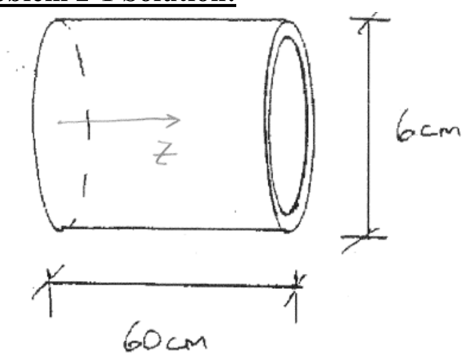
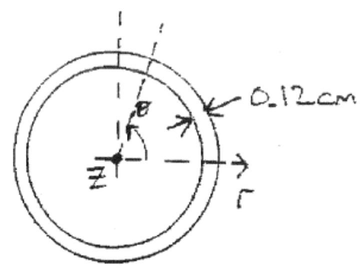
Lecture 2

The Concept of Strain

Problem 2-1: A thin- walled steel pipe of length 60 cm, diameter 6 cm, and wall thickness 0.12 cm is stretched 0.01 cm axially, expanded 0.001 cm in diameter, and twisted through 1^o. Determine the strain components of the pipe. Note that the shell with a diameter to thickness ratio of 50 is predominately in the membrane state. For plane stress there are only 3 components of the strain tensor.

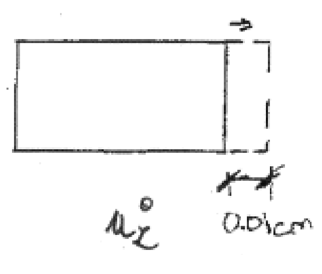
Problem 2-1 Solution:

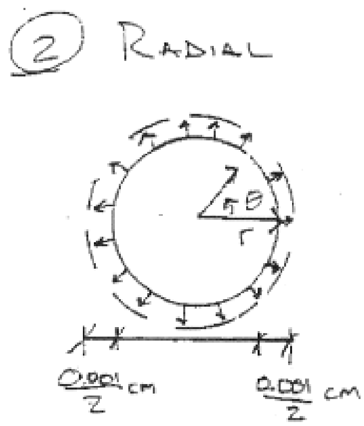


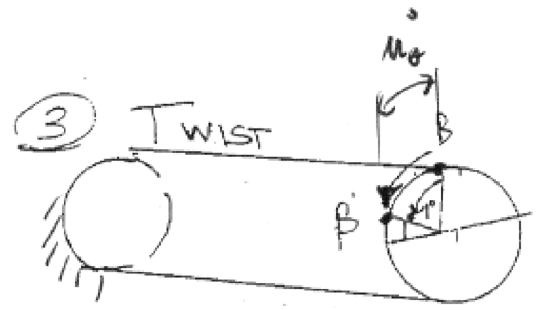


DEFORMED GEOMETRY:

(1) AXIAL







Evaluate deformation components at the end cross-section

$$u_r^{\circ} = 0.01/2cm = 0.0005cm$$

$$u_\theta^{\circ} = R\cdot\Delta\theta = 3\cdot(1^{\circ}\cdot\frac{\pi}{180^{\circ}})cm = 0.0523cm$$

$$u_z^{\circ} = 0.01cm$$

Evaluate deformation components across the whole pipe

u_r is constant

$$u_r = u_r^{\circ} = 0.0005cm$$

u_θ is linearly proportional to θ and r, independent of θ

$$u_\theta = u_\theta(z,r) = u_\theta^{\circ}\frac{z}{L}\frac{r}{R}$$

u_z is linearly proportional to z, independent of r and θ

$$u_z = u_z(z) = u_z^{\circ}\frac{z}{L}$$

Substitute the above equations of u_r, u_θ and u_z into the expression of six components of the strain tensor in cylindrical coordinate system, we have

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} = \frac{\partial u_r{}^{\mathrm{o}}}{\partial r} = 0$$

$$\varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = \frac{u_r{}^{\mathrm{o}}}{r} + \frac{1}{r}\cdot 0 = \frac{0.0005}{6/2} = 1.67\times 10^{-4}$$

