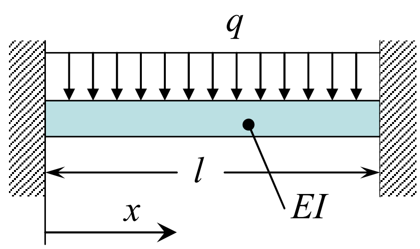
Lecture 5

Solution Method for Beam Deflection

Problem 5-1: Consider the clamped-clamped elastic beam loaded by a uniformly distributed line load q.



a) Formulate the boundary conditions.

b) Find the deflected shape of the beam using the direct integration method.

c) Find the maximum deflection magnitude and location.

d) Determine the location and magnitude of the maximum stress in the beam.

Problem 5-1 Solution:

(a) Boundary conditions

$$w(0) = w(l) = 0$$
$$w'(0) = w'(l) = 0$$

(b) Find the deflected shape use direct integration

We use the 4th order differentiated equation:

$$EIw^{\mathrm{IV}} = q$$

Integrate 3 times

$$EIw'(l) = 0 = \frac{ql^3}{6} + C_1\frac{l^2}{2} + C_2 l + C_3$$

Use B.C.

$$w'(0) = w'(l) = 0$$

We get

$$C_3 = 0$$

$$EIw'(l) = 0 = \frac{ql^3}{6} + C_1 \frac{l^2}{2} + C_2 l \qquad (1)$$

Integrate equation

$$EIw' = \frac{qx^3}{6} + C_1 \frac{x^2}{2} + C_2 x$$

We get

$$EIw = \frac{qx^4}{24} + C_1 \frac{x^3}{6} + C_2 \frac{x^2}{2} + C_4$$

Use B.C.

$$w(0) = w(l) = 0$$

We get

$$C_4 = 0$$

$$EIw(l) = 0 = \frac{ql^4}{24} + C_1 \frac{l^3}{6} + C_2 \frac{l^2}{2} \qquad (2)$$

Combine equations (1) and (2) to solve for C_1 and C_2

$$C_1 = -\frac{ql}{2}$$

$$C_2 = \frac{ql^2}{12}$$

Finally

$$\boxed{\begin{gathered} w(x) = \frac{qx^2}{24EI}\left(x^2 - 2lx + l^2\right) \\ \text{or} \\ w(l) = \frac{ql^4}{24EI}\left(\left(\frac{x}{l}\right)^4 - 2\left(\frac{x}{l}\right)^3 + \left(\frac{x}{l}\right)\right) \end{gathered}}$$

(c) The maximum deflection magnitude occurs at the mid-span

$$w_{\max}\Big|_{x=\frac{l}{2}} = \frac{q}{24EI}\left(\frac{l}{2}\right)^2\left(\left(\frac{l}{2}\right)^2 - 2l\left(\frac{l}{2}\right) + l^2\right)$$

$$\boxed{w_{\max}\Big|_{x=\frac{l}{2}} = \frac{ql^4}{384EI}}$$

(d) Determine location and magnitude of the maximum stress

Stress distribution in the beam is

$$\sigma = \frac{Mz}{I}$$

The maximum stress locates where moment and z magnitude are the maximum:

$$\sigma_{\max} = \frac{M_{\max} z_{\max}}{I}$$

Recall

$$M = -EIw''$$

Substitute $w(x)$ into the above equation

$$M = -q\left(\frac{x^2}{2} - \frac{lx}{2} + \frac{l^2}{12}\right)$$

Let's consider moment at $x = L/2$ and $x = 0$

$$M\Big|_{x=\frac{l}{2}} = -q\left(\frac{1}{2}\left(\frac{l}{2}\right)^2 - \frac{l}{2}\left(\frac{l}{2}\right) + \frac{l^2}{12}\right) = \frac{ql^2}{24}$$

$$M\Big|_{x=0} = -\frac{ql^2}{12}$$

$$M\Big|_{\max} = M\Big|_{x=0} = -\frac{ql^2}{12}$$

$$\boxed{\sigma_{\max} = -\frac{ql^2}{12}\frac{z_{\max}}{I}}$$

Problem 5-2: Calculate the second moment of inertia of the beam cross section for:

