Lecture 12: Fundamental Concepts in Structural Plasticity

Plastic properties of the material were already introduced briefly earlier in the present notes. The critical slenderness ratio of column is controlled by the yield stress of the material. The subsequent buckling of column in the plastic range requires the knowledge of the hardening curve. These two topics were described in Lecture 9. In Lecture 10 the concept of the ultimate strength of plates was introduced and it was shown that the yield stress is reached first along the supported or clamped edges and the plastic zones spread towards the plate center, leading to the loss of stiffness and strength. In the present lecture the above simple concepts will be extended and formalized to prepare around for the structural applications in terms of the limit analysis.

There are five basic concepts in the theory of plasticity:

- Yield condition
- Hardening curve
- Incompressibility
- Flow rule
- Loading/unloading criterion

All of the above concept will first be explained in the 1-D case and then extended to the general 3-D case.

12.1 Hardening Curve and Yield Curve

If we go to the lab and perform a standard tensile test on a round specimen or a flat dog-bone specimen made of steel or aluminum, most probably the engineering stress-strain curve will look like the one shown in Fig. (12.1a). The following features can be distinguished:

Point A - proportionality limit

Point B - 0.02% yield

Point C - arbitrary point on the hardening curve showing different trajectories on loading/unloading

Point D - fully unloaded specimen

For most of material the initial portion of the stress-strain curve is straight up to the proportionality limit, point A. From this stage on the stress-strain curve becomes slightly curved but there is no distinct yield point with a sudden change of slope. There is in international standard the yield stress is mapped by taking elastic slope with 0.02% strain

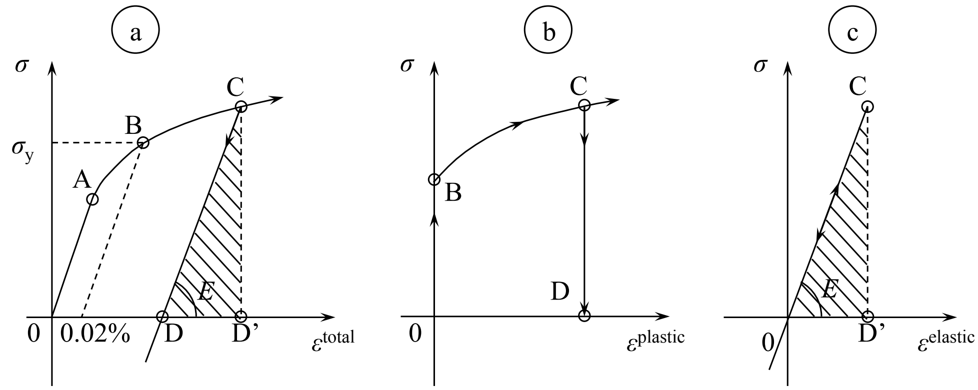

Figure 12.1: Elastic, plastic and total stress-strain curve.

($\epsilon = 0.0002$) offset strain. Upon loading, the material hardens and the stress is increasing with diminishing slop until the testing machine (either force or displacement controlled) is stopped. There are two possibilities. On unloading, meaning reversing the load or displacement of the cross-load of the testing machine, the unloading trajectory is straight. This is the elastic unloading where the slop of the stress-strain curve is equal to the initial slope, given by the Young's modulus. At point D the stress is zero but there is a residual plastic strain of the magnitude OD. The experiment on loading/unloading tell us that the *total strain* ϵ^{total} can be considered as the sum of the *plastic strain* $\epsilon^{\text{plastic}}$ and *elastic strain* $\epsilon^{\text{elastic}}$. Thus

$$\epsilon^{\text{total}} = \epsilon^{\text{plastic}} + \epsilon^{\text{elastic}} \tag{12.1}$$

The elastic component is not constant but depends on the current stress

$$\epsilon^{\text{elastic}} = \frac{\sigma}{E} \tag{12.2}$$

The plastic strain depends on how far a given specimen is loaded, and thus there is a difference between the total (measured) strain and known elastic strain. Various empirical formulas were suggested in the literature to fit the measured relation between the stress and the plastic strain. The most common is the swift hardening law

$$\sigma = A(\epsilon^{\text{plastic}} + \epsilon_o)^n \tag{12.3}$$

where A is the stress amplitude, n is the hardening exponent and ϵ_o is the strain shift parameter.

In many practical problems the magnitude of plastic strain is mud larger than the parameter ϵ_o, giving rise to a simpler power hardening law, extensively used in the literature.

$$\sigma = A\epsilon^n$$

For most metals the exponent n is the range of $n = 0.1 - 0.3$, and the amplitude can vary a lot, depending on the grade of steel. A description of the reverse loading and cycling plastic loading is beyond the scope of the present lecture notes.

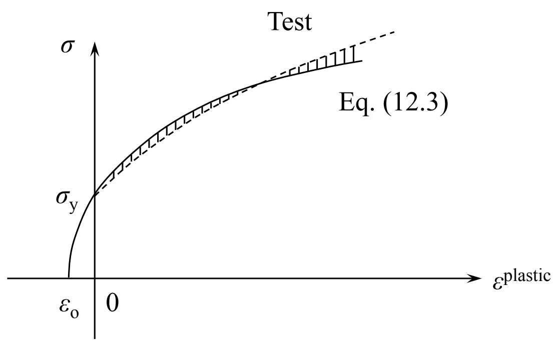

Figure 12.2: The experimentally measured stress-strain curve and the fit by the swift law.

Various other approximation of the actual stress-strain curve of the material are in common use and some of then are shown in Fig. (12.3).

