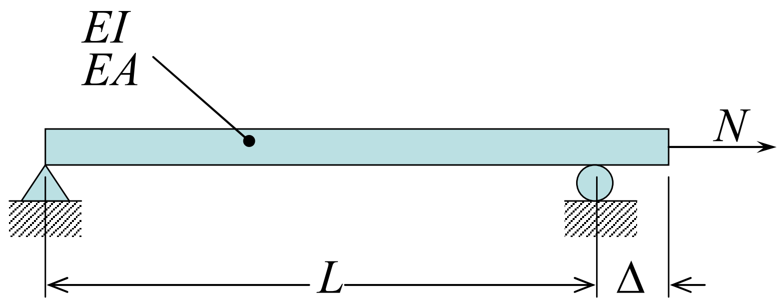
Lecture 6
Moderately Large Deflection Theory of Beams

Problem 6-1:

Part A: The department of Highways and Public Works of the state of California is in the process of improving the design of bridge overpasses to meet earthquake safety criteria. As a highly paid consultant to the project, you were asked to evaluate its soundness. You rush back to your lecture notes, and you model the overpass as a simply supported beam of span L with an overhang $\Delta=0.01L$. Assume that the distributed load is a sinusoidal function.



a) Calculate the maximum allowable midspan deflection $(w_o)_{critical}$ under which the beam will slide off its support.

Part B: Assume that the above design with an external axial force $N=0$ and $\Delta=0.01L$ has a safety factor of one. The design of earthquake resistant structures requires a safety factor of five, meaning that $(w_o)_{critical}$ must be increased by a factor of five without the bridge collapsing. Two possible design modifications were proposed. In the first one, the overhang is simply increased to Δ_{new}. In the second design, a tensile force N is applied to the bridge to increase its transverse stiffness and thus reduce the central deflection and the resulting motion of the support.

b) For the first proposed modification, what length Δ_{new} of the overhang will meet the requirement of a safety factor of five? Give your result in terms of the original Δ and other parameters if needed.

c) For the second design, what is the magnitude of the dimensionless tensile force N/EA that will give a safety factor equal to five?

d) Which design is better? Can you think of a third alternative design solution?

Problem 6-1 Solution:

Recall:

$$\frac{NL}{EA} = -\Delta + \int_0^L \frac{1}{2}\left(\frac{dw}{dx}\right)^2 dx$$

(a) Calculate the max deflection $(w_o)_{artical}$ at $L/2$ under which the beam will slide off its support

Assume $N = 0$

$$\Delta = \int_0^L \frac{1}{2}\left(\frac{dw}{dx}\right)^2 dx \qquad (1)$$

Since the applied load is a sin function, we can assume the deflected shape will also be a sine function

$$w(x) = w_o \sin\left(\frac{\pi x}{L}\right)$$

$$w'(x) = w_o\left(\frac{\pi}{L}\right)\cos\left(\frac{\pi x}{L}\right)$$

$$\left(w'(x)\right)^2 = w_o^{\,2}\left(\frac{\pi}{L}\right)^2 \cos^2\left(\frac{\pi x}{L}\right)$$

Substitute the above equation into equation (1), we have

$$\Delta = \int_0^L \frac{1}{2} w_o^{\,2}\left(\frac{\pi}{L}\right)^2 \cos^2\left(\frac{\pi x}{L}\right) dx$$

$$= \frac{1}{2} w_o^{\,2}\left(\frac{\pi}{L}\right)^2 \left[\frac{x}{2} + \frac{\sin(\pi x/L)}{4\pi/L}\right]\Bigg|_0^L$$

$$= \frac{1}{2} w_o^{\,2}\left(\frac{\pi}{L}\right)^2 \left[\frac{L}{2} + 0\right]$$

$$= \frac{1}{L}\left(\frac{w_o \pi}{2}\right)^2$$

$$\boxed{\Delta = \frac{1}{L}\left(\frac{w_o \pi}{2}\right)^2}$$

Thus, the maximum allowable midspan deflection is

$$w_o^{\,2} = \frac{4L\Delta}{\pi^2}$$

$$\boxed{w_o = \frac{2}{\pi}\sqrt{L\Delta}}$$

We're given $\Delta = 0.01L$

$$w_o = \frac{2}{\pi}\sqrt{\frac{L^2}{100}} = \frac{L}{5\pi}$$

$$\boxed{w_o = \frac{L}{5\pi}}$$

$$w_o = \frac{L}{5\pi}$$

(b) Case 1： increase Δ to meet safety factor of 5

$$5w_o = \frac{2}{\pi}\sqrt{L\Delta_{new}}$$

$$\Delta_{new} = \frac{1}{L}\left(\frac{5w_o\pi}{2}\right)^2$$

$$\Delta_{new} = \frac{1}{L}\left(\frac{5w_o\pi}{2}\right)^2 = 25\left[\frac{1}{L}\left(\frac{w_o\pi}{2}\right)^2\right]$$