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z} = \frac{u_z{}^{\mathrm{o}}}{L} = \frac{0.01}{60} = 1.67\times 10^{-4}$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{\partial u_r}{\partial \theta}\frac{1}{r} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) = \frac{1}{2}\left(0 + \frac{u_\theta{}^{\mathrm{o}} z}{LR} - \frac{u_\theta{}^{\mathrm{o}} z}{LR}\right) = 0$$

$$\varepsilon_{\theta z} = \frac{1}{2}\left(\frac{\partial u_z}{r\partial \theta} + \frac{\partial u_\theta}{\partial z}\right) = \frac{1}{2}\left(0 + \frac{u_\theta{}^{\mathrm{o}} r}{LR}\right) \approx \frac{1}{2}\frac{u_\theta{}^{\mathrm{o}} R}{LR} = \frac{1}{2}\frac{0.0523}{60} = 4.36\times 10^{-4}$$

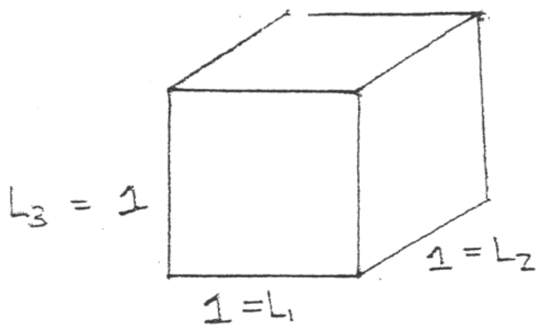
$$\varepsilon_{zr} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) = \frac{1}{2}\left(\frac{\partial u_r{}^{\mathrm{o}}}{\partial z} + \frac{\partial u_z(z)}{\partial r}\right) = 0$$

Summary

$$\boxed{\begin{aligned} &\varepsilon_{rr} = \varepsilon_{r\theta} = \varepsilon_{zr} = 0 \\ &\varepsilon_{zz} = 1.67\times 10^{-4} \\ &\varepsilon_{\theta\theta} = 1.67\times 10^{-4} \\ &\varepsilon_{z\theta} = 4.36\times 10^{-4} \end{aligned}}$$

Problem 2-2: Derive an expression for the change in volume of a unit volume element subjected to an arbitrary small strain tensor.

Problem 2-2 Solution:



A unit volume has sides of unit length

$$\text{Unit Volume} = 1 \times 1 \times 1 = 1[\text{length}]^3$$

Strain is defined by

$$\varepsilon = \frac{\Delta}{L}$$

If each side is deformed by δ , then the new length of each side is $1+\delta$

$$\varepsilon = \frac{\delta}{L}$$
$$\delta = L\varepsilon = 1 \times \varepsilon$$

Volume after deformation is:

$$\begin{aligned} V &= (1+\varepsilon_1)(1+\varepsilon_2)(1+\varepsilon_2) \\ &= 1+\varepsilon_1+\varepsilon_2+\varepsilon_3+\cancel{\varepsilon_1\varepsilon_2}+\cancel{\varepsilon_3\varepsilon_2}+\cancel{\varepsilon_3\varepsilon_1}+\cancel{\varepsilon_1\varepsilon_2\varepsilon_3} \end{aligned}$$

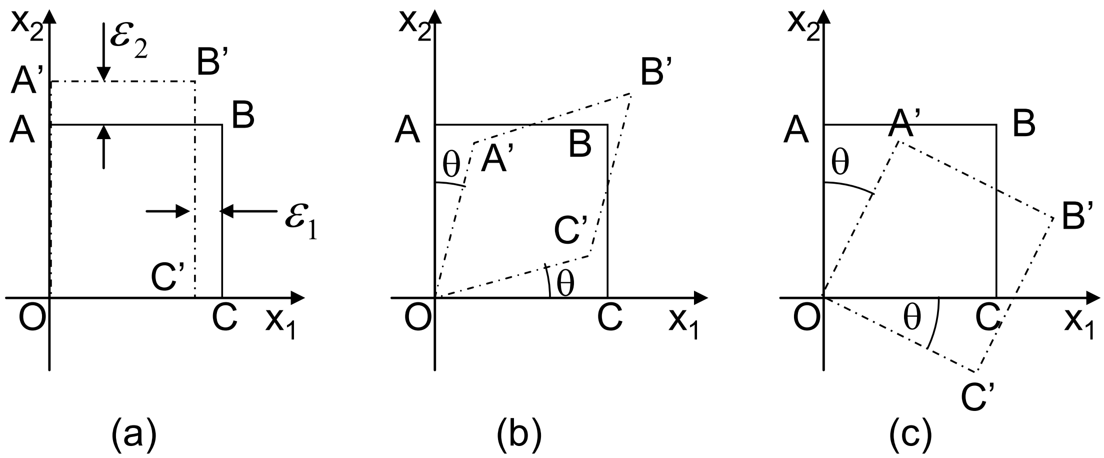
Because of **small** strain, we cancel all higher order terms