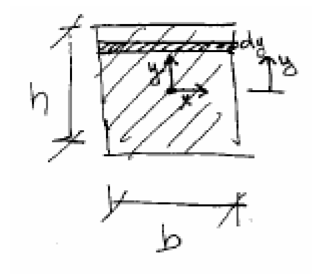
a) Solid rectangular cross section of width b and height h.

b) Thin-walled square box section of width and height b.

c) Solid circular cross section of radius r.

Problem 5-2 Solution:

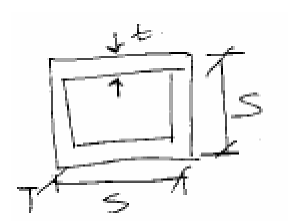
(a) Solid rectangular cross section of width b and height h.



$$I_x = \int_A y^2 dA$$
$$= \int_{-\frac{h}{2}}^{\frac{h}{2}} y^2 b dy$$
$$\boxed{I_x = \frac{bh^3}{12}}$$

(b) Thin-walled square box section of width and height b.

$$I_{\text{square}} = \frac{s^4}{12} - \frac{(s-2t)^4}{12}$$
$$= \frac{1}{12}\left[s^2 - (s-2t)^2\right]\left[s^2 + (s-2t)^2\right]$$
$$= \frac{1}{12}\left(4ts - 4t^2\right)\left(2s^2 - 4ts + 4t^2\right)$$



Eliminate higher order terms of t

$$I_{\text{square}} = \frac{1}{12}\left(4ts - \cancel{4t^2}\right)\left(2s^2 - 4ts + \cancel{4t^2}\right)$$

$$= \frac{1}{3}\left(2ts^3 - \cancel{4t^2s^2}\right)$$

$$\boxed{I_{\text{square}} = \frac{2}{3}tb^3}$$

(c) Thin-walled square box section of width and height b.

$$dA = rd\theta$$

$$I_x = \int_A y^2 dA$$

$$= \int_0^{2\pi}\int_0^r \left(r\sin\theta\right)^2 rdrd\theta$$

$$= \int_0^{2\pi} \frac{r^4}{4}\sin^2\theta drd\theta$$

$$= \frac{r^4}{4}\left(\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right)\Bigg|_0^{2\pi}$$

$$= \frac{\pi r^3}{4}$$

$$\boxed{J = I_x + I_y = \frac{\pi r^3}{2}}$$

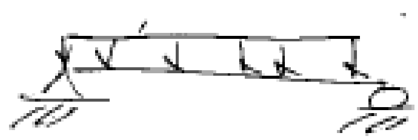
$$\boxed{I_x = I_y = \frac{\pi r^3}{4}}$$

Problem 5-3: In wood construction building codes the beam deflections cannot exceed L/360 where L is the length of the beam. Where do you think this requirement comes from? Choose a typical beam example and state clearly the formulation and your assumptions on the boundary conditions and loading. Using the deflection criteria estimate the fracture strain of the plaster board which is nailed directly to the ceiling beams (joist) in single home construction.

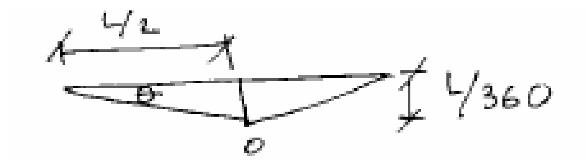
Problem 5-3 Solution:

The L/360 constraint is required to prevent the plaster board from cracking

We assume a simply supported beam with a distributed load

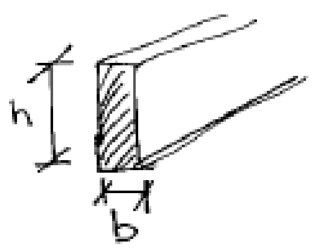


The deflected shape and subsequently curvature can be simplified as



$$\varepsilon_{xx} = \varepsilon_{xx}{}^{0} + z\frac{d\theta}{dx} = 0 + z\left(\frac{L/360}{L/2}\right) = z/180$$

