thus the uniaxial tension is producing the one-dimensional state of stress but three-dimensional state of strain

$$\sigma_{ij} = \begin{vmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \quad \epsilon_{ij} = \begin{vmatrix} \epsilon_{11} & 0 & 0 \\ 0 & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{33} \end{vmatrix} \quad (4.7)$$

We introduce now the above information into Eq. (4.6).

$$\sigma_{11} = 2\mu\epsilon_{11} + \chi(\epsilon_{11} - \nu\epsilon_{11} - \nu\epsilon_{11}) = E\epsilon_{11} \quad (4.8a)$$

$$\sigma_{22} = 2\mu(-\nu\epsilon_{11}) + \chi(\epsilon_{11} - \nu\epsilon_{11} - \nu\epsilon_{11}) = 0 \quad (4.8b)$$

and obtain two linear equations relating (χ, μ) with (E, ν)

$$2\mu + \chi(1 - 2\nu) = E \quad (4.9a)$$

$$-2\mu\nu + \chi(1 - 2\nu) = 0 \quad (4.9b)$$

Solving Eq.(4.9) for μ and χ gives

$$\mu = \frac{E}{2(1 - \nu)} \quad (4.10a)$$

$$\chi = \frac{E\nu}{(1 + \nu)(1 - 2\nu)} \quad (4.10b)$$

The general, 3-D elasticity law, expressed in terms of (E, ν) is

$$\sigma_{ij} = \frac{E}{1 + \nu} \left[\epsilon_{ij} + \frac{\nu}{1 - 2\nu} \epsilon_{kk} \delta_{ij} \right] \quad (4.11)$$

The mean stress p where $-p = \frac{1}{3}\sigma_{kk} = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})$ is called the hydrostatic pressure. At the same time the sum of the diagonal components of the strain tensor denotes the change of volume. Let us make the so-called “contraction” of the stress tensor in Eq.(4.11), meaning that $i = j = k$

$$\sigma_{kk} = \frac{E}{1 + \nu} \left[\epsilon_{kk} + \frac{\nu}{1 - 2\nu} \epsilon_{kk} \cdot 3 \right] \quad (4.12)$$

where $\delta_{kk} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3$. From the above equations the following relation is obtained between the hydrostatic pressure and volume change

$$-p = \kappa \frac{dV}{V} \quad (4.13)$$

where κ is the bulk modulus

$$\kappa = \frac{E}{3(1 - 2\nu)} \quad (4.14)$$

The elastic material is clearly compressible. It is the crystalline lattice that is compressed but on removal the forces returns to the original volume.

The inverted form of the 3-D Hook’s law is

$$\epsilon_{ij} = \frac{1 + \nu}{E} \sigma_{ij} - \frac{\nu}{E} \sigma_{kk} \delta_{ij} \quad (4.15)$$

which in terms of the components yields

$$\epsilon_{11} = \frac{1}{E} [\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})] \quad (4.16a)$$

$$\epsilon_{22} = \frac{1}{E} [\sigma_{22} - \nu(\sigma_{11} + \sigma_{33})] \quad (4.16b)$$

$$\epsilon_{33} = \frac{1}{E} [\sigma_{33} - \nu(\sigma_{11} + \sigma_{22})] \quad (4.16c)$$

$$\epsilon_{12} = \frac{1}{2\epsilon} \sigma_{12} \quad (4.16d)$$

$$\epsilon_{23} = \frac{1}{2\epsilon} \sigma_{23} \quad (4.16e)$$

$$\epsilon_{31} = \frac{1}{2\epsilon} \sigma_{31} \quad (4.16f)$$

where $G = \frac{E}{2(1+\nu)}$ is called the *shear modulus*. Eq.(4.16) illustrates the coupling of individual direct strains with all direct (diagonal) components of the stress tensor. At the same time there is no coupling in shear response. The shear strain is proportional to the corresponding shear stress.

4.2 Specification to the 2-D Continuum

Plane Stress

This is the state of stress that develops in thin plates and shells so it requires a careful consideration. The stress state in which $\sigma_{3j} = 0$, where the $x_3 = z$ axis is in the through thickness direction. The non-zero components of the stress tensor are:

$$\sigma_{ij} = \begin{vmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{vmatrix} \quad \epsilon_{ij} = \begin{vmatrix} \epsilon_{11} & \epsilon_{12} & 0 \\ \epsilon_{21} & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{33} \end{vmatrix} \quad (4.17)$$

where $i, j = 1, 2, 3$ and $\alpha, \beta = 1, 2$. Accordingly, $\sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33} = \sigma_{\gamma\gamma} + 0$. The 2-D elasticity law takes the following form in the tensor notation

$$\epsilon_{\alpha\beta} = \frac{1+\nu}{E} \sigma_{\alpha\beta} - \frac{\nu}{E} \sigma_{\gamma\gamma} \delta_{\alpha\beta} \quad (4.18)$$

It can be easily checked from Eq.(4.18) that in plane stress $\epsilon_{13} = \epsilon_{23} = 0$ but $\epsilon_{33} = -\frac{\nu}{E}(\sigma_{11} + \sigma_{22})$. The through-thickness component of the strain tensor is not zero. Because it does not enter the plane stress strain-displacement relation, its presence does not contribute to the solutions. It can only be determined afterwards from the known stresses σ_{11} and σ_{22} .