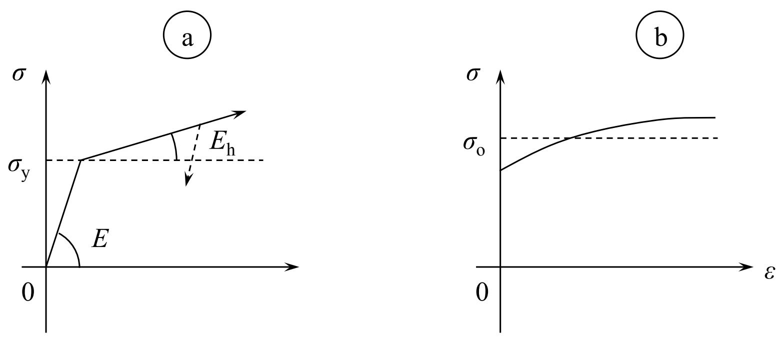

Figure 12.3: Elastic-linear hardening material (a) and rigid-plastic hardening material (b).

A further simplification is obtained by considering the average value σ_o of the stress-strain curve, illustrated in Fig. (12.3b). This concept gave rise to the concept of the rigid-perfectly plastic material characteristic time, depicted in Fig. (12.4).

The material model shown in Fig. (12.4) is adopted in the development of the *limit analysis* of structures. The extension of the concept of the hardening curve to the 3-D case will be presented later, after deriving the expression for the yield condition.

12.2 Loading/Unloading Condition

In the 1-D case the plastic flow rule is reduced to the following statement:

$$\dot{\epsilon}^{\mathrm{p}} > 0 \qquad \sigma = \sigma_o \tag{12.4a}$$

$$\dot{\epsilon}^{\mathrm{p}} < 0 \qquad \sigma = -\sigma_o \tag{12.4b}$$

$$\dot{\epsilon}^{\mathrm{p}} = 0 \qquad \sigma_o < \sigma < -\sigma_o \tag{12.4c}$$

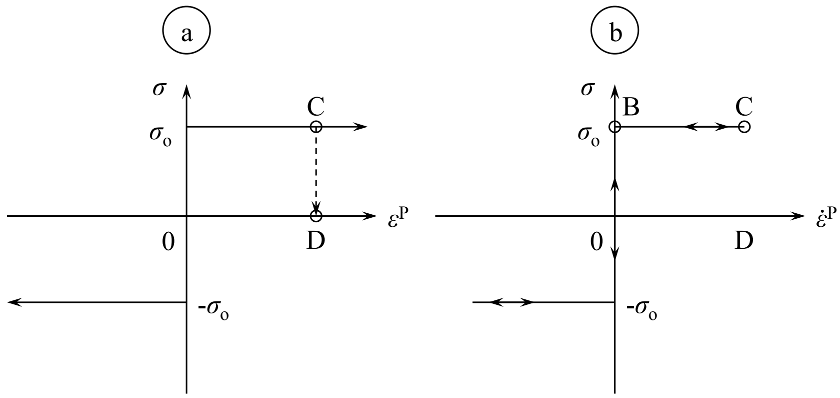

Figure 12.4: The flow stress $|\sigma_o|$ is assumed to be identical in tension and compression in the rigid-perfectly plastic material model.

In the case of unloading, the stress follow the path CD on the σ-ϵ^{p} graph. If the strain rate is an independent variable, the path of all unloading cases is the same CBO, as shown in Fig. (12.4).

12.3 Incompressibility

Numerous experiments performed over the past 100% have shown that metals are practically incompressible in the plastic range. Let's explore the consequences of this physical fact in the case of one-dimensional case. Denote the gauge length of the prismatic bar by l and its cross-sectional area by A. The current volume of the gauge section is $V = Al$. Incompressibility means that the volume must be unchanged or $\mathrm{d}V = 0$.

$$\mathrm{d}V = \mathrm{d}(Al) = \mathrm{d}Al + A\mathrm{d}l = 0 \tag{12.5}$$

From Eq. (12.5) we infer that the strain increment $\mathrm{d}\epsilon$ can be calculated either by tracking down the gauge length or the cross-sectional area

$$\mathrm{d}\epsilon = \frac{\mathrm{d}l}{l} = -\frac{\mathrm{d}A}{A} \tag{12.6}$$

Integrating the first part of Eq. (**??**)

$$\epsilon = \ln l + C_1 \tag{12.7}$$

The integration constant is obtained by requiring that the strain vanishes when the length l is equal to the gauge initial, reference length l_o, which gives $C = -\ln l_o$. Thus

$$\epsilon = \ln \frac{l}{l_o} \tag{12.8}$$

which is the logarithmic definition of strain, introduced in Lecture 2. Similarly, integrating the second part of Eq. (??) with the initial condition at $A = A_o$, $\epsilon = 0$, one gets

$$\epsilon = \ln \frac{A_o}{A} \tag{12.9}$$

In tension $l > l_o$ or $A < A_o$, so both Eqs. (??) and (??) gives the positive strain. In compression the strain is negative. The same is true for strain increments $\mathrm{d}\epsilon$ or strain rates

$$\dot{\epsilon} = \frac{\dot{l}}{l} \quad \text{or} \quad \dot{\epsilon} = -\frac{\dot{A}}{A} \tag{12.10}$$

From the above analysis follows a simple extension of the plastic incompressibility condition into the 3-D case. Consider an infinitesimal volume element $V = x_1 x_2 x_3$, Fig. (12.5).

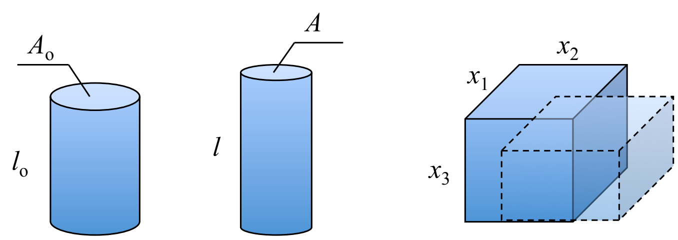

Figure 12.5: Undeformed and deformed 1-D and 3-D volume elements.