Volumetric Strain = change in volume per unit volume, the original volume of a unit volume element is 1

$$\varepsilon_v = \frac{V-V_o}{V_o} = \frac{\left(1+\varepsilon_1+\varepsilon_2+\varepsilon_3\right)-1}{1} = \varepsilon_1+\varepsilon_2+\varepsilon_3$$

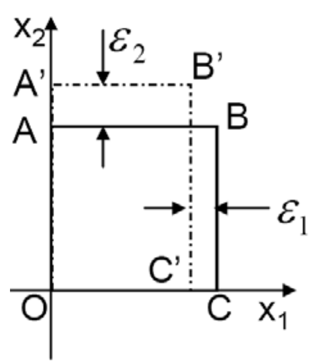
$$\boxed{\varepsilon_v = \varepsilon_1+\varepsilon_2+\varepsilon_3}$$

Problem 2-3: A unit square *OABC* is distorted to *OA'B'C'* in three ways, as shown in the figure below. In each case write down the displacement field (u_1, u_2) of every point in the square as a function of the location of that point (x_1, x_2) and the strain components ε_{ij}.



Problem 2-3 Solution:

(a)



Only **Axial deformation** in this case

$$u_1 = -\varepsilon_1 x_1$$
$$u_2 = \varepsilon_2 x_x$$

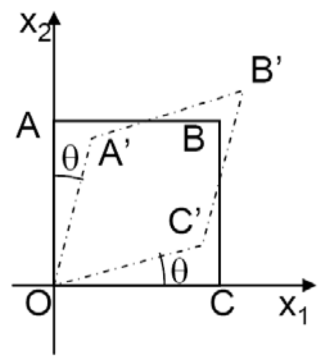
The strain components are

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} = -\varepsilon_1$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2} = \varepsilon_2$$

$$\varepsilon_{12} = \varepsilon_{21} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_1}{\partial x_2}\right) = 0$$

(b)



Only **pure shear** in this case

$$\tan\theta = \frac{u_1}{x_2}$$

$$\tan\theta = \frac{u_2}{x_1}$$

Deformation

$$u_1 = \theta x_2$$
$$u_2 = \theta x_1$$

Because of small angel $\theta, \tan\theta \approx \theta$

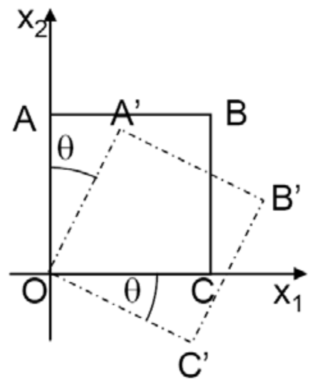
The strain components are

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} = 0$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2} = 0$$

$$\varepsilon_{12} = \varepsilon_{21} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_1}{\partial x_2}\right) = \theta$$

(c)



Only **rigid body rotation** in this case, there should be 0 strain everywhere

Deformation

$$u_1 = \theta x_2$$
$$u_2 = -\theta x_1$$

The strain components are

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} = 0$$
$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2} = 0$$
$$\varepsilon_{12} = \varepsilon_{21} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_1}{\partial x_2}\right) = \frac{1}{2}(\theta - \theta) = 0$$

Problem 2-4: The strain (plane strain) in a given point of a body is described by the 2x2 matrix below. Find components of the strain tensor $\varepsilon_{x'y'}$ in a new coordinate system rotated by the angle θ. Consider four cases $\theta = 45^o, -45^o, 60^o$ *and* -60^o.