Recall: $\Delta = \frac{1}{L}\left(\frac{w_o\pi}{2}\right)^2$

$$\boxed{\Delta_{new} = 25\Delta}$$

(c) Case 2: apply a tensile force N, so now $N \neq 0$

$$\frac{NL}{EA} = -\Delta + \int_0^L \frac{1}{2}\left(\frac{dw}{dx}\right)^2 dx$$

$$\frac{NL}{EA} = -\Delta + \frac{1}{L}\left(\frac{w_o\pi}{2}\right)^2$$

We want to calculate N that will give us the equivalent effect of applying a safety factor of 5, in which case

$$\frac{1}{L}\left(\frac{w_o\pi}{2}\right)^2 = 25\Delta$$

$$\frac{NL}{EA} = -\Delta + 25\Delta = 24\Delta$$

In the case of $\Delta = L/100$

$$\boxed{\frac{N}{EA} = \frac{24}{L}\frac{L}{100} = 0.24}$$

(d) Which design is better?

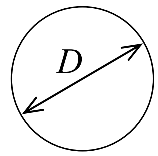
It is difficult to say which design is better. Each design has its advantage and disadvantages. For case 1, we will have a long overhang which may not be aesthetically pleasing. For case 2, it may be difficult to apply constantly a tensile force.

Other options include a stiffer simply supported beam, or add cables to suspend the bridge.

Problem 6-2:

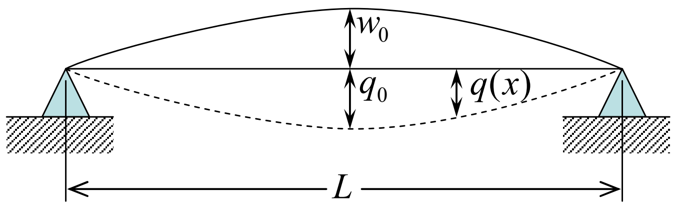
A long span aerial tramway steel cable of length L=1km is loaded by a hurricane wind with intensity q(x) sinusoidally distributed between the end stations. The cable deflects by wo=5m.

$$E = 2.1 \times 10^5 MPa$$
$$\sigma_y = 300MPa$$
$$D = 60mm$$



Cross-section of cable

$$q(x) = q_0 \sin\left(\frac{\pi x}{L}\right)$$



a) Calculate the resulting load intensity qo

b) Calculate the tension in the cable N.

c) Calculate the tensile stress.

d) Compare (c) with the yield stress, and determine the safety factor.

Problem 6-2 Solution:

Using the equation of equilibrium

$$EIw^{IV} - Nw'' = q$$

However, a cable has no bending stiffness, so our equation becomes:

$$-Nw'' = q$$

$$w'' = -\frac{q}{N} = -\frac{1}{N} q_o \sin\left(\frac{\pi x}{L}\right)$$

Integrate twice

$$w' = \frac{q_o}{N}\left(\frac{L}{\pi}\right)\cos\left(\frac{\pi x}{L}\right) + C_1$$

$$w = \frac{q_o}{N}\left(\frac{L}{\pi}\right)^2 \sin\left(\frac{\pi x}{L}\right) + C_1 x + C_2$$

Plug in the boundary conditions to solve for the constants

$$\begin{cases} w(0)=0 & w(0)=C_2=0 \\ w(L)=0 & w(L)=0+C_1L+C_2=0 \Rightarrow C_1=0 \end{cases}$$

$$\boxed{w(x)=\frac{q_o}{N}\left(\frac{L}{\pi}\right)^2 \sin\left(\frac{\pi x}{L}\right)}$$

a) Calculate the load intensity

The cable deflects by $w_o = 5m$ at the middle point $x = L/2$

$$w\left(\frac{L}{2}\right)=w_o=\frac{q_o}{N}\left(\frac{L}{\pi}\right)^2 \sin\left(\frac{\pi}{L}\frac{L}{2}\right)=\frac{q_o}{N}\left(\frac{L}{\pi}\right)^2$$

$$\boxed{q_o = N w_o\left(\frac{\pi}{L}\right)^2}$$

b) Calculate the tension on the cable(N)

