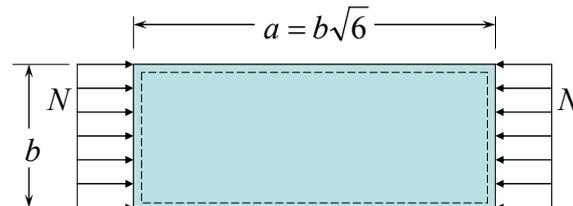


Lecture 11

Buckling of Plates and Sections

Problem 11-1:

A simply-supported rectangular plate is subjected to a uniaxial compressive load N , as shown in the sketch below.



- Calculate and compare buckling coefficients corresponding to the four first buckling modes as a function of a/b .
- Discuss the result for the specific value: $a/b = \sqrt{6}$

Problem 11-1 Solution:

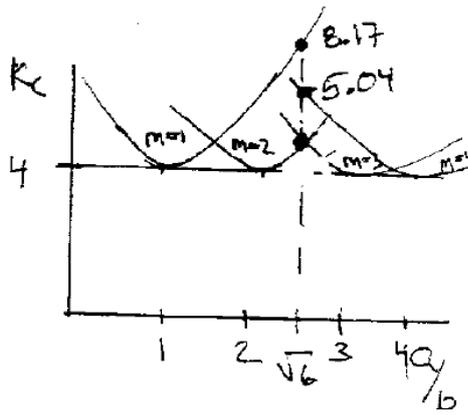
- Calculate and plot the buckling coefficients corresponding to the four first buckling modes as a function of a/b

$$K_c = \left(m \left(\frac{b}{a} \right) + \frac{1}{m} \left(\frac{a}{b} \right) \right)^2$$

- For $a/b = \sqrt{6}$

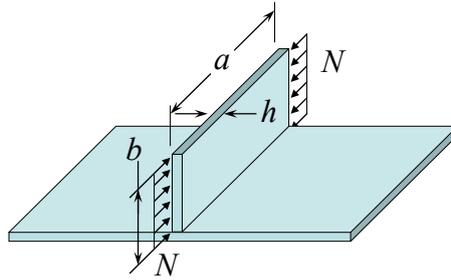
$$K_c = \left(\frac{m}{\sqrt{6}} + \frac{\sqrt{6}}{m} \right)^2$$

m	1	2	3	4
K_c	8.17	4.17	4.17	5.04



Problem 11-2:

A $h=10\text{mm}$ thick elastic flat bar stiffener is welded to a plate at the bottom.



- State the boundary conditions around all four edges.
- Calculate the total buckling load $P_c = N_c b$ of the stiffener, assuming that $a=1000\text{mm}$ and $b=200\text{mm}$. (Hint: use the graphical solution for k_c in the notes)
- Determine the length of the buckling half-wave.
- How much will the buckling load change if the boundary conditions at the loaded edges are changed from simply supported to clamped support?

Problem 11-2 Solution:

- Boundary condition for the unloaded edges are:

At the bottom, clamped boundary condition

$$w(x,0) = 0$$
$$w'(x,0) = 0$$

At the top, free boundary condition

$$w''(x,b) = 0$$
$$w'''(x,b) = 0$$

In addition, if loaded edges are:

Clamped

$$w(0,y) = w(a,y) = 0$$
$$w'(0,y) = w'(a,y) = 0$$

Simply supported

$$w(0, y) = w(a, y) = 0$$
$$w''(0, y) = w''(a, y) = 0$$

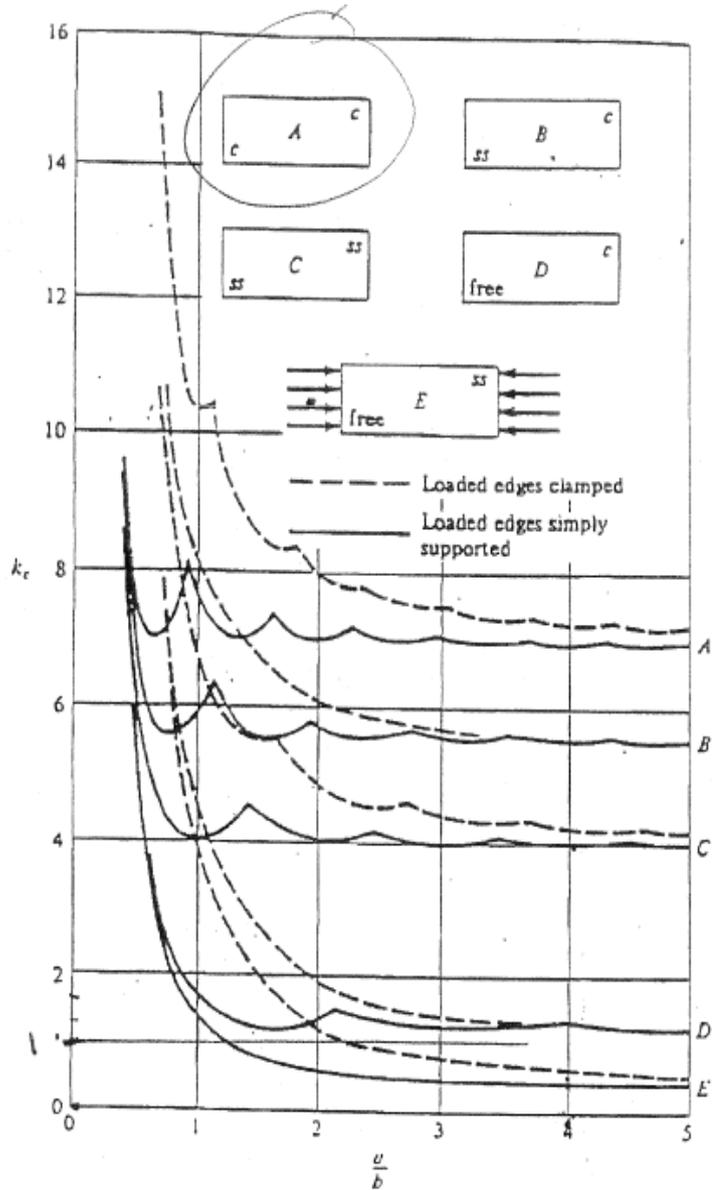
At the bottom, clamped boundary condition

$$w(x, 0) = 0$$
$$w'(x, 0) = 0$$

b) Calculate the total buckling load

$$P_c = N_c b$$
$$a = 1000mm$$
$$b = 200mm$$

Using the buckling coefficient chart on page 12



We are considering case D, $a/b = 5$. Notice that the cases for the loaded edges being clamped and simply supported are almost overlapping in buckling coefficients

$$K_c = 1.3$$

$$P_c = K_c \frac{\pi^2 D}{b} = 1.3 \frac{\pi^2 E \times (10 \times 10^{-3})^3}{200 \times 10^{-3} \times 12(1 - \nu^2)}$$

$$P_c = 5.35 \times 10^{-6} \frac{E}{(1 - \nu^2)}$$

- c) Determine the length of the buckling half-wave.

If edges are **simply-supported**:

From the plot in part (b), we can see that there are 3 waves in the m-direction.

Wavelength is equal to the length of one side divided by the number of waves in that direction

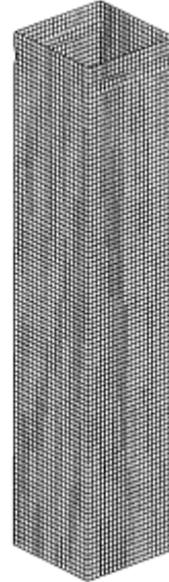
$$\lambda = \frac{a}{m} = \frac{1000}{3} = 333 \text{ mm}$$

If edges are **clamped**, it is difficult to read from the chart about how many waves (m) exists.

- d) As noted in part (b), the buckling coefficient for simply supported edges vs. clamped loaded edges is approximately the same. Thus the effect on the buckling load is insignificant if the boundary condition change from simply supported to clamped support.