The plastic incompressibility requires that

$$\begin{aligned} \mathrm{d}V &= \mathrm{d}(x_1 x_2 x_3) = \mathrm{d}x_1(x_2 x_3) + x_1 \mathrm{d}(x_2 x_3) \\ &= \mathrm{d}x_1 x_2 x_3 + x_1 \mathrm{d}x_2 x_3 + x_1 x_2 \mathrm{d}x_3 \end{aligned} \tag{12.11}$$

Dividing both sides of the above equation by the volume, one gets

$$\frac{\mathrm{d}x_1}{x_1} + \frac{\mathrm{d}x_2}{x_2} + \frac{\mathrm{d}x_3}{x_3} = 0 \tag{12.12}$$

or

$$\mathrm{d}\epsilon_{11} + \mathrm{d}\epsilon_{22} + \mathrm{d}\epsilon_{33} = 0, \quad \mathrm{d}\epsilon_{kk} = 0$$

Noting that $\mathrm{d}\epsilon_{11} = \dfrac{\partial \epsilon_{11}}{\partial t}\mathrm{d}t = \dot{\epsilon}_{11}\mathrm{d}t$, an alternative form of the incompressibility condition is

$$\dot{\epsilon}_{11} + \dot{\epsilon}_{22} + \dot{\epsilon}_{33} = 0, \quad \dot{\epsilon}_{kk} = 0 \tag{12.13}$$

The sum of the diagonal components of the strain rate tensor must vanish to ensure incompressibility. It follows from the flow rule (to be formulated later) that in uniaxial tension in x_1 direction the components $\dot{\epsilon}_{22} = \dot{\epsilon}_{33}$. Therefore $\dot{\epsilon}_{11} + 2\dot{\epsilon}_{22} = 0$ or $\dot{\epsilon}_{11} + 2\dot{\epsilon}_{33} = 0$. Finally we obtain

$$\dot{\epsilon}_{22} = -0.5\dot{\epsilon}_{11}, \quad \dot{\epsilon}_{33} = -0.5\dot{\epsilon}_{11} \tag{12.14}$$

The coefficient 0.5 can be interpreted as the Poisson ratio

$$\nu = -\frac{\dot{\epsilon}_{22}}{\dot{\epsilon}_{11}} = -\frac{\dot{\epsilon}_{33}}{\dot{\epsilon}_{11}} = 0.5 \tag{12.15}$$

We can conclude that plastic incompressibility requires that the Poisson ratio be equal to 1/2, which is different from the elastic Poisson ratio, equal ~ 0.3 for metals. Many other materials such as rubber, polymers and water are incompressible.

12.4 Yield Condition

From the previous section, the uniaxial yield condition under tension/compression in the x-direction is

$$\sigma_{11} = \pm\sigma_{\mathrm{y}} \tag{12.16}$$

In the general 3-D, all six components of the stress tensor contribute to yielding of the material. The von Mises yield condition takes the form

$$\frac{1}{2}[(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2] + 3(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2)] = \sigma_{\mathrm{y}}^2 \tag{12.17}$$

or in a short-hand notation

$$F(\sigma_{ij}) = \sigma_{\mathrm{y}}$$

The step-by-step derivation of the above equation is given in the next section. Here, several special cases are considered.

Principle coordinate system

All non-diagonal components of the stress tensor vanish, $\sigma_{12} = \sigma_{23} = \sigma_{31} = 0$. Then, Eq. (12.12) reduces to

$$(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2 = 2\sigma_{\mathrm{y}}^2 \tag{12.18}$$

where σ_1, σ_2 σ_3 are principal stresses. The graphical representation of Eq. (**??**) is the open ended cylinder normal to the octahedral plane, Fig. (12.6).

The equation of the straight line normal to the octahedral plane and passing through the origin is

$$\sigma_1 + \sigma_2 + \sigma_3 = 3p \tag{12.19}$$

where p is the hydrostatic pressure. Since the hydrostatic pressure does not have any effect on yielding, the yield surface is an open cylinder.

Plane stress

Substituting $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$ in Eq. (12.12), the plane stress yield condition becomes

$$\sigma_{11}^2 - \sigma_{11}\sigma_{22} + \sigma_{22}^2 + 3\sigma_{12}^2 = \sigma_{\mathrm{y}}^2 \tag{12.20}$$

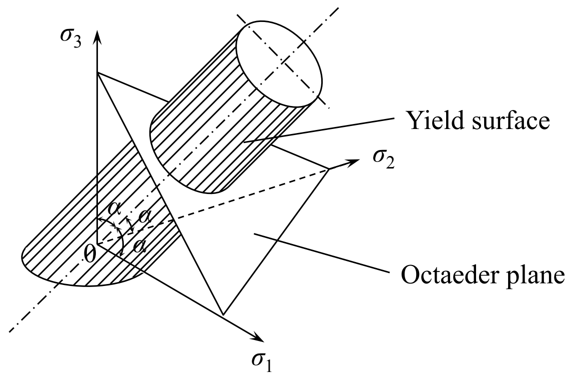

Figure 12.6: Representation of the von Mises yield condition in the space of principal stresses.

In particular, in pure shear $\sigma_{11} = \sigma_{22} = 0$ and $\sigma_{12} = \sigma_y/\sqrt{3}$. In the literature $\sigma_y/\sqrt{3} = k$ is called the yield stress in shear corresponding to the von Mises yield condition. In the principal coordinate system $\sigma_{12} = 0$ and the yield condition takes a simple form

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = \sigma_y^2 \tag{12.21}$$

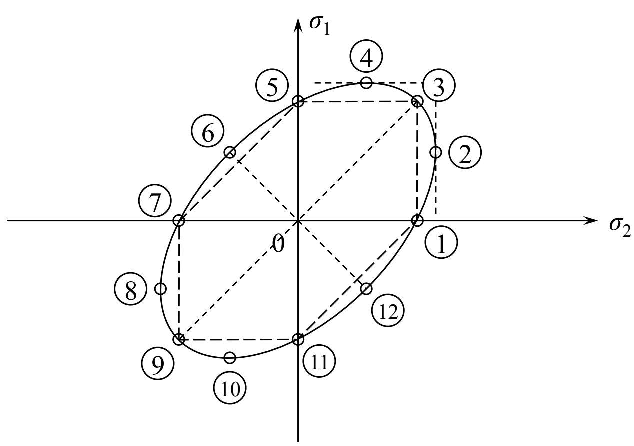

Figure 12.7: The von Mises ellipse in the principal coordinate system.