The maximum strain will be at the outmost fiber

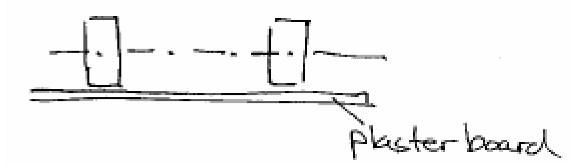


$$\text{at } z = h/2$$

$$\boxed{\varepsilon_{xx} = h/360}$$

Typically the wooden beams used in house construction are 2in×6in. Use those values

$$\boxed{\varepsilon_{xx} = \frac{6/2}{360} = 1/60}$$



The strain seen by the extreme fiber of the wood is also seen by the plaster board.

Problem 5-4: Given a beam with a "T" section subjected to pure bending shown in Figure 1, calculate:

a) the location of the neutral axis

b) the second moment of inertia

c) Find the shear stress distribution in the "T" section.

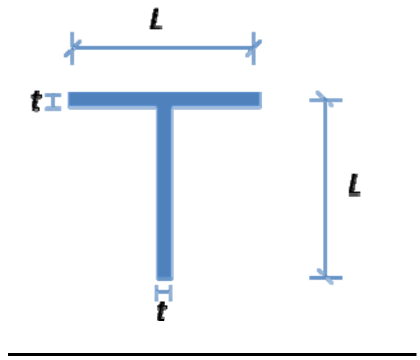
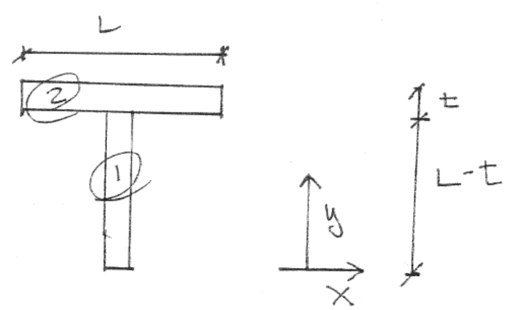


Figure 1

Problem 5-4 Solution:



a) The location of neutral axis is

$$\eta = \frac{\sum y_i A_i}{\sum A_i} = \frac{y_1 A_1 + y_2 A_2}{A_1 + A_2}$$

$$= \frac{\frac{1}{2}(L-t)(L-t)t + \left((L-t)+\frac{t}{2}\right)tL}{(L-t)t + Lt}$$

$$= \boxed{\frac{3L^2 - 3tL + t^2}{4L - 2t}}$$

If we assume $L >> t$

$$\eta \approx \frac{3L}{4}$$

b) If we assume $L >> t$,and don't consider stress variation over the thickness in the flange, the second moment of inertia is

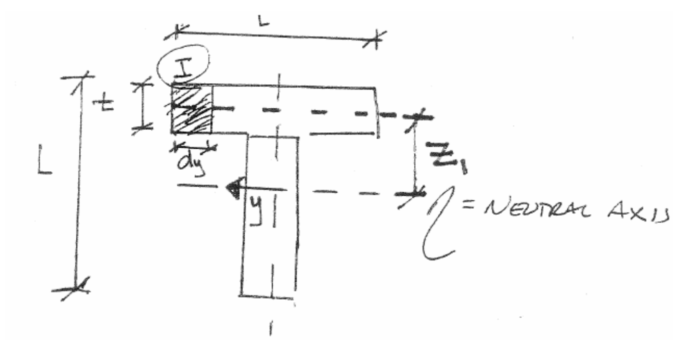
$$\begin{aligned} I &= \sum \left(I_i + A_i d_i \right) \\ &= \frac{tL^3}{12} + (Lt)\left((\eta - L)^2 \right) + \cancel{\frac{Lt^3}{12}} + (Lt)\left(\left(\eta - \frac{L}{2} \right)^2 \right) \\ &= \frac{tL^3}{12} + (Lt)\left(\left(\frac{3}{4}L - L \right)^2 \right) + (Lt)\left(\left(\frac{3}{4}L - \frac{L}{2} \right)^2 \right) \\ &= \frac{tL^3}{12} + \frac{tL^3}{8} \end{aligned}$$