Problem 11-3: A relatively short rectangular prismatic box column $b_1 \times b_2 \times h$ is subjected to a uniform axial compression. Take $b_1=40mm$, $b_2=60mm$, $h=1mm$. Then beam is made of an aluminum alloy with yield stress of 300MPa

- Calculate the total buckling load of the column.
- Consider a square column of the same cross-sectional perimeter. Which column is stronger: square or rectangular?
- Plot the load – shorting relation for that column consisting pre-buckling phase and post-buckling phase.



Bonus: Consider a rectangular prismatic box column of the same cross-sectional perimeter. Which column is stronger: square or rectangular?

Problem 11-3 Solution:

- Calculate the total buckling load

Using the derivation in lecture note page 17

$$P_c = 2K_c^{(2)} \frac{\pi^2 E}{12(1-\nu^2)} \frac{h_2^3}{b_2} \left(\frac{b_1}{h_2} \frac{h_1}{b_2} + 1 \right)$$

In our case,

$$h_1 = h_2 = h$$

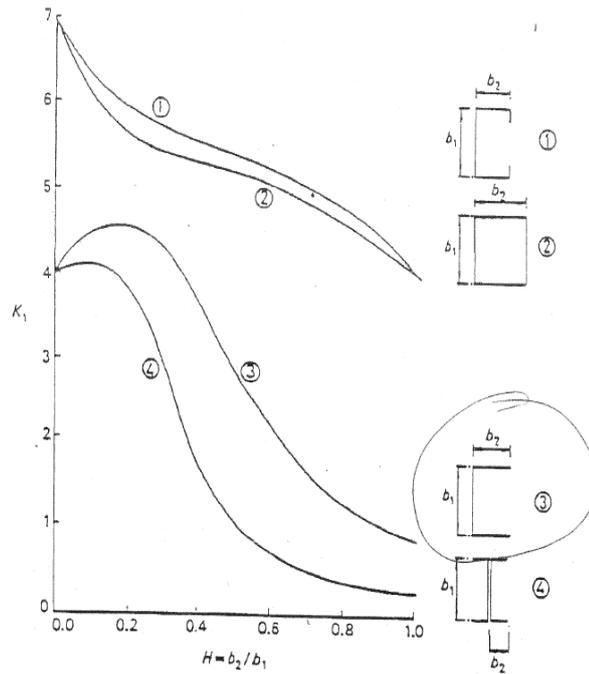
$$b_1 = b_2 = b$$

$$\Rightarrow P_c = 4K_c^{(2)} \frac{\pi^2 E}{12(1-\nu^2)} \frac{h^3}{b}$$

Use the plot in lecture note (see next page), when $b_1 = b_2$, $K_c = 4.1$

$$P_c = 4 \times 4.1 \times \frac{\pi^2 E}{12(1-\nu^2)} \frac{(1 \times 10^{-3})^3}{(40 \times 10^{-3})}$$

$$\boxed{P_c = 3.7 \times 10^{-7} \frac{E}{(1-\nu^2)}} [N]$$



For Aluminum alloy

$$E = 69 \text{ GPa}, \nu = 0.3$$

$$P_c = 3.7 \times 10^{-7} \frac{E}{(1-\nu^2)} = 3.7 \times 10^{-7} \frac{69 \times 10^9}{(1-0.3^2)} = 28 \text{ kN}$$

Buckling stress

$$\sigma_c = \frac{P_c}{4bh} = 175 \text{ MPa} < \sigma_y = 300 \text{ MPa}$$

This is elastic buckling.