By making contraction $\epsilon_{kk} = \frac{1-\nu}{E} \sigma_{kk}$, one can easily invert Eq.(4.18) in the form

$$\sigma_{\alpha\beta} = \frac{E}{1-\nu^2} [(1-\nu)\epsilon_{\alpha\beta} + \nu\epsilon_{\gamma\gamma}\delta_{\alpha\beta}] \quad (4.19)$$

The above equation is a starting point for deriving the elasticity law in generalized quantities for plates and shells. We shall return to that task later in this lecture. Before that, let's discuss three other important limiting cases

$$\sigma_{11} = \frac{E}{1-\nu^2}(\epsilon_{11} + \nu\epsilon_{22}) \quad (4.20a)$$

$$\sigma_{22} = \frac{E}{1-\nu^2}(\epsilon_{22} + \nu\epsilon_{11}) \quad (4.20b)$$

$$\sigma_{13} = \frac{E}{1+\nu}\epsilon_{12} \quad (4.20c)$$

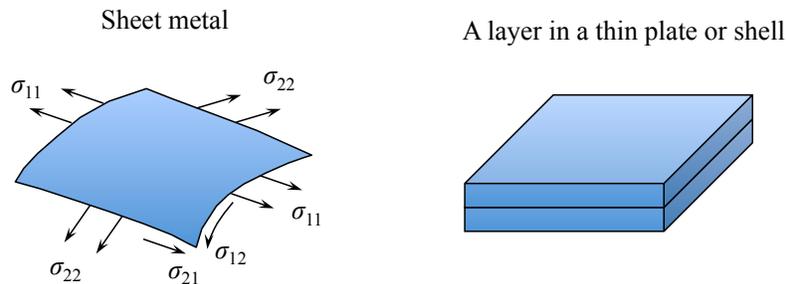


Figure 4.2: Examples of plane stress structures.

Plane strain holds whenever $\epsilon_{2j} = 0$. By imposing a constraint on $\epsilon_{22} = 0$, a reaction immediately develops in the direction as $\sigma_{22} \neq 0$.

The components of the strain and Eq.(4.8) stress tensors are

$$\epsilon_{ij} = \begin{vmatrix} \epsilon_{11} & \epsilon_{12} & 0 \\ \epsilon_{21} & \epsilon_{22} & 0 \\ 0 & 0 & 0 \end{vmatrix} \quad \sigma_{ij} = \begin{vmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{vmatrix} \quad (4.21)$$

Can you show that under the assumption of the plane strain, the reaction stress $\sigma_{33} = \nu(\sigma_{11} + \sigma_{22})$? The plane strain is encountered in many practical situations, such as cylindrical bending of a plate or wide beam.

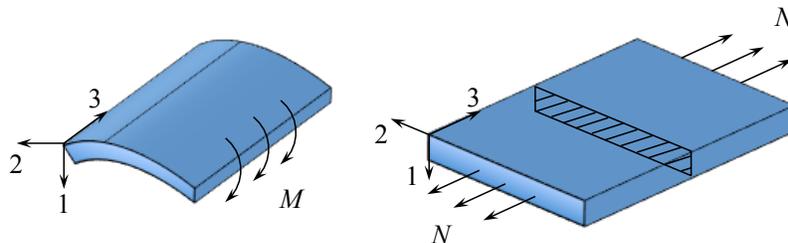


Figure 4.3: Tension of bending of a wide sheet/plate gives rise to plane strain.

Uniaxial Strain

Uniaxial strain is achieved when the displacement in two directions are constrained. For example, soil or granular materials are tested in a cylinder (called confinement) with a piston, Fig.(4.4). The uniaxial strain also develops in a compressed layer between two rigid plates. Also high velocity plate-to-plate impact products the one-dimensional strain. Here the only component of the strain tensor is the volumetric strain. The plate-to-plate experiments are conducted to establish the nonlinear compressibility of metals under very high hydrostatic loading $\sigma_{kk} = -3p$. Similarly, the plane wave in the 3-D space is generating a uniaxial strain.

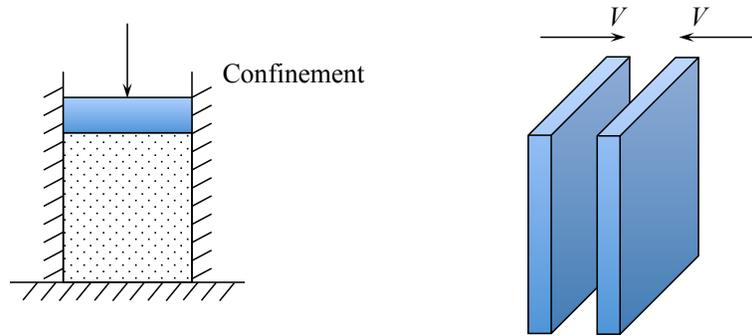


Figure 4.4: Examples of problems in which the strain state is uniaxial.

The components of the stress and strain tensor in the uniaxial strain are:

$$\sigma_{ij} = \begin{vmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{vmatrix} \quad \epsilon_{ij} = \begin{vmatrix} \epsilon_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \quad (4.22)$$

Where the reaction stresses are related to the active stress σ_{11} by $\sigma_{22} = \sigma_{33} = \frac{\nu(1+\nu)\sigma_{11}}{1-\nu^2}$. Can you prove that?

The *uniaxial stress* state was discussed earlier in this lecture when converting the Lamé' constants into the engineering constants (E, ν) .

4.3 Hook's Law in Generalized Quantities for Beams

There are three generalized forces in beams (M, N, ν) but only two generalized kinematic quantities (ϵ°, κ) . There is no generalized displacement on which the shear force could exert work. So the shear force is treated as a reaction in the elementary beam theory. This gives rise to some internal inconsistency in the beam theory, which will be enumerated in a separate section.