$$\boxed{I = \frac{5tL^3}{24}}$$

Shear stress distribution

First, let's define two parts of the cross section: Part I the horizontal part and Part II the vertical part

Part I



$$Q_I = \int z_I dA$$

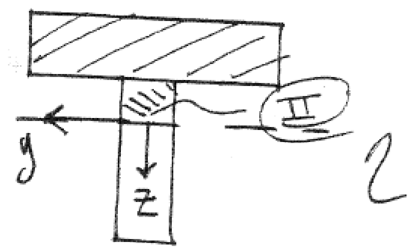
where z_I is the moment are to the shaded region I

$$z_I = L - \frac{t}{2} - \eta = \text{constant}$$
$$\mathrm{d}A = t\mathrm{d}y$$

$$Q_I = \int_y^{\frac{L}{2}} \left(L - \frac{t}{2} - \eta \right) t\mathrm{d}y$$

$$\boxed{Q_I = t\left(L - \frac{t}{2} - \eta \right)\left(\frac{L}{2} - y \right)}$$

Part II



$$Q_{II} = 2Q_I\big|_{y=0} + \int z_{II} dA$$

where z_{II} is the moment are to the shaded region II

$$z_I = z \quad \mathrm{d}A = t\mathrm{dz}$$

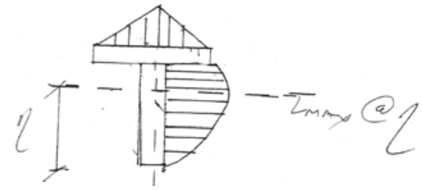
$$Q_I = 2Q_I\big|_{y=0} + \int_z^{L-t-\eta} zt\mathrm{dz}$$

$$\boxed{Q_{II} = t\left(L - \frac{t}{2} - \eta \right)\frac{L}{2} + \frac{t}{2}\left[\left(L - t - \eta \right) - z^2 \right]}$$

check $Q_{II}\big|_{z=\eta} = 0,$ yes!

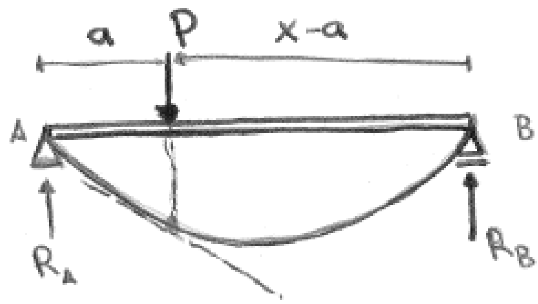
$$\tau = \frac{VQ}{It}$$

Finally, the shear stress distribution is showed as below picture



Problem 5-5: Continuity Condition

Solve the problem of a simply-simply supported beam loaded by a point force acting at eh symmetry plane, but at a distance *a* from the left support



In the notes of lecture 5 the solution of this problem was outlined, but not completed,

(a) Complete the derivation by calculating all four integration constants

(b) Proof that all continuity conditions are satisfied at $x=a$

(c) Show that in the limiting case of $a=L/2$ the solution is identical to one that was derived in class and you were asked to memorize

Problem 5-5 Solution:

(a) The reaction force are calculated from moment equilibrium

$$R_A = P\frac{b}{l} = P\left(1-\frac{a}{l}\right)$$

$$R_B = P\frac{a}{l} \tag{1}$$

The corresponding bending moments and shear forces are

$$M(x) = \begin{cases} R_A x = \dfrac{Pbx}{l} & 0 < x < a \\ R_B(l-x) = \dfrac{Pa(l-x)}{l} & a < x < L \end{cases}$$

$$V(x) = \begin{cases} \dfrac{Pb}{l} & 0 < x < a \\ -\dfrac{Pa}{l} & a < x < L \end{cases} \tag{2}$$