b) Crippling load of the column

Use derivation in page 2 of lecture notes 16

$$\frac{\sigma_{ult}}{\sigma_y} = \frac{1.9}{\beta}$$

where

$$\frac{1}{\beta} = \sqrt{\frac{E}{\sigma_y}} \frac{t}{b}$$

In our case

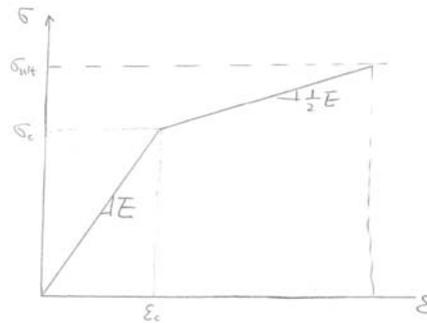
$$E = 69GPa, \sigma_y = 300MPa, \frac{t}{b} = \frac{1}{40} \Rightarrow \frac{1}{\beta} = \sqrt{\frac{69 \times 10^9}{300 \times 10^6}} \times \frac{1}{40} = 0.379$$

$$\sigma_{ult} = 1.9 \times 0.379 \times \sigma_y = 216.1MPa$$

$$P_{ult} = \sigma_{ult} \cdot 4bh = 36kN$$

- c) We know that in post-buckling phase, Young's modulus reduces to half of the pre-buckling phase.

$$E_{post} = \frac{1}{2} E_{pre}$$



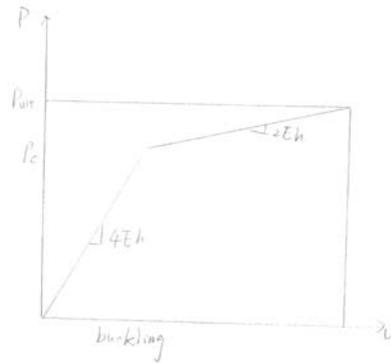
In a half-wave length b , for **pre-buckling** phase

$$u_{pre} = \varepsilon_{pre} b = \frac{\sigma}{E_{pre}} b = \frac{P/4bh}{E_{pre}} b = \frac{P}{4E_{pre}h}$$

where $E_{pre} = 69GPa$, $h = 1mm$

for **post-buckling** phase

$$u_{post} = \varepsilon_{post} b = \frac{\sigma}{E_{post}} b = \frac{P/4bh}{\frac{1}{2} E_{pre}} b = \frac{P}{2E_{pre}h}$$



Bonus:

For a rectangular prismatic box, assume $b_1 = 48\text{mm}$, $b_2 = 32\text{mm}$, so that

$b_2/b_1 = 0.67$, using the plot $K_c = 5$

$$\begin{aligned}
 h_1 &= h_2 = h \\
 \Rightarrow P_c &= 2K_c^{(2)} \frac{\pi^2 E}{12(1-\nu^2)} \frac{h_2^3}{b_2} \left(\frac{b_1}{h_2} \frac{h_1}{b_2} + 1 \right) \\
 P_c &= 2K_c^{(2)} \frac{\pi^2 E}{12(1-\nu^2)} \frac{h^3}{b_2} \left(\frac{b_1}{b_2} + 1 \right) \\
 P_c &= 2 \times 5 \times \frac{\pi^2 E}{12(1-\nu^2)} \frac{(1 \times 10^{-3})^3}{48 \times 10^{-3}} \left(\frac{32 \times 10^{-3}}{48 \times 10^{-3}} + 1 \right) \\
 \boxed{P_c} &= 2.9 \times 10^{-7} \frac{E}{(1-\nu^2)}
 \end{aligned}$$

Compare with square prismatic box column

$$\boxed{P_c} = 3.7 \times 10^{-7} \frac{E}{(1-\nu^2)}$$

Square prismatic box column is stronger

Problem 11-4: Buckling of a box section

A relatively slender, thin-walled square box column is subjected to axial compression. The column is simply supported at its ends.

What is the combination of the geometrical parameters (length L , wall thickness h and width of the flange b) so that the critical Euler buckling load will be equal to the critical local plate buckling load. Explain all the assumptions that you made in your derivation

Problem 11-4 Solution:

The critical Euler buckling load is

$$P_c^{Euler} = \frac{\pi EI}{l^2}$$

where $I = \frac{2}{3}b^3h$

For a slender column, $a \gg b$, so $k_c = 4$, critical plate buckling load is

$$P_c^{plate} = 4k_c \frac{\pi^2 D}{b}$$

where $D = \frac{Eh^3}{12(1-\nu^2)}$

Equating two critical loads $P_c^{plate} = P_c^{Euler}$, this gives us

$$\frac{2}{3} \frac{\pi Eb^3h}{l^2} = \frac{4k_c \pi^2 Eh^3}{12(1-\nu^2)b}$$

Rearrange the above equation, we have

$$\left(\frac{b^2}{hl}\right)^2 = \frac{2}{1-\nu^2} = \frac{2}{1-0.3^2} = 2.2$$
$$\Rightarrow \frac{b^2}{hl} = 1.5$$