The starting point in the derivation of the elasticity law for beams is the Euler-Bernoulli hypothesis,

$$\epsilon(z) = \epsilon^\circ + z\kappa \quad (4.23)$$

and the one-dimensional Hook law, Eq.(4.1), and the definition of the bending moment and axial force in the beam, Eqs.(3.36a-3.36c). Let's calculate first the axial force N

$$\begin{aligned} N &= \int_A \sigma_{xx} \, dA = \int_A E\epsilon_{xx} \, dA = E \int_A (\epsilon^\circ + z\kappa) \, dA \\ &= E \int_A \epsilon^\circ \, dA + E \int_A \kappa z \, dA = E\epsilon^\circ \int_A \, dA + E\kappa \int_A z \, dA \end{aligned} \quad (4.24)$$

Note that the strain of the middle axis ϵ° and the curvature of the beam axis are independent of the z -coordinate and could be brought in front of the respective integrals. Also $Q = \int_A z \, dA$ is the static (first) moment of inertia of the cross-section. From the definition of the neutral axis, $Q = 0$. The expression for the axial force reduces then to

$$N = EA\epsilon^\circ \quad (4.25)$$

where EA is called the *axial rigidity* of the beam. We calculate next the bending moment in a similar way

$$\begin{aligned} N &= \int_A \sigma_{xx} z \, dA = \int_A E(\epsilon^\circ + z\kappa) z \, dA \\ &= E\epsilon^\circ \int_A z \, dA + E\kappa \int_A z^2 \, dA \end{aligned} \quad (4.26)$$

Because the first term involving the static moment of inertia vanishes, and the expression for the bending moment becomes

$$M = EI\kappa \quad (4.27)$$

where EI is called the bending rigidity and

$$I = \int_A z^2 \, dA \quad (4.28)$$

is the second moment of inertia. For the rectangular cross-section ($b \times h$)

$$I = \frac{bh^3}{12} \quad (4.29)$$

The significance of the above derivation is that the bending response is uncoupled from the axial response and vice versa. This property allows to derive the famous *stress formula* for beams. This is indeed one line derivation

$$\begin{aligned} \sigma &= E\epsilon = E(\epsilon^\circ + z\kappa) = E \left(\frac{N}{EA} + \frac{Mz}{EI} \right) \\ \sigma(z) &= \frac{N}{A} + \frac{Mz}{I} \end{aligned} \quad (4.30)$$

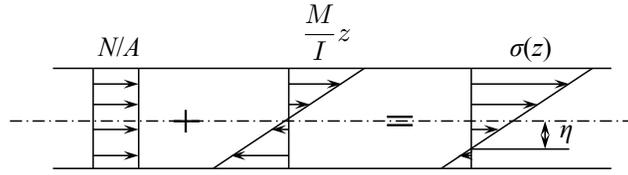


Figure 4.5: Linear distribution of stresses along the height of the beam.

Both axial force and bending moment contribute to the stress distribution along the height of the beam, as illustrated in Fig.(4.5).

From Eq.(4.30) one can calculate the point $z = \eta$ where the stresses become zero

$$\eta = -\frac{I}{A} \frac{N}{M} = -\rho^2 \frac{N}{M} \quad (4.31)$$

where ρ is the moment of giration of the cross-section defined by $I = \rho^2 A$. The position of the zero stress axis depends on the ratio of axial force to bending moment. If $\eta < h$, where h is the thickness of a rectangular section beam, the zero stress point is inside the beam boundary, there is a *bending dominated* response. The tension dominated response is when η is several times larger than h .

4.4 Inconsistencies in the Elementary Beam Theory

The equations presented in Section 3.6 under the ADVANCED TOPIC were derived without any approximate assumption. In order for the beam to be in equilibrium, shear force V must be present, when the beam is under *pure bending* (uniform bending over the length of the beam). It is the shear stress σ_{xz} that give rise to the shear force, according to the definition, Eq.(3.45). Therefore any inconsistencies must come from the strain-displacement relations as well as constitutive equations, where some approximations were introduced.

The presence of the shear stresses $\sigma_{xz} = \sigma_{13}$ means that shear strains $\epsilon_{13} = \epsilon_{xz}$ must develop according to Eq.(4.16).

$$\epsilon_{xz}(z) = \frac{\sigma_{xz}(z)}{2G} \quad (4.32)$$

The shear strain is defined as

$$\epsilon_{xz} = \frac{1}{2} \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) \quad (4.33)$$

The Euler-Bernoulli assumption tells us that the shear strain vanishes. Then, Eq.(4.32) is violated because the LH is zero while the RH is not. Suppose for a while that $\epsilon_{xz} = 0$. Then

$$\frac{\partial u_x}{\partial z} = -\frac{\partial w(x)}{\partial x} = -\theta(x) \quad (4.34)$$

where $u_z = w(x)$ is independent of the coordinate z . Integrating Eq.(4.34) one gets

$$u_x(z) = u^o - z\theta \quad (4.35)$$

which is equivalent to the plane-remain-plane and normal-remain-normal hypothesis, introduced in Lecture 2. Assume now that the out-of-plane strain is a certain given function of z . Performing the integration of Eq.(4.32) in a similar way as before, one gets

$$u_x(z) = u^\circ - z\theta + \int \epsilon_{xz}(z) dz \quad (4.36)$$

It transpires from the above results that deformed section are not flat but are *warped* instead. The amount of warping is given by the third term in Eq.(4.36).

Can we estimate the amount of warping? Yes, but we have to go ahead of the presented material and quote the solution for the deflected slope θ of the beam. Let's settle on the simplest case of a clamped cantilever beam loaded at its tip by the point force P

$$\theta = \frac{Pl^2}{2EI} \quad (4.37)$$

This solution will be derived in Lecture 5.

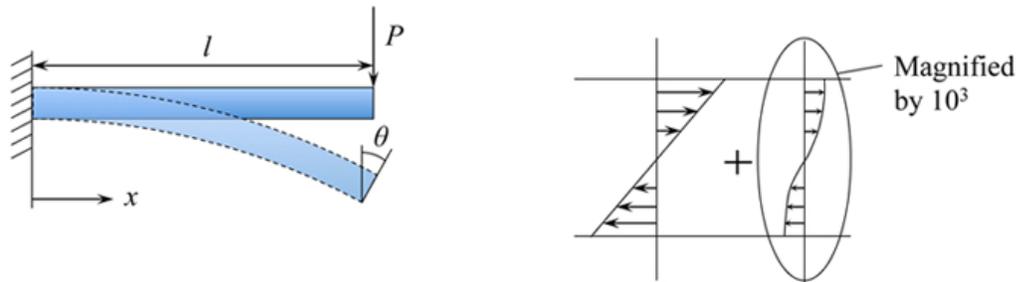


Figure 4.6: Warping of the end section of the cantilever beam.