Integrating governing equation

$$-EI\frac{d^2w}{dx^2}=M\left(x\right) \tag{3}$$

Combing with equation (2), we have

$$\begin{aligned}-EIw^{I}&=\frac{Pbx^3}{6l}+C_1x+C_2 && 0<x<a\\ -EIw^{II}&=\frac{Pa}{l}\left(\frac{lx^2}{2}-\frac{x^3}{6}\right)+C_2x+C_4 && a<x<L\end{aligned} \tag{4}$$

Boundary conditions and continuity conditions

$$\begin{aligned}&w\left(0\right)=w\left(l\right)=0\\ &w^{I}\left(a\right)=w^{II}\left(a\right)\\ &\left.\frac{dw^{I}}{dx}\right|_{x=a}=\left.\frac{dw^{II}}{dx}\right|_{x=a}\end{aligned} \tag{5}$$

Substitute (4) into (5) , we have

$$\begin{cases}C_2=0\\ \dfrac{Pal^3}{3}+C_3l+C_4=0\\ \dfrac{Pba^3}{6l}+C_1a=\dfrac{Pa}{l}\left(\dfrac{la^2}{2}-\dfrac{a^3}{6}\right)+C_3a+C_4\\ \dfrac{Pba^2}{2l}+C_1=\dfrac{Pa}{l}\left(la-\dfrac{a^2}{2}\right)+C_3\end{cases} \tag{6}$$

Solve for (6), we get the four unknown integration constants

$$\begin{aligned}&C_1=\frac{Pa}{6l}\left(a-l\right)\left(2l+a\right)\\ &C_2=0\\ &C_3=-\frac{Pa^2}{2}-\frac{Pl\left(l-a\right)\left(2l+a\right)a}{6l}\\ &C_4=\frac{Pa^3}{6}\end{aligned}$$

(b) Check continuity condition

(i) Check continuity of moment M at $x=a$, referring equation (2)

$$\left[M\right]=M^{+}-M^{-}=\frac{Pa\left(l-a\right)}{l}-\frac{Pba}{l}=0$$

(ii) Check continuity of shear force V at $x=a$, referring equation (2)

$$\left[V\right]=V^{+}-V^{-}=\frac{Pb}{l}-\left(-\frac{Pa}{l}\right)=P$$

(iii) Check continuity of angle θ at $x=a$

$$\theta^{-}=\left.\frac{dw^{I}}{dx}\right|_{x=a}=\frac{Pba^{2}}{2l}+\frac{Pa}{6l}\left(a-l\right)\left(2l+a\right)$$

$$\theta^{+}=\left.\frac{dw^{II}}{dx}\right|_{x=a}=\frac{Pa}{l}\left(la-\frac{a^{2}}{2}\right)-\frac{Pa^{2}}{2}-\frac{P\left(l-a\right)\left(2l+a\right)a}{6l}$$

$$\left[\theta\right]=\theta^{+}-\theta^{-}=0$$

(iv) Check continuity of deflection w

$$w^{-}=\frac{Pba^{3}}{6l}+\frac{Pa^{2}}{6l}\left(a-l\right)\left(2l+a\right)$$

$$w^{+}=\frac{Pa}{l}\left(\frac{la}{2}-\frac{a^{3}}{6}\right)-\frac{Pa^{3}}{2}-\frac{P\left(l-a\right)\left(2l+a\right)a^{2}}{6l}+\frac{Pa^{3}}{6}$$

$$\left[w\right]=w^{+}-w^{-}=0$$

All continuity conditions are satisfied.