The graphical representation of Eq. (**??**) is the ellipse shown in Fig. (12.7). Several important stress states can be identified in Fig. (12.7).

Point 1 and 2	– Uniaxial tension, $\sigma_1 = \sigma_2 = \sigma_y$
Point 7 and 11	– Uniaxial compression, $\sigma_1 = \sigma_2 = -\sigma_y$
Point 3	– Equi-biaxial tension, $\sigma_1 = \sigma_2$
Point 9	– Equi-biaxial compression, $-\sigma_1 = -\sigma_2$
Points 2, 4, 8 and 10	– Plain strain, $\sigma_1 = \frac{2}{\sqrt{3}}\sigma_y$
Points 6 and 12	– Pure shear, $\sigma_1 = -\sigma_2$

The concept of the plane strain will be explained in the section dealing with the flow rule.

Equivalent stress and equivalent strain rate

In the finite element analysis the concept of the *equivalent stress* $\bar{\sigma}$ or the von Mises stress is used. It is defined by in terms of principal stresses

$$\bar{\sigma} = \frac{1}{2}[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2] \tag{12.22}$$

The equivalent stress $\bar{\sigma}(\sigma_{ij})$ is the square root of the left hand side of Eq. (12.12). Having defined the equivalent stress, the energy conjugate equivalent strain rate can be evaluated from

$$\bar{\sigma}\dot{\bar{\epsilon}} = \sigma_{ij}\dot{\epsilon}_{ij} \tag{12.23}$$

and is given by

$$\dot{\bar{\epsilon}} = \left\{\frac{2}{9}\left[(\dot{\epsilon}_{11} - \dot{\epsilon}_{22})^2 + (\dot{\epsilon}_{22} - \dot{\epsilon}_{33})^2 + (\dot{\epsilon}_{33} - \dot{\epsilon}_{11})^2\right]\right\}^{1/2} \tag{12.24}$$

The equivalent strain is obtained from integrating in time the equivalent strain rate

$$\bar{\epsilon} = \int \dot{\bar{\epsilon}}\, \mathrm{d}t \tag{12.25}$$

12.5 Isotropie and Kinematic Hardening

It should be noted that in the case of uniaxial stress, $\sigma_2 = \sigma_3 = 0$ and Eq. (12.14) reduces to $\bar{\sigma} = \sigma_1$. Likewise, for uniaxial stress $\dot{\epsilon}_2 = -0.5\epsilon_1$ and $\dot{\epsilon}_3 = -0.5\epsilon_1$ and the equivalent strain rate becomes equal to $\dot{\bar{\epsilon}} = \dot{\epsilon}_1$. Then, according to Eq. (**??**), $\bar{\epsilon}_1 = \epsilon_1$. The hypothesis of the isotropic hardening is that the size of the instantaneous yield condition, represented by the radius of the cylinder (Fig. (12.6)) is a function of the intensity of the plastic strain defined by the equivalent plastic strain $\bar{\epsilon}$. Thus

$$\bar{\sigma} = \sigma_{\mathrm{y}}(\bar{\epsilon}) \tag{12.26}$$

The hardening function $\sigma_{\mathrm{y}}(\bar{\epsilon})$ is determined from a single test, such as a uniaxial tension. In this case

$$\bar{\sigma} = \sigma_1 = \sigma_{\mathrm{y}}(\bar{\epsilon}) = \sigma_{\mathrm{y}}(\epsilon_1) \tag{12.27}$$

Thus the form of the function $\sigma_{\mathrm{y}}(\bar{\epsilon})$ is identical to the hardening curve obtained from the tensile experiment. If the tensile test is fit by the power hardening law, the equivalent stress is a power function of the equivalent strain

$$\bar{\sigma} = A\bar{\epsilon}^n \tag{12.28}$$

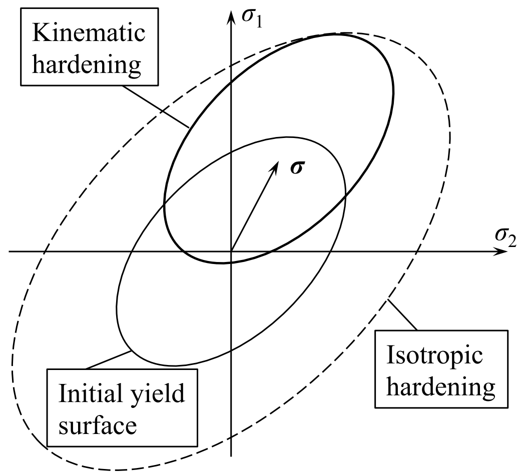

Figure 12.8: Comparison of the isotropic and kinematic hardening under plane stress.

The above function often serves as an input to many general purpose finite element codes. A graphical representation of the 3-D hardening rule is a uniform growth of the initial yield ellipse with equivalent strain $\bar{\epsilon}$, Fig. (12.8).

In the case of kinematic hardening the size of the initial yield surface remains the same, but the center of the ellipse is shifted, see Fig. (12.8). The coordinates of the center of the ellipse is called the *back stress*. The concept of the kinematic hardening is important for reverse and cyclic loading. It will not be further pursued in the present lecture notes.

12.6 Flow Rule

The simplest form of the associated flow rule for a rigid perfectly plastic material is given by

$$\dot{\epsilon}_{ij} = \dot{\lambda}\frac{\partial F(\sigma_{ij})}{\partial \sigma_{ij}} \tag{12.29}$$

where the function $F(\sigma_{ij})$ is defined by Eq. (12.12), and $\dot{\lambda}$ is the scalar multiplication factor. Equation (12.15) determines uniquely the direction of the strain rate vector, which is always directed normal to the yield surface at a given stress point. In the case of plane stress, the two components of the strain rate vector are

$$\dot{\epsilon}_1 = \dot{\lambda}(2\sigma_1 - \sigma_2) \tag{12.30a}$$

$$\dot{\epsilon}_2 = \dot{\lambda}(2\sigma_2 - \sigma_1) \tag{12.30b}$$

The magnitudes of the components $\dot{\epsilon}_1$ and $\dot{\epsilon}_2$ are undetermined, but the ratio, which defines the direction $\dot{\epsilon}/\epsilon_2$, is uniquely determined.