Another result needed is the distribution of shear stresses across the height of the beam. For the rectangular section beam ($b \times h$), the shear stress is a parabolic function of z

$$\sigma_{xz}(z) = \frac{3P}{2A} \left[1 - \frac{z^2}{(h/2)^2} \right] \quad (4.38)$$

The corresponding strain is calculated from Eq.(4.32). Assume that there is no axial force, $N = 0$, so from Eq.(4.25) $\epsilon^\circ = 0$ and $u^\circ = 0$. After integration, the displacement profile defined by Eq.(4.36) becomes

$$u_x(z) = -\frac{Pl^2}{2EI}z + \frac{1}{2\epsilon} \frac{3P}{2A} \left[z - \frac{z^3}{3(h/2)^2} \right] \quad (4.39)$$

In order to quantify the correction of the displacement field due to warping, let's calculate the maximum values of the two terms at $z = -\frac{h}{2}$. The first term arising from the Euler-Bernoulli assumption gives

$$u_x^I(z = -\frac{h}{2}) = \frac{\rho l^2}{2EI} \frac{h}{2} \quad (4.40)$$

The second correction term is

$$u_x^{\text{II}}(z = -\frac{h}{2}) = \frac{1}{2G} \frac{P h}{A} \quad (4.41)$$

The ratio of the two terms is

$$\left| \frac{u_x^{\text{II}}}{u_x^{\text{I}}} \right| = \frac{E}{2G} \frac{I}{A l^2} = \frac{E}{2G} \left(\frac{\rho}{l} \right)^2 \quad (4.42)$$

where ρ is the radius of giration of the cross-section. For a rectangular cross-section ($b \times h$),

$$\rho^2 = \frac{I}{A} = \frac{b h^3}{12 b h} = \frac{h^2}{12} \quad (4.43)$$

The ratio $E/2G$ is

$$\frac{E}{2G} = \frac{E}{\frac{2}{3} E} = (1 + \nu) \quad (4.44)$$

Then, the relative amplitude of warping from Eq.(4.42) is

$$\frac{u_x^{\text{II}}}{u_x^{\text{I}}} = \frac{1 + \nu}{12} \left(\frac{h}{l} \right)^2 \quad (4.45)$$

For a typical beam with $\frac{l}{h} = 20$, the above ratio becomes 0.25×10^{-3} !!! In order to compare the plane and wrapped cross-section, the amount of warping had to be magnified thousand times, see Fig.(4.6). It can be concluded that the effect of warping is of an order of 0.1% and can be safely neglected in the engineering beam theory. In other words the “rein” of the Euler-Bernoulli assumption is unchallenged.

Another inconsistency of the elementary beam theory is that the uniaxial stress gives rise to the tri-axial strain state. In particular, from the 3-D constitutive equation, the strain components

$$\epsilon_{yy} = \epsilon_{zz} = -\frac{\nu}{E} \sigma_{xx} \quad (4.46)$$

Let's take as an example the same cantilever beam with a tip load. The bending moment at root of the beam is $M = Pl$, and from the stress formula,

$$\sigma_{xx} = \frac{Pl}{I} z \quad (4.47)$$

From the definition $\epsilon_{yy} = \frac{du_y}{dy}$, and after integrating with respect to y , one gets

$$u_y = -\frac{Pl\nu}{IE} zy \quad (4.48)$$

The maximum displacement occurs at $z = \frac{h}{2}$ and $y = \frac{b}{2}$. Making use of the beam deflection formula (see Lecture 5)

$$\delta = \frac{Pl^3}{3EI} \text{ or } \frac{Pl}{EI} = \frac{3\delta}{l^2} \quad (4.49)$$

the formula for the maximum displacement of a beam, normalized with respect to the beam thickness becomes

$$\frac{(u_y)_{\max}}{h} = \frac{3}{4} \nu \left(\frac{\delta}{h} \right) \left(\frac{h}{l} \right)^2 \quad (4.50)$$

What is the range of the normalized beam deflections δ ? The beam deflects elastically until the most stressed fibers reach yield of the materials, $\sigma_{xx}|_{z=\frac{h}{2}} = \sigma_y$.

Then, from the stress formula

$$\sigma_y = \frac{Plh}{I} \frac{h}{2} \quad (4.51)$$

Combining the above expression with the beam deflection formula, Eq.(4.49), the estimate for the maximum elastic tip displacement

$$\frac{\delta}{h} = \frac{2}{3} \frac{\sigma_y}{E} \left(\frac{l}{h} \right)^2 \quad (4.52)$$

Combining Eqs.(4.50) and (4.52), the expression for the maximum normalized displacement of the corner of the cross-section becomes

$$\frac{(u_y)_{\max}}{h} = \frac{\nu}{2} \frac{\sigma_y}{E} \quad (4.53)$$

With realistic values $\nu = \frac{1}{3}$ and $\frac{\sigma_y}{E} = 10^{-3}$, the amount of maximum change of the width of the beam is 0.1 % of the beam height. Such a tiny change in the cross-sectional dimension has no practical effect on the beam solution. A similar analysis can be performed to estimate the change in the height of the beam.

When the signs of z and y coordinates is properly taken into account, the present calculations predict the following change in the shape of the cross-section.

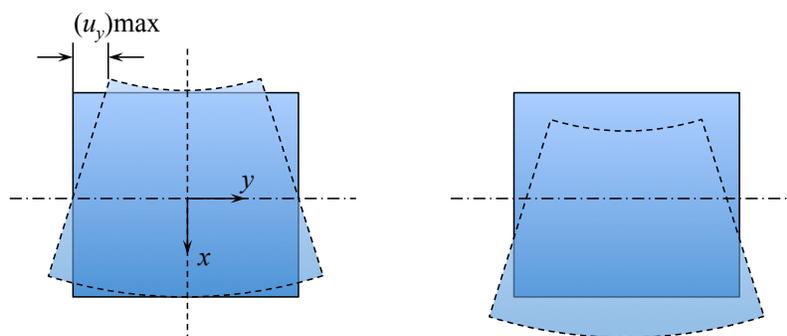


Figure 4.7: Predicted (left) and actual “anticlastic” deformed cross-section of the beam subjected to pure bending. Note that the deflections were magnified by a factor of 10^4 .