Using $\frac{N}{EA}=\frac{1}{L}\int_0^L \frac{1}{2}\left(\frac{dw}{dx}\right)^2 dx$ and $w(x)=\frac{q_o}{N}\left(\frac{L}{\pi}\right)^2 \sin\left(\frac{\pi x}{L}\right)$

$$w'(x)=\frac{q_o}{N}\left(\frac{L}{\pi}\right)^2 \frac{\pi}{L}\cos\left(\frac{\pi x}{L}\right)$$

$$\left(w'(x)\right)^2=\left(\frac{q_o L}{N\pi}\right)^2 \cos^2\left(\frac{\pi x}{L}\right)$$

$$\begin{aligned} N &= \frac{EA}{2L}\int_0^L \left(\frac{q_o L}{N\pi}\right)^2 \cos^2\left(\frac{\pi x}{L}\right) dx \\ &= \frac{EAL}{2}\left(\frac{q_o}{N\pi}\right)^2 \left[\frac{x}{2}+\frac{\sin(2x\pi/L)}{4\pi/L}\right]_0^L \\ &= \frac{EAL}{2}\left(\frac{q_o}{N\pi}\right)^2 \left(\frac{L}{2}+0\right) \\ N^3 &= \frac{EA}{4}\left(\frac{q_o L}{\pi}\right)^2 \end{aligned}$$

$$\boxed{N = \left(EA\left(\frac{q_o L}{2\pi} \right)^2 \right)^{1/3}}$$

Given $w_o = 5m$, then $q_o = Nw_o\left(\frac{\pi}{L}\right)^2 = 5N\left(\frac{\pi}{L}\right)^2$

$$N^3 = \frac{EA}{4}\left(\frac{q_o L}{\pi}\right)^2 = \frac{EA}{4}\left(\frac{5N\left(\frac{\pi}{L}\right)^2 L}{\pi}\right)^2 = \frac{25EA}{4}\left(\frac{L}{\pi}\right)^2 N^2\left(\frac{\pi}{L}\right)^4$$

$$N = \frac{25EA}{4}\left(\frac{\pi}{L}\right)^2 = \frac{25}{4}\left(2.1\times10^5\right)\times10^6\left[\frac{\pi}{4}\left(60\times10^{-3}\right)^2\right]\left(\frac{\pi}{1\times10^3}\right)^2$$

$$\boxed{N = 36626\text{N}}$$

c) Calculate the tension stress on the cable

$$\sigma = \frac{N}{A} = \frac{36626}{\frac{\pi}{4}\left(60\times10^{-3}\right)^2} = 12.95\text{MPa}$$

$$\boxed{\sigma = 12.95\text{MPa}}$$

d) Compare with the yield stress and determine the safety factor

$$\text{safety factor} = \frac{\text{yield stress}}{\text{working stress}} = \frac{300}{12.95}$$

$$\boxed{\text{safety factor} = 23.17}$$

Problem 6-3:

Plot the dimensionless deflections (w_o/L) versus the dimensionless line load for both bending and membrane (cable) solutions over a slender beam. At what dimensionless deflections will the bending and membrane solutions be equal, assuming a length to thickness ratio equal to 10?

Problem 6-3 Solution:

Recall bending and membrane solutions:

Pure Bending

$$w(x) = \frac{P_o}{EI\left(\frac{\pi}{L}\right)^4} \sin\frac{\pi x}{L}$$

at $x = \frac{L}{2}$

$$w\left(\frac{L}{2}\right) = w_o = \frac{P_o}{EI\left(\frac{\pi}{L}\right)^4}$$

$$w_o = \frac{P_o}{EI}\left(\frac{L}{\pi}\right)^4$$

where $I = \frac{h^4}{12}$

$$w_o = \frac{P_o}{E\frac{h^4}{12}}\left(\frac{L}{\pi}\right)^4 = L\left(\frac{P_o L}{Eh^2}\right)\left(\frac{12L^2}{\pi^4 h^2}\right)$$

$$\boxed{\frac{w_o}{L} = \left(\frac{P_o L}{Eh^2}\right)\left(\frac{12L^2}{\pi^4 h^2}\right)}$$

Membrane

$$w(x) = \frac{P_o}{N\left(\frac{\pi}{L}\right)^2} \sin\frac{\pi x}{L}$$

at $x = \frac{L}{2}$

$$w\left(\frac{L}{2}\right) = w_o = \frac{P_o}{N\left(\frac{\pi}{L}\right)^2}$$

$$w_o = \frac{P_o}{N}\left(\frac{L}{\pi}\right)^2$$

where $N = \left(EA\left(\frac{q_o L}{2\pi}\right)^2\right)^{1/3}$ (Problem 6-2)

so $N = \left(Eh^2\left(\frac{q_o L}{2\pi}\right)^2\right)^{1/3}$

$$w_o = \frac{P_o}{N}\left(\frac{L}{\pi}\right)^2 = \frac{P_o}{\left(Eh^2\left(\frac{q_o L}{2\pi}\right)^2\right)^{1/3}}\left(\frac{L}{\pi}\right)^2$$

$$= L\left(\frac{P_o L}{\pi^2}\right)\left(\frac{1}{E}\left(\frac{2\pi}{q_o L h}\right)^2\right)^{1/3}$$

where $q_o = P_o$

rearrange the above expression

$$\boxed{\frac{w_o}{L} = \left(\frac{P_o L}{Eh^2}\right)^{1/3}\left(\frac{4}{\pi^4}\right)^{1/3}}$$