Finally

$$\boxed{b^2 = 1.5hl}$$

Problem 11-5: Thin walled prismatic box

Consider a thin-walled prismatic box structure of length $a = 2b$, where b is the width of the cross-section. The box is put in the universal testing machine and is subjected to a compressive load.

- Calculate the ultimate compressive load of the box structure.
- Would the ultimate load change if the member were twice as long?
- What will the most weight efficient way to increase the ultimate load by a factor of 2?

Would your solution depend on the b/t ratio or on the magnitude of the effective width?

Problem 11-5 Solution:

- a) Von Karman theory of effective width

$$b_{eff} = 1.9 \sqrt{\frac{E}{\sigma_y}} t$$

If $b \geq b_{eff}$, we have

$$\frac{\sigma_{eff}}{\sigma_y} = \frac{b_{eff}}{b}$$
$$\sigma_{eff} = \frac{b_{eff}}{b} \sigma_y = 1.9 \sqrt{E \sigma_y} \frac{t}{b}$$

The ultimate compressive load is

$$P_{ult} = \sigma_{ult} \cdot 4bt = 7.6t^2 \sqrt{E \sigma_y}$$

or

$$P_{ult} = \sigma_{ult} \cdot 4bt = \frac{b_{eff}}{b} \sigma_y \cdot 4bt$$

$$\boxed{P_{ult} = 4bt \sigma_y \frac{b_{eff}}{b}}$$

If $b < b_{eff}$, we have

$$\boxed{P_{ult} = 4\sigma_y bt}$$

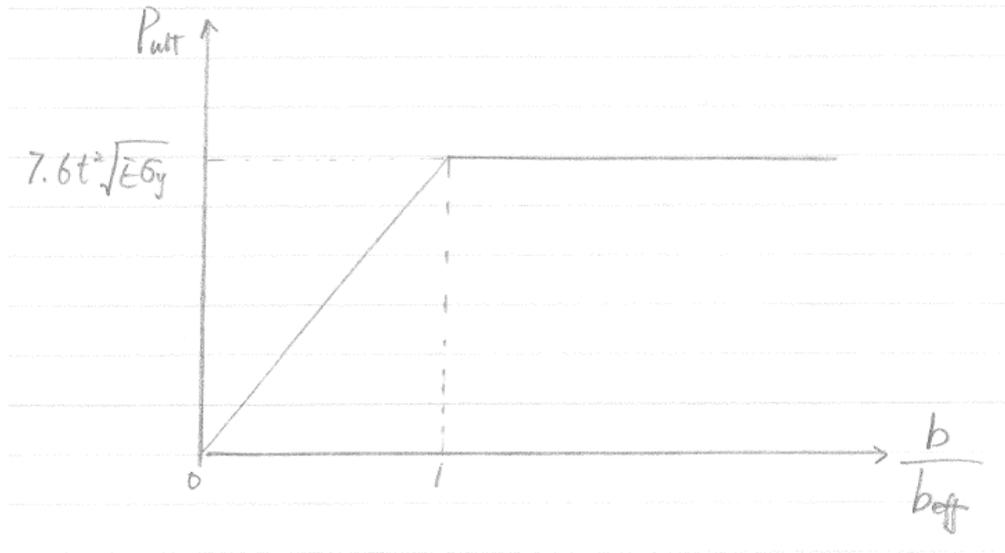
To sum up

$$P_{ult} = \begin{cases} 4bt \sigma_y \frac{b_{eff}}{b}, & b \geq b_{eff} \\ 4\sigma_y bt, & b < b_{eff} \end{cases}$$