The anticlastic deformation can be easily seen by bending a rubber eraser, which is a very short beam. We can conclude the present section that the internal inconsistencies of the beam theory do not produce any significant errors in engineering applications. Therefore, one can safely assume that the cross-section of the beam does not deform and only moves as a rigid body with the increasing beam deflections.

ADVANCED TOPIC

4.5 Derivation of Constitutive Equations for Plates

For convenience, the set of equations necessary to derive the elasticity law for plates is summarized below.

Hook's law in plane stress reads:

$$\sigma_{\alpha\beta} = \frac{E}{1-\nu^2} [(1-\nu)\epsilon_{\alpha\beta} + \nu\epsilon_{\gamma\gamma}\delta_{\alpha\beta}] \quad (4.54)$$

In terms of components:

$$\sigma_{xx} = \frac{E}{1-\nu^2} (\epsilon_{xx} + \nu\epsilon_{yy}) \quad (4.55a)$$

$$\sigma_{yy} = \frac{E}{1-\nu^2} (\epsilon_{yy} + \nu\epsilon_{xx}) \quad (4.55b)$$

$$\sigma_{xy} = \frac{E}{1+\nu} \epsilon_{xy} \quad (4.55c)$$

Here, strain tensor can be obtained from the strain-displacement relations:

$$\epsilon_{\alpha\beta} = \epsilon_{\alpha\beta}^{\circ} + z\kappa_{\alpha\beta} \quad (4.56)$$

Now, define the tensor of bending moment:

$$M_{\alpha\beta} \equiv \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha\beta} z \, dz \quad (4.57)$$

and the tensor of axial force (membrane force):

$$N_{\alpha\beta} \equiv \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha\beta} \, dz \quad (4.58)$$

Bending Moments and Bending Energy

The bending moment $M_{\alpha\beta}$ is now calculated by substituting Eq.(4.54) with Eq.(4.57)

$$\begin{aligned} M_{\alpha\beta} &= \frac{E}{1-\nu^2} \int_{-\frac{h}{2}}^{\frac{h}{2}} [(1-\nu)\epsilon_{\alpha\beta} + \nu\epsilon_{\gamma\gamma}\delta_{\alpha\beta}] z \, dz \\ &= \frac{E}{1-\nu^2} [(1-\nu)\epsilon_{\alpha\beta}^{\circ} + \nu\epsilon_{\gamma\gamma}^{\circ}\delta_{\alpha\beta}] \int_{-\frac{h}{2}}^{\frac{h}{2}} z \, dz \\ &\quad + \frac{E}{1-\nu^2} [(1-\nu)\kappa_{\alpha\beta} + \nu\kappa_{\gamma\gamma}\delta_{\alpha\beta}] \int_{-\frac{h}{2}}^{\frac{h}{2}} z^2 \, dz \\ &= \frac{Eh^3}{12(1-\nu^2)} [(1-\nu)\kappa_{\alpha\beta} + \nu\kappa_{\gamma\gamma}\delta_{\alpha\beta}] \end{aligned} \quad (4.59)$$

Note that the term $\int_{-\frac{h}{2}}^{\frac{h}{2}} z \, dz$ is zero, as shown in the case of beams. Therefore there are no mid-surface strains $\epsilon_{\alpha\beta}^o$ entering the moment-curvature relation.

Here we define the bending rigidity of a plate D as follows:

$$D = \frac{Eh^3}{12(1-\nu^2)} \quad (4.60)$$

Now, one gets the moment-curvature relations in the tensorial form

$$\boxed{M_{\alpha\beta} = D[(1-\nu)\kappa_{\alpha\beta} + \nu\kappa_{\gamma\gamma}\delta_{\alpha\beta}]} \quad (4.61)$$

$$M_{\alpha\beta} = \begin{vmatrix} M_{11} & M_{22} \\ M_{21} & M_{22} \end{vmatrix} \quad (4.62)$$

where $M_{12} = M_{21}$ due to symmetry. In the expanded notation,

$$M_{11} = D(\kappa_{11} + \nu\kappa_{22}) \quad (4.63a)$$

$$M_{22} = D(\kappa_{22} + \nu\kappa_{11}) \quad (4.63b)$$

$$M_{12} = D(1-\nu)\kappa_{12} \quad (4.63c)$$

One-dimensional Bending Energy Density

Here, we use the hat notation for a function of certain argument, such as:

$$\begin{aligned} M_{11} &= \hat{M}_{11}(\kappa_{11}) \\ &= D\kappa_{11} \end{aligned} \quad (4.64)$$

Then, the bending energy density \bar{U}_b reads:

$$\begin{aligned} \bar{U}_b &= \int_0^{\bar{\kappa}} \hat{M}_{11}(\kappa_{11}) \, d\kappa_{11} \\ &= D \int_0^{\bar{\kappa}_{11}} \kappa_{11} \, d\kappa_{11} \\ &= \frac{1}{2} D (\bar{\kappa}_{11})^2 \end{aligned} \quad (4.65)$$

$$\bar{U}_b = \frac{1}{2} M_{11} \bar{\kappa}_{11} \quad (4.66)$$

General Case

General definition of the bending energy density reads:

$$\bar{U}_b = \oint M_{\alpha\beta} \, d\kappa_{\alpha\beta} \quad (4.67)$$

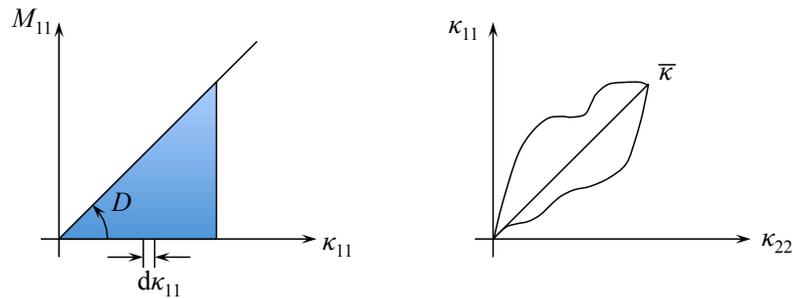


Figure 4.8: In one dimension the energy density is the area under the linear moment-curvature plot. In the multi-axial case the final value can be reached along the straight or nonlinear path.

where the symbol \oint denotes integration along a certain loading path.