Let's call $\frac{w_o}{L} = y, \frac{P_o L}{Eh^2} = x$

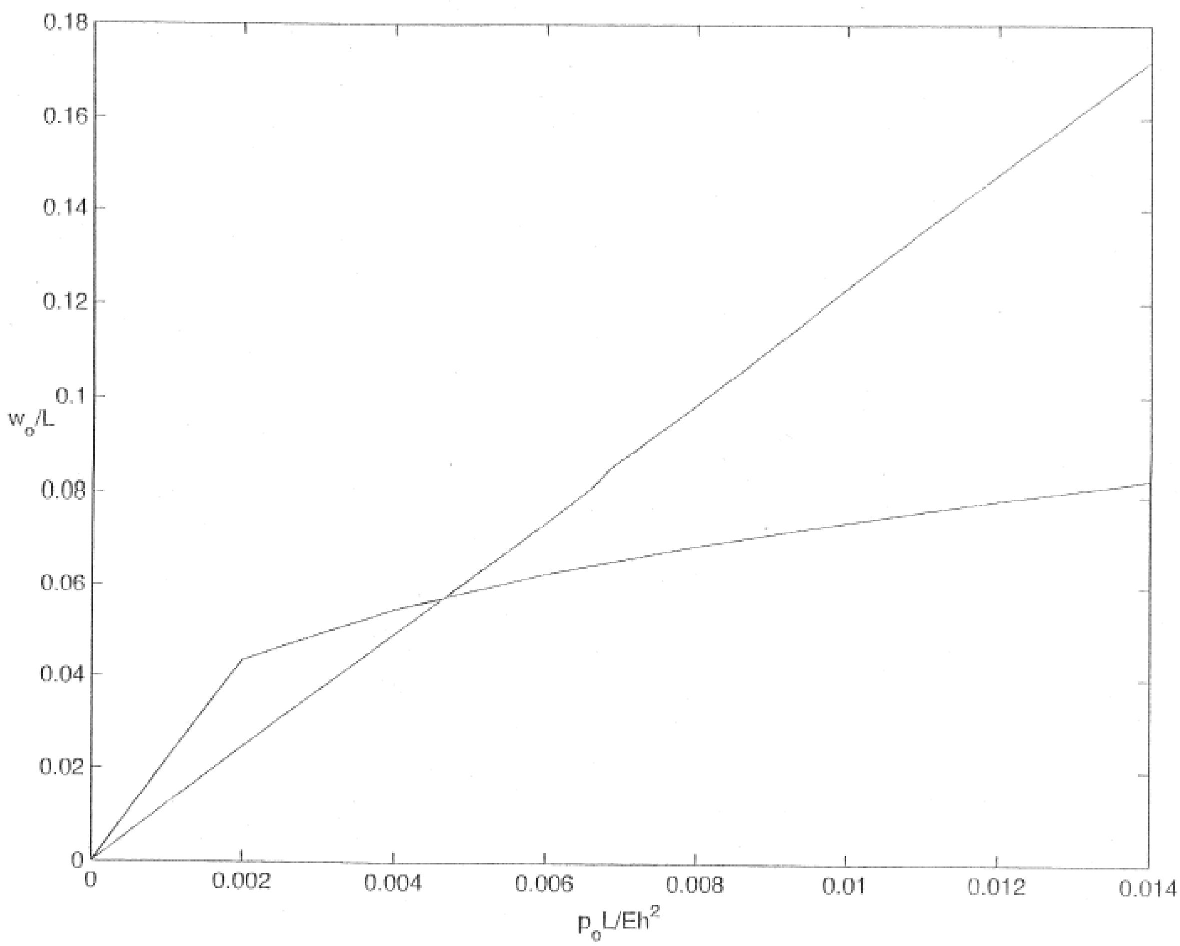
We want to plot

Bending $y = \left(\frac{12L^2}{\pi^4 h^2} \right) x$

Membrane $y = \left(\frac{4}{\pi^4} \right)^{1/3} x^{1/3}$

Use a length to thickness ratio equal to 10: $\frac{L}{h} = 10$

Bending $y = 12.32x$
Membrane $y = 0.345x^{1/3}$



At what dimensionless deflections will the bending and membrane solutions be equal?

$$\left.\frac{w_o}{L}\right|_{bending} = \left.\frac{w_o}{L}\right|_{membrane}$$

$$12.32x = 0.345x^{1/3}$$

$$x = 0.005$$

$$\frac{w_o}{L} = 12.32 \times 0.005 = 0.0577$$

So at $\frac{P_o L}{Eh^2} = 0.005$, the bending and membrane solutions will be equal, where the dimensionless deflections $\frac{w_o}{L} = 0.0577$