Note: when $b = b_{eff}$, the expression for $b \geq b_{eff}$ becomes

$$P_{ult}|_{b=b_{eff}} = 4bt\sigma_y \frac{b_{eff}}{b} \Big|_{b=b_{eff}} = 4\sigma_y bt$$

So P_{ult} is continuous at $b = b_{eff}$



b) The ultimate load is independent of length a . So the ultimate load P_{ult} won't change if member is twice as long.

c) If $b < b_{eff}$, the most efficient way is to increase length b until b reaches b_{eff}

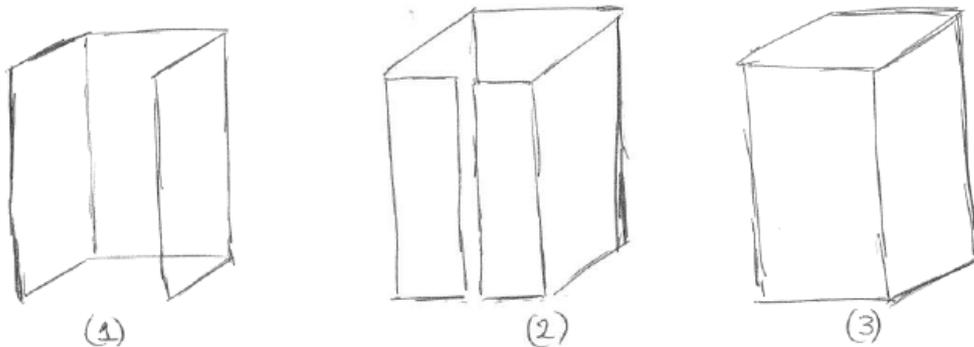
If $b > b_{eff}$, the most efficient way is to increase thickness t

Recall $P_{ult} = 7.6t^2\sqrt{E\sigma_y}$, increase t to $\sqrt{2}t$ will increase P_{ult} by a factor of 2

Problem 11-6: Buckling of a channel

Consider a plain channel (as supposed to lipped channel) with sides of equal width ($b_1 = b_2$).

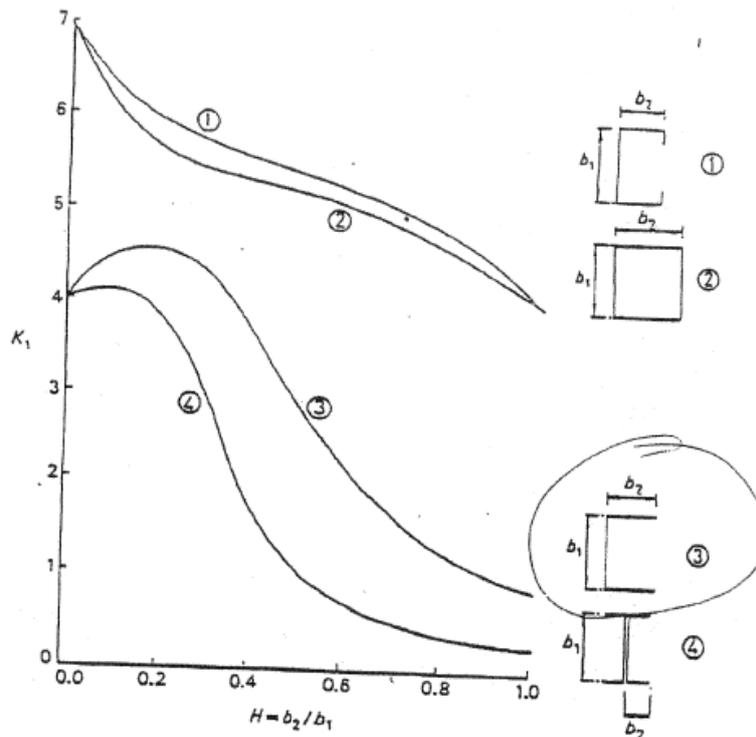
Both loaded edges are simply supported. The length of the column (a) is equal to $5b$. Consider the three cases shown in the figure below, all sections are thin-walled.