(c) Referring to equation (4)

$$w^{I}=\frac{1}{-EI}\left(\frac{Pbx^{3}}{6l}+C_{1}x+C_{2}\right)$$

$$a=\frac{L}{2},b=\frac{L}{2}$$

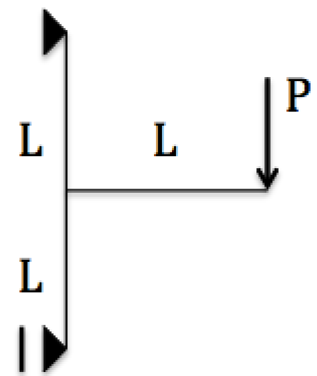
$$\boxed{\begin{aligned}w^{I}&=\frac{Px}{48EI}\left(3l^{2}-4x^{2}\right)\\ w_{o}&=\left.w^{I}\right|_{x=\frac{L}{2}}=\frac{PL^{3}}{48EI}\end{aligned}}$$

They are the same as was derived in lecture notes.

Problem 5-6:

Another problem to test your knowledge on continuity conditions.

A system of two identical beams shown in the figure below is a statically determined problem. The beams are rigidly welded, so that the angle remains 90 degrees.



Determine

(a) The distribution of bending moment and shear forces in both segments.

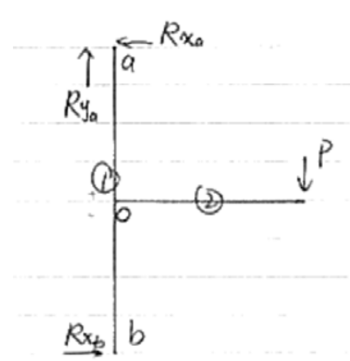
(b) Find the deflected shape of both beams and make sketch.

(c) Find the relation between the tip-- - load and the vertical displacement of the tip.(Hint: assume the rotations to be very small.)

(Note: Do not solve this problem using the Castigliano's theorem, which is much simpler, but has not been covered yet.)

Problem 5-6 Solution:

(a) Free body diagram as below, in which R_{xa}, R_{xb} and R_{ya} are reaction forces