Let's calculate the energy density stored when the curvature reaches a given value $\bar{\kappa}_{\alpha\beta}$ along a straight loading path:

$$\kappa_{\alpha\beta} = \eta \bar{\kappa}_{\alpha\beta} \quad (4.68a)$$

$$d\kappa_{\alpha\beta} = \bar{\kappa}_{\alpha\beta} d\eta \quad (4.68b)$$

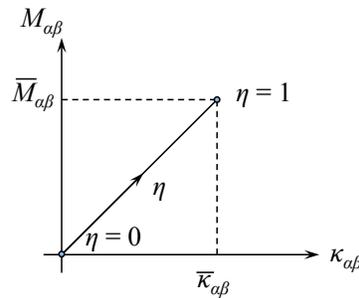


Figure 4.9: The straight loading path in the 3-dimensional space of bending moments..

From the linearity of the moment-curvature relation, Eq.(4.61), it follows that

$$\begin{aligned} M_{\alpha\beta} &= \hat{M}_{\alpha\beta}(\kappa_{\alpha\beta}) \\ &= \hat{M}_{\alpha\beta}(\eta \bar{\kappa}_{\alpha\beta}) \\ &= \eta \hat{M}_{\alpha\beta}(\bar{\kappa}_{\alpha\beta}) \end{aligned} \quad (4.69)$$

where $\hat{M}_{\alpha\beta}(\kappa_{\alpha\beta})$ is a homogenous function of degree one.

$$\begin{aligned}
 \bar{U}_b &= \oint \hat{M}_{\alpha\beta}(\kappa_{\alpha\beta}) d\kappa_{\alpha\beta} \\
 &= \int_0^1 \eta \hat{M}_{\alpha\beta}(\bar{\kappa}_{\alpha\beta}) \bar{\kappa}_{\alpha\beta} d\eta \\
 &= \hat{M}_{\alpha\beta}(\bar{\kappa}_{\alpha\beta}) \bar{\kappa}_{\alpha\beta} \int_0^1 \eta d\eta \\
 &= \frac{1}{2} \hat{M}_{\alpha\beta}(\bar{\kappa}_{\alpha\beta}) \bar{\kappa}_{\alpha\beta} \\
 &= \frac{1}{2} M_{\alpha\beta} \bar{\kappa}_{\alpha\beta}
 \end{aligned} \tag{4.70}$$

Now, the bending energy density reads

$$\begin{aligned}
 \bar{U}_b &= \frac{D}{2} [(1 - \nu) \bar{\kappa}_{\alpha\beta} + \nu \bar{\kappa}_{\gamma\gamma} \delta_{\alpha\beta}] \bar{\kappa}_{\alpha\beta} \\
 &= \frac{D}{2} [(1 - \nu) \bar{\kappa}_{\alpha\beta} \bar{\kappa}_{\alpha\beta} + \nu \bar{\kappa}_{\gamma\gamma} \bar{\kappa}_{\alpha\beta} \delta_{\alpha\beta}] \\
 &= \frac{D}{2} [(1 - \nu) \bar{\kappa}_{\alpha\beta} \bar{\kappa}_{\alpha\beta} - \nu (\kappa_{\gamma\gamma})^2]
 \end{aligned} \tag{4.71}$$

The bending energy density expressed in terms of components are:

$$\begin{aligned}
 \bar{U}_b &= \frac{D}{2} \{(1 - \nu)[(\bar{\kappa}_{11})^2 + 2(\bar{\kappa}_{12})^2 + (\bar{\kappa}_{22})^2] + \nu(\bar{\kappa}_{11} + \bar{\kappa}_{22})^2\} \\
 &= \frac{D}{2} \{(1 - \nu)[(\bar{\kappa}_{11} + \bar{\kappa}_{22})^2 - 2\bar{\kappa}_{11}\bar{\kappa}_{22} + 2(\bar{\kappa}_{12})^2] + \nu(\bar{\kappa}_{11} + \bar{\kappa}_{22})^2\} \\
 &= \frac{D}{2} \{(\bar{\kappa}_{11} + \bar{\kappa}_{22})^2 - 2\bar{\kappa}_{11}\bar{\kappa}_{22} + 2(\bar{\kappa}_{12})^2 - \nu[-2\bar{\kappa}_{11}\bar{\kappa}_{22} + 2(\bar{\kappa}_{12})^2]\} \\
 &= \frac{D}{2} \{(\bar{\kappa}_{11} + \bar{\kappa}_{22})^2 - 2\bar{\kappa}_{11}\bar{\kappa}_{22} + 2(\bar{\kappa}_{12})^2 - \nu[-2\bar{\kappa}_{11}\bar{\kappa}_{22} + 2(\bar{\kappa}_{12})^2]\} \\
 &= \frac{D}{2} \{(\bar{\kappa}_{11} + \bar{\kappa}_{22})^2 + 2(1 - \nu)[- \bar{\kappa}_{11}\bar{\kappa}_{22} + (\bar{\kappa}_{12})^2]\}
 \end{aligned} \tag{4.72}$$

$$\boxed{\bar{U}_b = \frac{D}{2} \{(\bar{\kappa}_{11} + \bar{\kappa}_{22})^2 - 2(1 - \nu) [\bar{\kappa}_{11}\bar{\kappa}_{22} - (\bar{\kappa}_{12})^2]\}} \tag{4.73}$$

The term in the square brackets is the Gaussian curvature, κ_G , introduced in Lecture 2, Eq.(2.62). Should the Gaussian curvature vanish, as it is often the case in plates, then the bending energy density assumes a very simple form $\bar{U}_b = \frac{1}{2}D(\bar{\kappa}_{11} + \bar{\kappa}_{22})^2$.