In particular, under the transverse plain strain $\dot{\epsilon}_2 = 0$, so $\sigma_1 = 2\sigma_2$ and $\sigma_1 = \dfrac{2}{\sqrt{3}}\sigma_{\mathrm{y}}$.

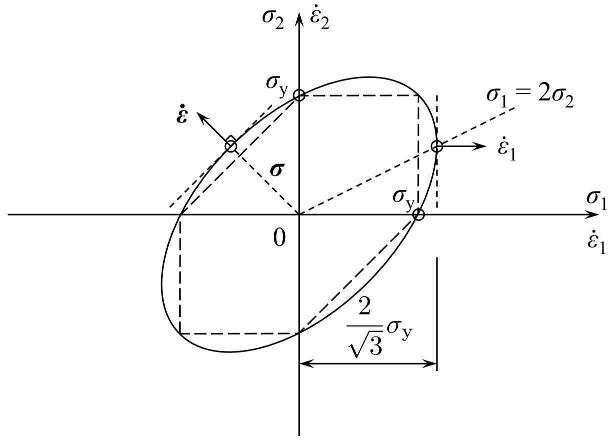

Figure 12.9: The strain rate vector is always normal to the yield surface.

ADVANCED TOPIC

12.7 Derivation of the Yield Condition from First Principles

The analysis starts from stating the stress-strain relations for the elastic material, covered in Lecture 4. The general Hook's law for the isotropic material is

$$\epsilon_{ij} = \frac{1}{E}[(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}] \tag{12.31}$$

The elastic constitutive equation can also be written in an alternative form, separately for the distortional and dilatational part

$$e_{ij} = \frac{1+\nu}{E}s_{ij} \quad - \quad \text{distorsion} \tag{12.32a}$$

$$\epsilon_{kk} = \frac{1-2\nu}{E}\sigma_{kk} \quad - \quad \text{dilatation} \tag{12.32b}$$

The next step is to invoke the basic property of the elastic material that the strain energy density $\bar{U}$, defined by

$$\bar{U} = \oint \sigma_{ij}\, \mathrm{d}\epsilon_{ij} \tag{12.33}$$

does not depend on the loading path of the above line integral but only on the final state. Thus, evaluating the strain energy on the proportional (straight) loading path, one gets

$$\bar{U} = \frac{1}{2}\sigma_{ij}\epsilon_{ij} \tag{12.34}$$

The next step is to prove that the strain energy density can be decomposed into the distortional and dilatational part. This is done by recalling the definition of the stress deviator s_{ij} and strain deviator e_{ij}

$$\sigma_{ij} = s_{ij} + \frac{1}{3}\sigma_{kk}\delta_{ij} \tag{12.35a}$$

$$\epsilon_{ij} = e_{ij} + \frac{1}{3}\epsilon_{kk}\delta_{ij} \tag{12.35b}$$

Introducing Eq. (12.20) into Eq. (12.19), there will be four terms in the expression for $\bar{U}$

$$2\bar{U} = s_{ij}e_{ij} + s_{ij}\frac{1}{3}\epsilon_{kk}\delta_{ij} + \frac{1}{3}\sigma_{kk}\delta_{ij}e_{ij} + \frac{1}{3}\sigma_{kk}\delta_{ij}\frac{1}{3}\epsilon_{kk}\delta_{ij} \tag{12.36}$$

Note that $s_{ij}\delta_{ij} = s_{ii} = 0$ from the definition, Eq. (12.20). Likewise $e_{ij}\delta_{ij} = e_{jj} = 0$, also from the definition. Therefore the second and third term of Eq. (**??**) vanish and the energy density becomes

$$\bar{U} = \frac{1}{2}s_{ij}e_{ij} + \frac{1}{6}\sigma_{kk}\epsilon_{ll} = \bar{U}_{\text{dist}} + \bar{U}_{\text{dil}} \tag{12.37}$$

Attention is focused on the distortional energy, which with the help of the elasticity law Eq. (12.17) can be put into the form

$$\bar{U}_{\text{dist}} = \frac{1+\nu}{2E}s_{ij}s_{ij} \tag{12.38}$$

The product $s_{ij}s_{ij}$ can be expressed in terms of the components of the stress tensor

$$\begin{aligned} s_{ij}s_{ij} &= (\sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij})(\sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}) \\ &= \sigma_{ij}\sigma_{ij} - \frac{1}{3}\sigma_{ij}\sigma_{kk}\delta_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}\sigma_{ij} + \frac{1}{9}\sigma_{kk}\sigma_{kk}\delta_{ij}\delta_{ij} \\ &= \sigma_{ij}\sigma_{ij} - \frac{2}{3}\sigma_{kk}\sigma_{kk} + \frac{1}{3}\sigma_{kk}\sigma_{kk} \end{aligned}$$

The final result is

$$\bar{U} = \frac{1+\nu}{2E}(\sigma_{ij}\sigma_{ij} - \frac{1}{3}\sigma_{kk}\sigma_{kk}) \tag{12.39}$$

In 1904 the Polish professor Maximilian Tytus Huber proposed a hypothesis that yielding of the material occurs when the distortional energy density reaches a critical value

$$\sigma_{ij}\sigma_{ij} - \frac{1}{3}\sigma_{kk}\sigma_{kk} = C \tag{12.40}$$

where C is the material constant that must be determined from tests. The calibration is performed using the uni-axial tension test for which the components of the stress tensor are

$$\sigma_{ij} = \begin{vmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \tag{12.41}$$

From Eq. (12.21) we get

$$\sigma_{11}\sigma_{11} - \frac{1}{3}\sigma_{11}\sigma_{11} = \frac{2}{3}\sigma_{11}\sigma_{11} = C \tag{12.42}$$

Yielding occurs when $\sigma_{11} = \sigma_{\mathrm{y}}$ so $C = \frac{2}{3}\sigma_{\mathrm{y}}^2$. The most general form of the Huber yield condition is

$$(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + 6(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2) = 2\sigma_{\mathrm{y}}^2 \tag{12.43}$$

which was the starting point of the analysis of various special cases in section 12.4. A similar form of the yield condition for plane stress was derived by von Mises in 1913, based on plastic slip consideration and was later extended to the 3-D case by Hencky. The present form is reformed to in the literature as the Huber-Mises-Hencky yield criterion, called von Mises for short.

