# Lecture 4

# Development of Constitutive Equations for Continuum, Beams and Plates

**Problem 4-1: Summation Convention**

Hooke's law, a constitutive equation for a linear, elastic material, can be written in general form as:

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$$ where λ and μ are Làme constants.

a) Expand Hooke's Law. How many independent equations are there?

b) Express $\lambda$ and $\mu$ in terms of Young's Modulus, $E$, and Poisson's ratio, $v$.

c) Where does the symmetry of the stress strain tensor come from?

d) Re-write the Hooke's law in terms of $E$ and $v$.

e) Specify the constitutive equations to the case of plane stress.

**Problem 4-1 Solution:**

**a)** Hooke's law

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$$

where

$$\delta_{ij} = \begin{cases} 1, \mathrm{i} = \mathrm{j} \\ 0, \mathrm{i} \neq \mathrm{j} \end{cases}$$

For i=1

$$\sigma_{11} = \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2\mu \varepsilon_{11}$$
$$\sigma_{12} = 2\mu \varepsilon_{12}$$
$$\sigma_{13} = 2\mu \varepsilon_{13}$$

For i=2

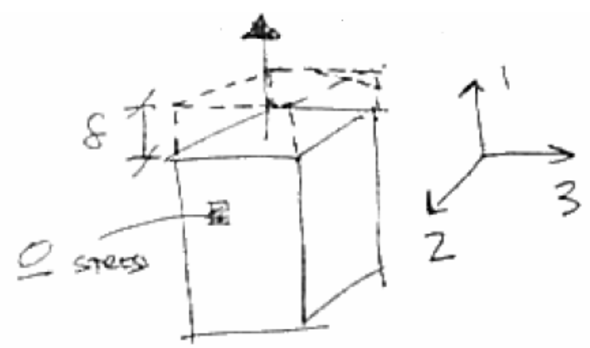
$$\sigma_{21} = 2\mu\varepsilon_{21}$$
$$\sigma_{22} = \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2\mu\varepsilon_{22}$$
$$\sigma_{23} = 2\mu\varepsilon_{23}$$

For i=3

$$\sigma_{31} = 2\mu\varepsilon_{31}$$
$$\sigma_{32} = 2\mu\varepsilon_{32}$$
$$\sigma_{33} = \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2\mu\varepsilon_{33}$$

**b) Express $\lambda$, $\mu$ in terms of E, ν**

We need to solve for $\lambda$, $\mu$. Let's assume a plane stress state in a uniaxial test




$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Note: Assume $\sigma_{11}$=0 because it is a slender bar

Substitute the stress components into the inversed form of the constitutive equations

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} \qquad (1)$$

We have

$$\varepsilon_{11} = \frac{1}{E}\sigma_{11}$$
$$\varepsilon_{22} = -\frac{\nu}{E}\sigma_{11} = -\nu\varepsilon_{11} \quad (2)$$
$$\varepsilon_{33} = -\frac{\nu}{E}\sigma_{11} = -\nu\varepsilon_{11}$$

From the constitutive equations

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} \quad (3)$$

We have

$$\sigma_{22} = \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2\mu\varepsilon_{22} \quad (4)$$

$$\sigma_{11} = \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2\mu\varepsilon_{11} \quad (5)$$

Combining equation (2) and (4), we have

$$\sigma_{22} = \lambda(\varepsilon_{11} - \nu\varepsilon_{11} - \nu\varepsilon_{11}) + 2\mu(-\nu\varepsilon_{11}) = 0 \quad (6)$$

$$\lambda = \frac{2\mu\nu}{1-2\nu} \quad (7)$$

Combining equation (2) and (5), we have

$$\sigma_{11} = \lambda(\varepsilon_{11} - \nu\varepsilon_{22} - \nu\varepsilon_{33}) + 2\mu\varepsilon_{11} = E\varepsilon_{11} \quad (8)$$