Moment equilibrium about point a

$$R_{xb} \cdot 2L - PL = 0$$

$$\Rightarrow R_{xb} = \frac{P}{2}$$

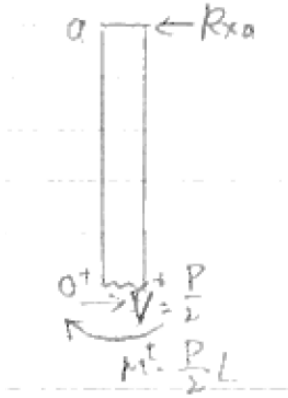
Horizontal and vertical forces balance

$$R_{ya} = P$$

$$R_{xa} = R_{xb} = \frac{P}{2}$$

Beam 1

Let's look at cross section O^+

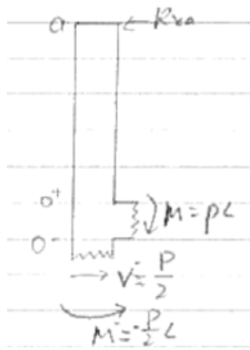


Where

$$V^+ = \frac{P}{2}$$

$$M^+ = \frac{P}{2}L$$

Also, let's look at cross section O^-

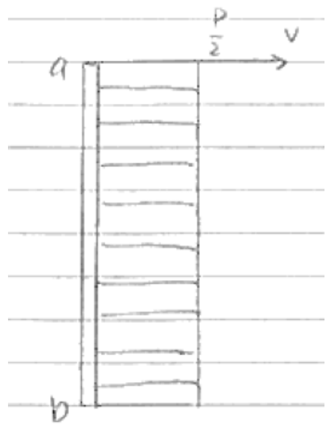


Where

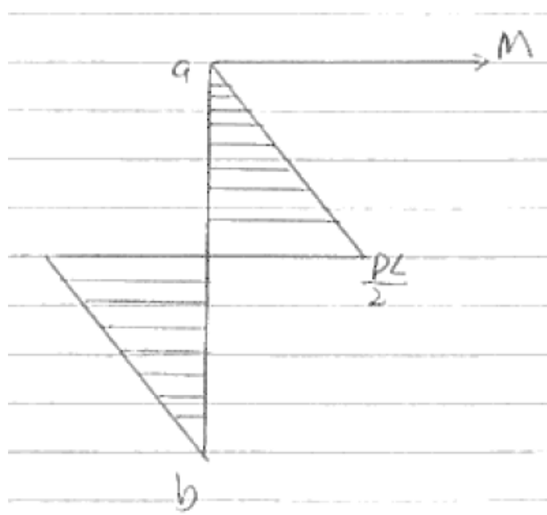
$$V^{-} = \frac{P}{2}$$

$$M^{-} = -\frac{P}{2}L$$

Distribution of shear force

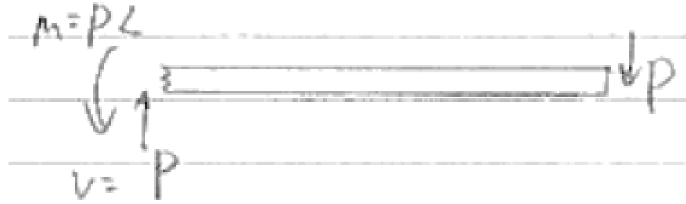


Distribution of moment



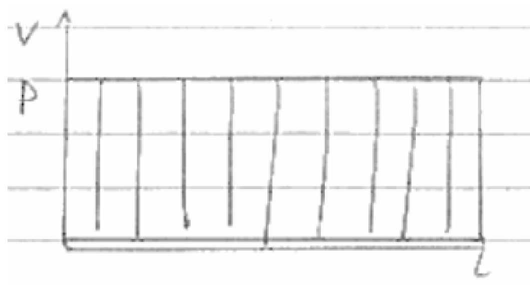
Beam 2

At the left cross-section

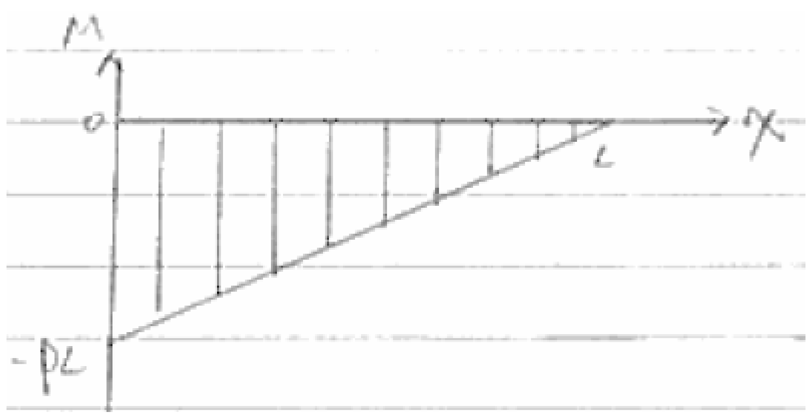


$$V = P$$
$$M = PL$$

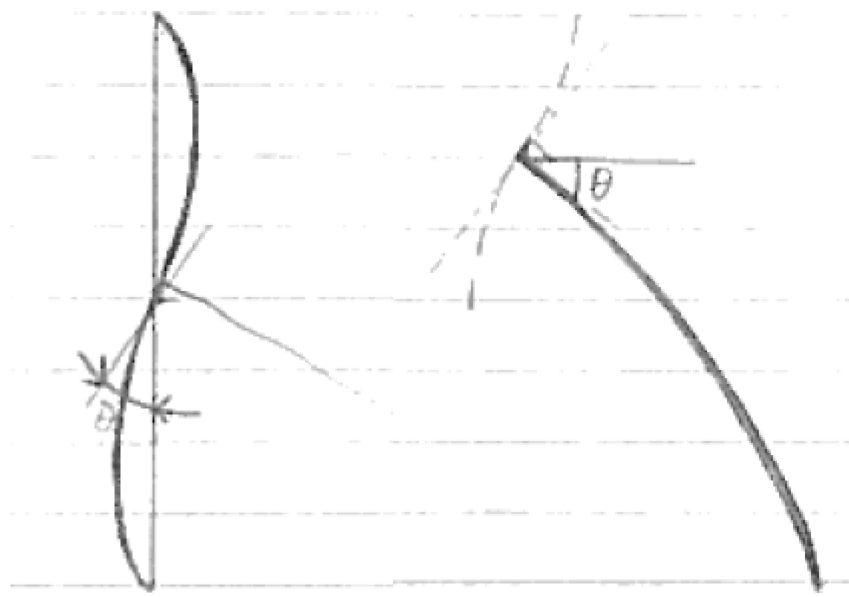
Distribution of shear force



Distribution of moment



(b) Deflection is anti-symmetric for beam 1, at point O, $w = 0$, deflection shapes are



(c) From moment distribution of beam 1, we know

$$M_1 = \frac{P}{2}(L - y) \qquad 0 < y < L$$

Governing equation

$$-EI\frac{d^2 w_1}{dy^2} = M_1 = \frac{P}{2}(L - y) \qquad 0 < y < L$$

Integrate twice

$$w_1 = -\frac{P}{2EI}\left(\frac{Ly^2}{2} - \frac{y^3}{6}\right) + C_1 y + C_2 \qquad 0 < y < L$$

Apply boundary conditions

$$w_1\big|_{y=0} = w_1\big|_{y=L} = 0$$

$$\Rightarrow C_1 = \frac{PL^2}{6EI}, C_2 = 0$$