END OF ADVANCED TOPIC

12.8 Tresca Yield Condition

The stress state in uni-axial tension of a bar depends on the orientation of the plane on which the stresses are resolved. In Lecture 3 it was shown that the shear stress τ on the plane inclined to the horizontal plane by the angle α is

$$\tau = \frac{1}{2}\sigma_{11}\sin 2\alpha \tag{12.44}$$

where σ_{11} is the uniaxial tensile stress, see Fig. (12.10).

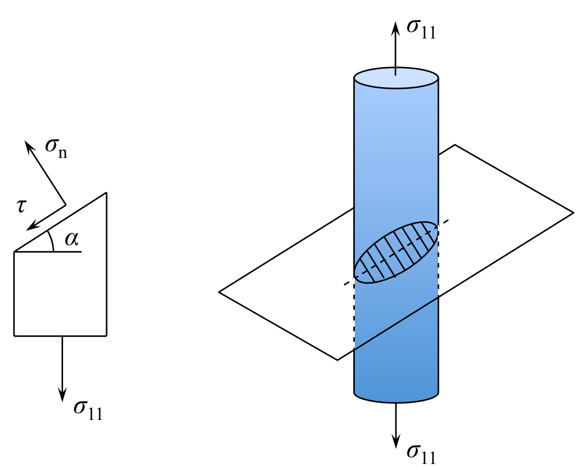

Figure 12.10: Shear and normal stresses at an arbitrary cut.

The maximum shear occurs when $\sin 2\alpha = 1$ or $\alpha = \frac{\pi}{4}$. Thus in uniaxial tension

$$\tau_{\max} = \frac{\sigma_{11}}{2} \tag{12.45}$$

Extending the analysis to the 3-D case (see for example Fung) the maximum shear stresses on three shear planes are

$$\tau_1 = \frac{|\sigma_1 - \sigma_2|}{2}, \quad \tau_2 = \frac{|\sigma_2 - \sigma_1|}{2}, \quad \tau_3 = \frac{|\sigma_3 - \sigma_1|}{2} \tag{12.46}$$

where σ_1, σ_2, σ_3 are principal stresses. In 1860 the French scientist and engineer Henri Tresca put up a hypothesis that yielding of the material occurs when the maximum shear stress reaches a critical value τ_c

$$\tau_o = \max\left\{\frac{|\sigma_1 - \sigma_2|}{2}, \frac{|\sigma_2 - \sigma_3|}{2}, \frac{|\sigma_3 - \sigma_1|}{2}\right\} \tag{12.47}$$

The unknown constant can be calibrated from the uniaxial test for which Eq. (12.24) holds. Therefore at yield $\tau_o = \sigma_y/2$ and the Tresca yield condition takes the form

$$\max\{|\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_3 - \sigma_1|\} = \sigma_y \tag{12.48}$$

In the space of principal stresses the Tresca yield condition is represented by a prismatic open-ended tube, whose intersection with the octahedral plane is a regular hexagon, see Fig. (12.11).

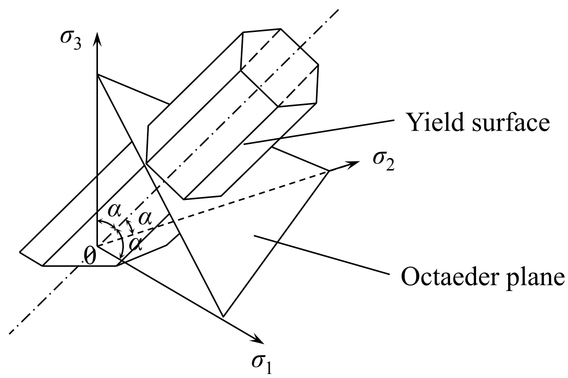

Figure 12.11: Representation of the Tresca yield condition in the space of principal stresses.

For plane stress, the intersection of the prismatic tube with the plane $\sigma_3 = 0$ forms a familiar Tresca hexagon, shown in Fig. (12.12).

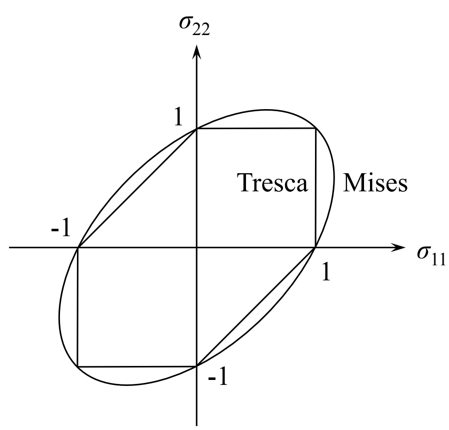

Figure 12.12: Tresca hexagon inscribed into the von Mises ellipse.

The effect of the *hydrostatic pressure* on yielding can be easily assessed by considering $\sigma_1 = \sigma_2 = \sigma_3 = p$. Then

$$\sigma_1 - \sigma_2 = 0 \tag{12.49a}$$

$$\sigma_2 - \sigma_3 = 0 \tag{12.49b}$$

$$\sigma_3 - \sigma_1 = 0 \tag{12.49c}$$

Under this stress state both von Mises yield criterion (Eq. (**??**)) and the Tresca criterion (Eq. (**??**)) predict that there will be no yielding.