$$\lambda(1-2\nu) + 2\mu = E \quad (9)$$

Substitute (7) into (9), solve for $\mu$

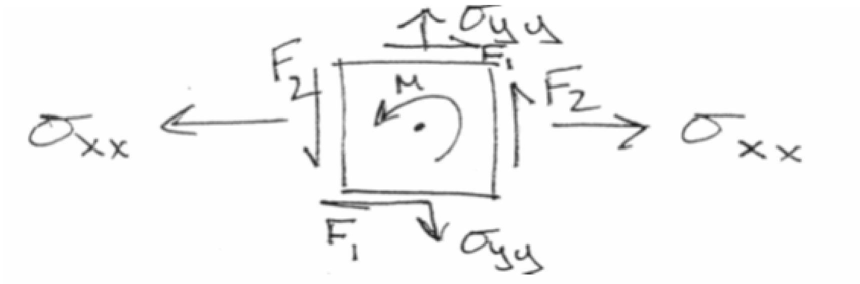
$$\boxed{\mu = \frac{E}{2(\nu+1)} = G} \quad (10)$$

Substitute (10) into (7), solve for $\lambda$

$$\boxed{\lambda = \frac{E\nu}{(1-2\nu)(1+\nu)}} \tag{11}$$

**c) Symmetry of the stress tensor**

Let's take an element in 2D(a unit square)




We know the normal forces are equal in magnitude and opposite in direction at opposite faces, so as the shear forces.

$$\begin{aligned}
\sigma_{xx} &= \sigma_{xx} \\
\sigma_{yy} &= \sigma_{yy} \\
F_1 &= F_1 \\
F_2 &= F_2
\end{aligned}$$

Now let's prove $F_1 = F_2$, take moments about the center

$$\sum \mathrm{M} = F_2 \times \frac{1}{2} + F_2 \times \frac{1}{2} - F_1 \times \frac{1}{2} - F_1 \times \frac{1}{2} = 0$$

$$F_2 = F_1$$

$$\boxed{\sigma_{xy} = \sigma_{yx}}$$

**d) Rewrite Hook's law in terms of $E$ and $\nu$**

Substitute equation (10) and (11) in part b into Hook's law, we have:

$$\sigma_{ij} = \lambda \varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} = \frac{E\nu}{(1-2\nu)(1+\nu)}\varepsilon_{kk}\delta_{ij} + 2\frac{E}{2(\nu+1)}\varepsilon_{ij}$$

$$\boxed{\sigma_{ij} = \frac{E}{(1+\nu)}\left[\frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij} + \varepsilon_{ij}\right]}$$

**e) For Plane stress**

$$\sigma_{11} = \frac{E}{1-\nu^2}\left(\varepsilon_{11} + \nu\varepsilon_{22}\right)$$

$$\sigma_{22} = \frac{E}{1-\nu^2}\left(\varepsilon_{22} + \nu\varepsilon_{11}\right)$$

$$\sigma_{12} = \frac{E}{1+\nu}\varepsilon_{12}$$

**Problem 4-2: Inverting constitutive equations**

The original form of the constitutive equation is to express stress $\sigma_{ij}$ in terms of strain. Invert the 3D constitutive equation and the 2D (plane stress) constitutive equation, meaning that strain $\varepsilon_{ij}$ will be expressed in terms of stresses. The starting point of this problem is Eq. 4.19 for the 3D case in the printed lecture notes. For the 2D case you can use Eq. 4.32 as a starting point.

**Problem 4-2 Solution:**

(1) The **3D** constitutive equations for isotropic linear elastic materials expressed in terms of $E, \nu$ is

$$\sigma_{ij} = \frac{E}{(1+\nu)}\left[\frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij} + \varepsilon_{ij}\right] \tag{1}$$

By making contraction $i = j = k$ , we have

$$\sigma_{kk} = \frac{E}{(1+\nu)}\left[\frac{\nu}{1-2\nu}\varepsilon_{kk}\cdot 3 + \varepsilon_{kk}\right] \tag{2}$$

where $\delta_{kk} = \delta_{11} + \delta_{22} + \delta_{33} = 3$

Express $\varepsilon_{kk}$ in terms of $\sigma_{kk}, E$ and $\nu$

$$\varepsilon_{kk} = \frac{1-2\nu}{E}\sigma_{kk} \tag{3}$$

Substitute equation (3) into equation (1), we have

$$\sigma_{ij} = \frac{E}{(1+\nu)}\left[\frac{\nu}{1-2\nu}\left(\frac{1-2\nu}{E}\sigma_{kk}\right)\delta_{ij} + \varepsilon_{ij}\right] \tag{4}$$