Total Bending Energy

The total bending energy is the integral of the bending energy density over the area of plate:

$$U_b = \int_S \bar{U}_b dA \tag{4.74}$$

Membrane Forces and Membrane Energy

The axial force can be calculated in a similar way as before

$$\begin{aligned}
 N_{\alpha\beta} &= \frac{E}{1-\nu^2} \int_{-\frac{h}{2}}^{\frac{h}{2}} [(1-\nu)\epsilon_{\alpha\beta} + \nu\epsilon_{\gamma\gamma}\delta_{\alpha\beta}] dz \\
 &= \frac{E}{1-\nu^2} \int_{-\frac{h}{2}}^{\frac{h}{2}} [(1-\nu)\epsilon_{\alpha\beta}^{\circ} + \nu\epsilon_{\gamma\gamma}^{\circ}\delta_{\alpha\beta}] dz \\
 &\quad + \frac{E}{1-\nu^2} \int_{-\frac{h}{2}}^{\frac{h}{2}} [(1-\nu)\kappa_{\alpha\beta} + \nu\kappa_{\gamma\gamma}\delta_{\alpha\beta}] z dz \\
 &= \frac{E}{1-\nu^2} [(1-\nu)\epsilon_{\alpha\beta}^{\circ} + \nu\epsilon_{\gamma\gamma}^{\circ}\delta_{\alpha\beta}] \int_{-\frac{h}{2}}^{\frac{h}{2}} dz \\
 &\quad + \frac{E}{1-\nu^2} [(1-\nu)\kappa_{\alpha\beta} + \nu\kappa_{\gamma\gamma}\delta_{\alpha\beta}] \int_{-\frac{h}{2}}^{\frac{h}{2}} z dz \\
 &= \frac{Eh}{1-\nu^2} [(1-\nu)\epsilon_{\alpha\beta}^{\circ} + \nu\epsilon_{\gamma\gamma}^{\circ}\delta_{\alpha\beta}]
 \end{aligned} \tag{4.75}$$

The integral $\int_{-\frac{h}{2}}^{\frac{h}{2}} z dz$ is zero which means that there is no coupling between the membrane force and curvatures.

Here we define the axial rigidity of a plate C as follows:

$$C = \frac{Eh}{1-\nu^2} \tag{4.76}$$

Now, one gets the membrane force-extension relation in the tensor notation:

$$\boxed{N_{\alpha\beta} = C[(1-\nu)\epsilon_{\alpha\beta}^{\circ} + \nu\epsilon_{\gamma\gamma}^{\circ}\delta_{\alpha\beta}]} \tag{4.77}$$

$$N_{\alpha\beta} = \begin{vmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{vmatrix} \tag{4.78}$$

where $N_{12} = N_{21}$ due to symmetry. In components,

$$N_{11} = C(\epsilon_{11}^{\circ} + \nu\epsilon_{22}^{\circ}) \tag{4.79a}$$

$$N_{22} = C(\epsilon_{22}^{\circ} + \nu\epsilon_{11}^{\circ}) \tag{4.79b}$$

$$N_{12} = C(1-\nu)\epsilon_{11}^{\circ} \tag{4.79c}$$

Membrane Energy Density

Using the similar definition used in the calculation of the bending energy density, the extension energy (membrane energy) reads:

$$\bar{U}_m = \oint N_{\alpha\beta} d\epsilon_{\alpha\beta}^{\circ} \tag{4.80}$$

Let's calculate the energy stored when the extension reaches a given value $\bar{\epsilon}_{\alpha\beta}^{\circ}$. Consider a straight path:

$$\epsilon_{\alpha\beta}^{\circ} = \eta \bar{\epsilon}_{\alpha\beta}^{\circ} \quad (4.81a)$$

$$d\epsilon_{\alpha\beta}^{\circ} = \bar{\epsilon}_{\alpha\beta}^{\circ} d\eta \quad (4.81b)$$

$$\begin{aligned} N_{\alpha\beta} &= \hat{N}_{\alpha\beta}(\epsilon_{\alpha\beta}^{\circ}) \\ &= \hat{N}_{\alpha\beta}(\eta \bar{\epsilon}_{\alpha\beta}^{\circ}) \\ &= \eta \hat{N}_{\alpha\beta}(\bar{\epsilon}_{\alpha\beta}^{\circ}) \end{aligned} \quad (4.82)$$

where $\hat{N}_{\alpha\beta}(\bar{\epsilon}_{\alpha\beta}^{\circ})$ is a homogenous function of degree one.

$$\begin{aligned} \bar{U}_m &= \int_0^{\bar{\epsilon}_{\alpha\beta}^{\circ}} \hat{N}_{\alpha\beta}(\epsilon_{\alpha\beta}^{\circ}) d\epsilon_{\alpha\beta}^{\circ} \\ &= \int_0^1 \eta \hat{N}_{\alpha\beta}(\bar{\epsilon}_{\alpha\beta}^{\circ}) \bar{\epsilon}_{\alpha\beta}^{\circ} d\eta \\ &= \frac{1}{2} \hat{N}_{\alpha\beta}(\bar{\epsilon}_{\alpha\beta}^{\circ}) \bar{\epsilon}_{\alpha\beta}^{\circ} \\ &= \frac{1}{2} N_{\alpha\beta} \bar{\epsilon}_{\alpha\beta}^{\circ} \end{aligned} \quad (4.83)$$