12.9 Experimental Validation

The validity of the von Mises and Tresca yield criteria and their comparison has been the subject of extensive research over the past century. The easiest way to generate the complex state of stress is to perform tension/compression/torsion tests of thin-walled tubes, sometimes with added internal pressure. The results from the literature are collected in Fig. (12.13) where the experimental points represent a combination of the measured two principal stresses causing yielding. There is a fair amount of spread of the data so that there is no clear winner between the two competing theories. After all, the physics behind both approaches is similar: shear stresses (Tresca) produces shape distortion, and shape distortion (von Mises) can only be achieved through the action of shear stresses (in a rotated coordinate system). The maximum difference between the von Mises and Tresca yield curve occurs at the transverse plane strain and is equal to $(2/r_3 - 1) = 0.15$.

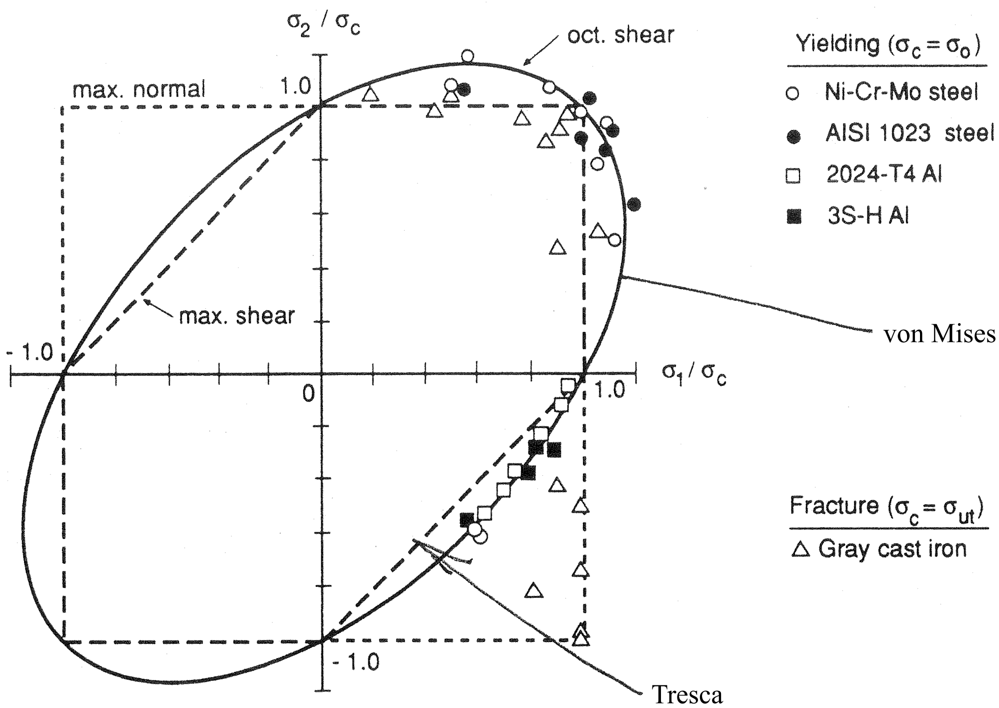

Figure 12.13: Plane stress failure loci for three criteria. These are compared with biaxial yield data for ductile steels and aluminum alloys, and also with biaxial fracture data for gray cast iron.

Quasi-brittle materials, such as cast iron behave differently in tension and compression. They can be modeled by the pressure dependent or normal stress dependent (Coulomb-Mohr) failure criterion. The comparison of theory with experimental data is shown in Fig. (12.14).

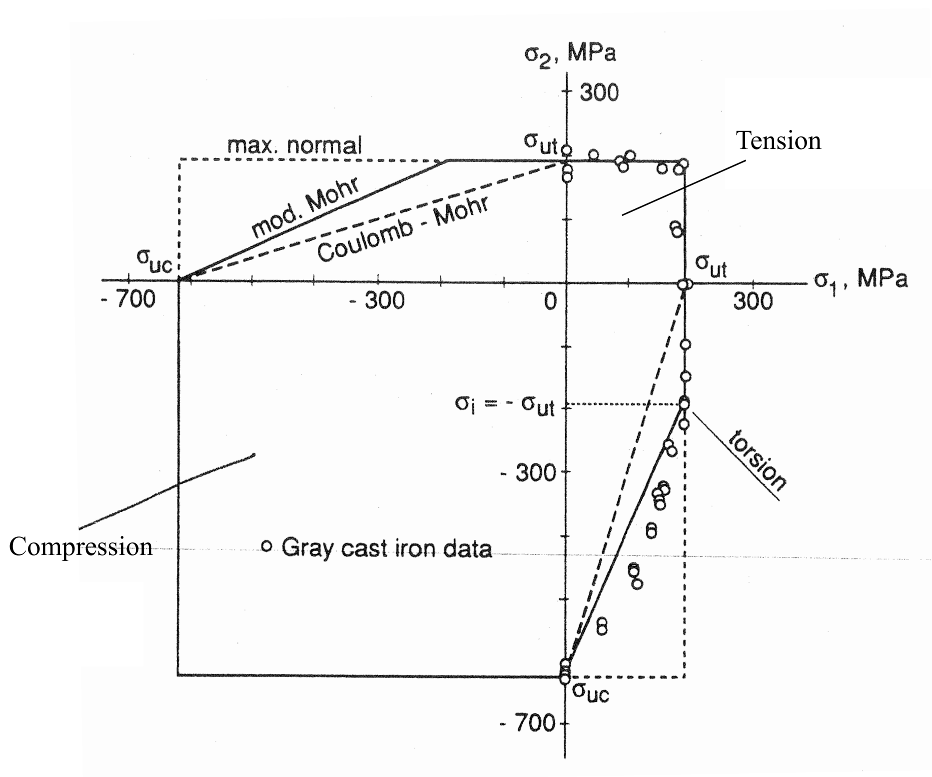

Figure 12.14: Biaxial fracture data of gray cast iron compared to various fracture criteria.