$$\Rightarrow w_1 = -\frac{P}{2EI}\left(\frac{Ly^2}{2} - \frac{y^3}{6}\right) + \frac{PL^2}{6EI}y \qquad 0 < y < L$$

At y=0

$$\boxed{\theta = \frac{PL^2}{6EI}}$$

Vertical displacement of point O

$$u_o = \int_{x=L}^{o} \varepsilon dy = \int_{x=L}^{o} \frac{P}{EA} dy = \frac{PL}{EA}$$

Integrate governing equation of beam 2

$$-EI\frac{d^2 w_2}{dy^2} = M_2 = -P(L-x)$$

We have

$$-EI\frac{dw_2}{dx} = -P\left(Lx - \frac{x^2}{2}\right) + C_1$$

$$-EIw_2 = P\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) + C_1 x + C_2$$

Use angle continuity condition

$$-EI\frac{dw_2}{dx}\bigg|_{x=0} = \theta_1\big|_{x=0} = \frac{PL^2}{6EI}$$

$$\Rightarrow C_1 = -\frac{PL^2}{6}$$

$$w_2\big|_{x=0} = u_0 = \frac{PL}{EA}$$

$$\Rightarrow C_2 = -\frac{PIL}{A}$$

Finally

$$w_2 = \frac{P}{EI}\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) + \frac{PL^2}{6EI}x + \frac{PL}{EA}$$

At x=L, the tip deflection is

$$\boxed{w_p = \frac{PL^3}{2EI} + \frac{PL}{EA}}$$

Now, let's compare the magnitude of the two terms in w_p, for a square cross-section

$$\frac{w_p{}^{II}}{w_p{}^{I}} = \frac{PL}{EA} \Big/ \frac{PL^3}{2EI} = \frac{2I}{LA} = \frac{2\cdot\frac{h^3}{12}}{Lh^2} = \frac{1}{6}\left(\frac{h}{l}\right)^2$$

For $h/L = 1/20$, $w_p{}^{II} / w_p{}^{I} = 0.25\times10^{-3}$. $w_p{}^{II}$, which is the deflection caused by compression, is three orders of magnitudes smaller than $w_p{}^{I}$, which is the deflection caused by bending.