Now, the extension energy reads:

$$\begin{aligned} \tilde{U}_m &= \frac{C}{2} [(1 - \nu) \bar{\epsilon}_{\alpha\beta}^{\circ} + \nu \bar{\epsilon}_{\gamma\gamma}^{\circ} \delta_{\alpha\beta}] \bar{\epsilon}_{\alpha\beta}^{\circ} \\ &= \frac{C}{2} [(1 - \nu) \bar{\epsilon}_{\alpha\beta}^{\circ} \bar{\epsilon}_{\alpha\beta}^{\circ} + \nu (\bar{\epsilon}_{\gamma\gamma}^{\circ})^2] \end{aligned} \quad (4.84)$$

The extension energy density expressed in terms of components is:

$$\begin{aligned} \bar{U}_m &= \frac{C}{2} \{ (1 - \nu) [(\bar{\epsilon}_{11}^{\circ})^2 + 2(\bar{\epsilon}_{12}^{\circ})^2 + (\bar{\epsilon}_{22}^{\circ})^2] + \nu (\bar{\epsilon}_{11}^{\circ} + \bar{\epsilon}_{22}^{\circ})^2 \} \\ &= \frac{C}{2} \{ (1 - \nu) [(\bar{\epsilon}_{11}^{\circ} + \bar{\epsilon}_{22}^{\circ})^2 - 2\bar{\epsilon}_{11}^{\circ} \bar{\epsilon}_{22}^{\circ} + 2(\bar{\epsilon}_{12}^{\circ})^2] + \nu (\bar{\epsilon}_{11}^{\circ} + \bar{\epsilon}_{22}^{\circ})^2 \} \\ &= \frac{C}{2} \{ (\bar{\epsilon}_{11}^{\circ} + \bar{\epsilon}_{22}^{\circ})^2 - 2\bar{\epsilon}_{11}^{\circ} \bar{\epsilon}_{22}^{\circ} + 2(\bar{\epsilon}_{12}^{\circ})^2 - \nu [-2\bar{\epsilon}_{11}^{\circ} \bar{\epsilon}_{22}^{\circ} + 2(\bar{\epsilon}_{12}^{\circ})^2] \} \\ &= \frac{C}{2} \{ (\bar{\epsilon}_{11}^{\circ} + \bar{\epsilon}_{22}^{\circ})^2 + 2(1 - \nu) [-\bar{\epsilon}_{11}^{\circ} \bar{\epsilon}_{22}^{\circ} + (\bar{\epsilon}_{12}^{\circ})^2] \} \end{aligned} \quad (4.85)$$

$$\boxed{\bar{U}_m = \frac{C}{2} \{ (\bar{\epsilon}_{11}^{\circ} + \bar{\epsilon}_{22}^{\circ})^2 - 2(1 - \nu) [\bar{\epsilon}_{11}^{\circ} \bar{\epsilon}_{22}^{\circ} - (\bar{\epsilon}_{12}^{\circ})^2] \}} \quad (4.86)$$

The total membrane energy is the integral of the membrane energy density over the area of plate:

$$U_m = \int_S \bar{U}_m dS \quad (4.87)$$

END OF ADVANCED TOPIC

4.6 Stress Formula for Plates

In the section on beams, it was shown that the profile of axial stress can be determined from the known bending moment M and axial force N , see Eq. (4.30). A similar procedure can be developed for plates by comparing Eqs (4.61-4.77) with Eq. (4.54). The stress-strain curve for the plane stress can be expressed in terms of the middle surface strain tensor $\epsilon_{\alpha\beta}^{\circ}$ and curvature tensor $\kappa_{\alpha\beta}$ by combining Eqs. (4.54) and (4.56).

$$\begin{aligned}\sigma_{\alpha\beta} = & \frac{E}{1-\nu^2}[(1-\nu)\epsilon_{\alpha\beta}^{\circ} + \nu\epsilon_{\gamma\gamma}^{\circ}\delta_{\alpha\beta}] \\ & + \frac{E}{1-\nu^2}[(1-\nu)\kappa_{\alpha\beta} + \nu\kappa_{\gamma\gamma}\delta_{\alpha\beta}]z\end{aligned}\quad (4.88)$$

From the moment-curvature relation, Eq. (4.61):

$$(1-\nu)\kappa_{\alpha\beta} + \nu\kappa_{\gamma\gamma}\delta_{\alpha\beta} = \frac{M_{\alpha\beta}}{D}\quad (4.89)$$

Similarly, from Eq. (4.72)

$$(1-\nu)\epsilon_{\alpha\beta}^{\circ} + \nu\epsilon_{\gamma\gamma}^{\circ}\delta_{\alpha\beta} = \frac{N_{\alpha\beta}}{C}\quad (4.90)$$

where $D = \frac{Eh^3}{12(1-\nu^2)}$ is the bending rigidity, and $C = \frac{Eh}{1-\nu^2}$ is the axial rigidity of the plate.

From the above system, one gets

$$\sigma_{\alpha\beta} = \frac{Ez}{1-\nu^2} \frac{M_{\alpha\beta}}{D} + \frac{E}{1-\nu^2} \frac{N_{\alpha\beta}}{C}\quad (4.91)$$

or using the definitions of D and C

$$\sigma_{\alpha\beta} = \frac{N_{\alpha\beta}}{h} + \frac{zM_{\alpha\beta}}{h^3/12}\quad (4.92)$$

The above equation is dimensionally correct, because both $N_{\alpha\beta}$ and $M_{\alpha\beta}$ are respective quantities per unit length. In particular stress in the case of cylindrical bending is

$$\sigma_{xx} = \frac{N_{xx}}{h} + \frac{zM_{xx}}{h^3/12}\quad (4.93)$$

Multiplying both the numerators and denominators of the two terms above by b yields

$$\sigma_{xx} = \frac{N_{xx}b}{hb} + \frac{zM_{xx}b}{bh^3/12}\quad (4.94)$$

Now, observing that $N_{xx}b = N$ is the beam axial force, $bM_{xx} = M$ is the beam bending moment, $hb = A$ is the cross-section of the rectangular section beam, and $\frac{bh^3}{12}$ is the moment of inertia, the familiar beam stress formula is obtained

$$\sigma = \frac{N}{A} + \frac{Mz}{I}\quad (4.95)$$

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Fall 2013

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