12.10 Example of the Design against First Yield

Safety of pressure vessels and piping systems is critical in design of offshore, chemical and nuclear installation. The simplest problem in this class of structures is a thick pipe loaded by an internal pressure p. The tube is assumed to be infinitely long and the internal and external radii are denoted respectively by a and b. In the cylindrical coordinate system (r, θ, z), $\sigma_{zz} = 0$ for the open-ended short tube and $\sigma_{rr} = \sigma_r$ and $\sigma_{\theta\theta} = \sigma_\theta$ are the principal radial and circumferential stresses. The material is elastic up to the point of the first yield. The objective is to determine the location where the first yield occurs and the corresponding critical pressure p_y.

The governing equation is derived by writing down three groups of equations:

Geometrical relation:

$$\epsilon_r = \frac{\mathrm{d}}{\mathrm{d}r}u, \quad \epsilon_\theta = \frac{u}{r} \tag{12.50}$$

where u is the radial component of the displacement vector, $u = u_r$. The hoop component is zero because of axial symmetry.

Equilibrium:

$$\frac{\mathrm{d}}{\mathrm{d}r}\sigma_r + \frac{\sigma_r - \sigma\theta}{r} = 0 \tag{12.51}$$

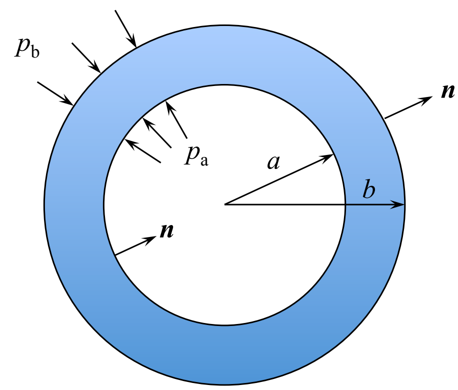

Figure 12.15: Expansion of a thick cylinder by an internal pressure.

Elasticity law:

$$\sigma_r = \frac{E}{1-\nu^2}(\epsilon_r + \nu\epsilon_\theta) \tag{12.52a}$$

$$\sigma_\theta = \frac{E}{1-\nu^2}(\epsilon_\theta + \nu\epsilon_r) \tag{12.52b}$$

There are five equations for five unknowns, σ_r, σ_θ, ϵ_r, ϵ_θ and u. Solving the above system for u, one gets

$$r^2\frac{\mathrm{d}^2}{\mathrm{d}r^2}u + r\frac{\mathrm{d}}{\mathrm{d}r}u - u = 0 \tag{12.53}$$

The solution of this equation is

$$u(r) = C_1 r + \frac{C_2}{r} \tag{12.54}$$

where C_1 and C_2 are integration constants to be determined from the boundary conditions. The stress and displacement boundary condition for this problem are

$$(T - \sigma_r) = 0 \quad \text{or} \quad \delta u = 0 \tag{12.55}$$

In the case of pressure loading, the stress boundary condition applies:

$$\text{at } r = a \quad \sigma_r = -p_a \tag{12.56a}$$

$$\text{at } r = b \quad \sigma_r = -p_b \tag{12.56b}$$

The minus sign appears because the surface traction T, which in our case is pressure loading, acts in the opposite direction to the unit normal vectors $\boldsymbol{n}$, see Fig. (12.15). In the present case of internal pressure, $\sigma_r(r = a) = -p$ and $\sigma_r(r = b) = 0$. The radial stress is calculated from Eqs. (12.28) and (12.30)

$$\sigma_r = \frac{E}{1-\nu^2}\left[(1+\nu)C_1 - (1-\nu)\frac{C_2}{r^2}\right] \tag{12.57}$$

The integration constants can be easily calculated from two boundary conditions, and the final solution for the stresses is

$$\sigma_r(r) = \frac{a^2 p}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right) \tag{12.58a}$$

$$\sigma_\theta(r) = \frac{a^2 p}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right) \tag{12.58b}$$

Eliminating the term $(b/r)^2$ between the above two equations gives the straight line profile of the stresses, shown in Fig. (12.16).

$$\sigma_r + \sigma_\theta = 2p\frac{1}{\left(\frac{b}{a}\right)^2 - 1} \tag{12.59}$$

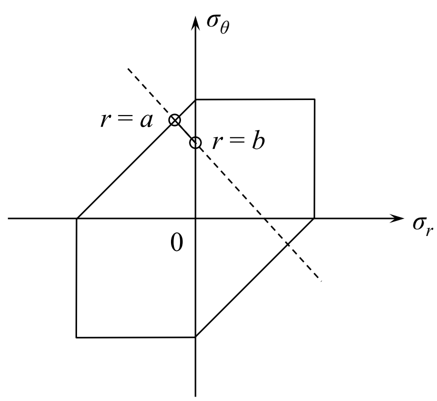

Figure 12.16: The stress profile across the thickness of the cylinder.

It is seen that the stress profile is entirely in the second quadrant and the tube reaches yield at $r = a$, for which the stresses are

$$\sigma_r = -p \tag{12.60a}$$

$$\sigma_\theta = p\frac{b^2 + a^2}{b^2 - a^2} \tag{12.60b}$$

In the case of the Tresca yield condition

$$|\sigma_\theta - \sigma_r| = \sigma_{\mathrm{y}} \tag{12.61}$$

The dimensionless yield pressure is

$$\frac{p}{\sigma_{\mathrm{y}}} = \frac{1}{2}\left[1 - \left(\frac{a}{b}\right)^2\right] \tag{12.62}$$

The von Mises yield condition predicts

$$\frac{p}{\sigma_{\mathrm{y}}} = \frac{1 - (a/b)^2}{\sqrt{3 + (a/b)^4}} \tag{12.63}$$

For example, if $\frac{a}{b} = \frac{1}{2}$, the first yield pressure according to the von Mises yield condition is $\frac{p_{\text{y}}}{\sigma_{\text{y}}} = \frac{3}{7}$ while the Tresca yield criterion predicts $\frac{p_{\text{y}}}{\sigma_{\text{y}}} = \frac{3}{8}$. The difference between the above two cases is 14%.