Straight forwardly, the above equation gives

$$\boxed{\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}} \tag{5}$$

which is the inverted form of **3D** Hook's law

(2) The 2**D** constitutive equations for plane stress in terms of $E, \nu$ is

$$\sigma_{\alpha\beta} = \frac{E}{1-\nu^2}\left[(1-\nu)\varepsilon_{\alpha\beta} + \nu\varepsilon_{\gamma\gamma}\delta_{\alpha\beta}\right] \tag{1}$$

Where

$$\delta_{\alpha\beta} = \begin{cases} 1, \alpha = \beta \\ 0, \alpha \neq \beta \end{cases} \tag{2}$$

Since

$$\delta_{\alpha\alpha} = \delta_{11} + \delta_{11} = 2 \tag{3}$$

making contraction of equation (1): $\alpha = \beta$

$$\sigma_{\alpha\alpha} = \frac{E}{(1+\nu)}\left[(1-\nu)\varepsilon_{\alpha\alpha} + 2\nu\varepsilon_{\gamma\gamma}\right] \tag{4}$$

Replacing $\alpha$ with $\gamma$ , Straight forwardly,

$$\varepsilon_{\gamma\gamma} = \frac{1-\nu}{E}\sigma_{\gamma\gamma} \tag{5}$$

Substitute (5) into (1), we have

$$\sigma_{\alpha\beta} = \frac{E}{1-\nu^2}\left[(1-\nu)\varepsilon_{\alpha\beta} + \nu\left(\frac{1-\nu}{E}\right)\sigma_{\gamma\gamma}\delta_{\alpha\beta}\right] \tag{6}$$

Rearrange (6), we have the inverted form

$$\boxed{\varepsilon_{\alpha\beta} = \frac{1+\nu}{E}\sigma_{\alpha\beta} - \frac{\nu}{E}\sigma_{\gamma\gamma}\delta_{\alpha\beta}} \tag{7}$$

**Problem 4-3: Stress and strain deviator**

Defining the stress deviator $s_{ij}$

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$$

and the strain deviator $e_{ij}$

$$e_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij}$$

Convert the constitutive equation into two separate equations, one for the spherical part and other for the distortional part. The spherical part gives a relation between the hydrostatic pressure and the change of volume (Eq. 4.21). For the distortional part the equation was not given in the notes, so we are asking you to find it.

**Problem 4-3 Solution:**

From the lecture notes, express $\sigma_{kk}$ in terms of $\varepsilon_{kk}$ , we have

$$\sigma_{kk} = \frac{E}{1-2\nu}\varepsilon_{kk} \tag{1}$$

From definition of stress deviator

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} \tag{2}$$

We have

$$\sigma_{ij} = s_{ij} + \frac{1}{3}\sigma_{kk}\delta_{ij} \tag{3}$$

From definition of strain deviator

$$e_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij} \tag{4}$$

We have

$$\varepsilon_{ij} = e_{ij} + \frac{1}{3}\varepsilon_{kk}\delta_{ij} \tag{5}$$

Substitute equation (3) and (5) into 3D constitutive equation

$$\sigma_{ij} = \frac{E}{(1+\nu)}\left[\frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij} + \varepsilon_{ij}\right] \tag{6}$$

We have

$$s_{ij} + \frac{1}{3}\sigma_{kk}\delta_{ij} = \frac{E}{(1+\nu)}\left[\frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij} + \left(e_{ij} + \frac{1}{3}\varepsilon_{kk}\delta_{ij}\right)\right] \tag{7}$$

Combining equation (1) with equation (7)

$$s_{ij} + \frac{1}{3}\left(\frac{E}{1-2\nu}\varepsilon_{kk}\right)\delta_{ij} = \frac{E}{(1+\nu)}\left[\frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij} + \left(e_{ij} + \frac{1}{3}\varepsilon_{kk}\delta_{ij}\right)\right] \tag{8}$$

Finally,

$$\boxed{s_{ij} = \frac{E}{1+\nu}e_{ij}} \tag{9}$$

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