- a) Calculate the total buckling load of open section; figure (1).
- b) Calculate the buckling load if two lips are added figure (2).
- c) Finally, calculate the buckling load of the section where two lips are welded, meaning that it becomes a square prismatic section; figure (3).
- d) Discuss the results found in (a), (b) and (c).

Problem 11-6 Solution:

We will use the above figure in the following deductions



Assumption

Upon uniform compression, the stresses on the adjacent plates (flanges) are the same

$$\sigma_c^{(1)} = \sigma_c^{(2)}$$

a) For a plain channel, use case ③ in the figure, we also know in this case

$$k_c^{(1)} = \frac{2}{\sqrt{\beta_o}} + \frac{2 + 4.8H}{\beta_o^2}$$

$$\text{where } \beta_o = \sqrt{(1 + 15H^3)}, \quad H = \frac{b_2}{b_1}$$

in our case $b_1 = b_2$

$$H = \frac{b_2}{b_1} = 1, \quad \beta_o = 4$$

$$k_c^{(1)} = 0.925$$

Critical stresses on the adjacent plates

$$\sigma_c^{(1)} = k_c^{(1)} \frac{\pi^2 D_1}{b_1^2 h_1}$$

$$\sigma_c^{(1)} = \sigma_c^{(2)} = k_c^{(2)} \frac{\pi^2 D_2}{b_2^2 h_2}$$

Total buckling load is the sum of critical loads on all plates

$$\begin{aligned} P_c &= P_1 + 2P_2 \\ &= \sigma_c^{(1)} b_1 h_1 + 2\sigma_c^{(2)} b_2 h_2 \end{aligned}$$

We know $\sigma_c^{(1)} = \sigma_c^{(2)}$, $b_1 = b_2 = b$, $h_1 = h_2 = h$

$$\begin{aligned} P_c &= \sigma_c^{(1)} bh + 2\sigma_c^{(1)} bh \\ &= 3\sigma_c^{(1)} bh \\ &= 3k_c^{(1)} \frac{\pi^2 D}{b} \\ &= 3 \times 0.925 \times \frac{\pi^2 E h^3}{12b(1-\nu^2)} \end{aligned}$$

$$\boxed{P_c = 0.23 \frac{\pi^2 E h^3}{b(1-\nu^2)}}$$

b) For a lipped channel, use case ① in the figure, we also know in this case

$$k_c^{(1)} = 7 - \frac{1.8H}{0.15 + H} - 1.43H^3 \approx 4$$

Similar to part a), total buckling load is the sum of critical loads on all plates

$$\begin{aligned} P_c &= P_1 + 2P_2 \\ &= 3k_c^{(1)} \frac{\pi^2 D}{b} \\ &= 3 \times 4 \times \frac{\pi^2 E h^3}{12b(1-\nu^2)} \end{aligned}$$

$$\boxed{P_c = \frac{\pi^2 E h^3}{b(1-\nu^2)}}$$

c) For box section

$$k_c^{(1)} = 4$$

$$P_c = 4k_c^{(1)} \frac{\pi^2 D}{b}$$

$$\boxed{P_c = 1.3 \frac{\pi^2 E h^3}{b(1-\nu^2)}}$$

d) Let's calculate load-bearing per unit volume in a unit perimeter length for each case

$$\text{Case a): } \frac{P_c}{3bh} = \frac{0.23 \frac{\pi^2 E h^3}{b(1-\nu^2)}}{3bh} = 0.077 \frac{\pi^2 E h^2}{b^2(1-\nu^2)}$$

$$\text{Case b): } \frac{P_c}{4bh} = \frac{\frac{\pi^2 E h^3}{b(1-\nu^2)}}{4bh} = 0.25 \frac{\pi^2 E h^2}{b^2(1-\nu^2)}$$

$$\text{Case c): } \frac{P_c}{4bh} = \frac{1.3 \frac{\pi^2 E h^3}{b(1-\nu^2)}}{4bh} = 0.33 \frac{\pi^2 E h^2}{b^2(1-\nu^2)}$$

The box section is the stronger while the plain channel is the weakest.

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