$$\varepsilon_{xy} = \begin{bmatrix} 0 & 0.05 \\ 0.05 & 0 \end{bmatrix}$$

Problem 2-4 Solution:

Transformation equations for plane strain, from old coordinate system xy to new coordinate system x'y'

$$\varepsilon_{x'x'} = \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} + \frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\cos(2\theta) + \varepsilon_{xy}\sin(2\theta)$$
$$\varepsilon_{y'y'} = \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} - \frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\cos(2\theta) - \varepsilon_{xy}\sin(2\theta)$$
$$\varepsilon_{x'y'} = -\frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\sin(2\theta) + \varepsilon_{xy}\cos(2\theta)$$

Given strain components:

$$\varepsilon_x = \varepsilon_y = 0$$
$$\varepsilon_{xy} = 0.05$$

Simplify the transformation equations:

$$\varepsilon_{x'x'} = \varepsilon_{xy}\sin(2\theta)$$
$$\varepsilon_{y'y'} = -\varepsilon_{xy}\sin(2\theta)$$
$$\varepsilon_{x'y'} = \varepsilon_{xy}\cos(2\theta)$$

Substitute the stain components into the simplified transformation equations:

$\theta = 45^{\circ}$

$$\varepsilon_{x'y'} = \begin{bmatrix} 0.05 & 0 \\ 0 & -0.05 \end{bmatrix}$$

$\theta = -45^{\circ}$

$$\varepsilon_{x'y'} = -\begin{bmatrix} 0.05 & 0 \\ 0 & -0.05 \end{bmatrix}$$

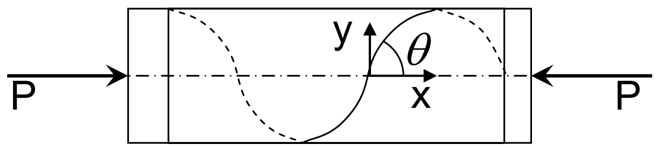
$\theta = 60^{\circ}$

$$\varepsilon_{x'y'} = \begin{bmatrix} 0.04 & -0.025 \\ -0.025 & -0.04 \end{bmatrix}$$

$\theta = -60^{\circ}$

$$\varepsilon_{x'y'} = \begin{bmatrix} -0.04 & -0.025 \\ -0.025 & 0.04 \end{bmatrix}$$

Problem 2-5: A cylindrical pipe of 160-mm outside diameter and 10-mm thickness, spirally welded at an angle of $\theta=40^o$ with the axial (x) direction, is subjected to an axial compressive load of P=150 kN through the rigid end plates (see below). Determine the normal force $\sigma_{x'}$ and shearing stresses $\tau_{x'y'}$ acting simultaneously in the plane of the weld.



Problem 2-5 Solution:

Calculate stress in given coordinate system

Compressive force in x-direction

$$\sigma_{xx} = -\frac{P}{A} = -\frac{150kN}{\pi(80^2 - 70^2)(10^{-3})^2 cm^2}$$

The stress components are

$$\begin{aligned} \sigma_{xx} &= -32MPa \\ \sigma_{yy} &= 0 \\ \sigma_{xy} &= 0 \end{aligned}$$

Transformation equations for plane strain

$$\begin{aligned} \varepsilon_{x'x'} &= \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} + \frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\cos(2\theta) + \varepsilon_{xy}\sin(2\theta) \\ \varepsilon_{y'y'} &= \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} - \frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\cos(2\theta) - \varepsilon_{xy}\sin(2\theta) \\ \varepsilon_{xy} &= -\frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\sin(2\theta) + \varepsilon_{xy}\cos(2\theta) \end{aligned}$$

Substitute the stain components and $\theta = 40^{\circ}$ into the simplified transformation equations:

$$\boxed{\begin{aligned} \sigma_{x'x'} &= -18.77MPa \\ \sigma_{y'y'} &= -13.22MPa \\ \sigma_{z'z'} &= -15.76MPa \end{aligned}}$$

Problem 2-6: A displacement field in a body is given by:

$$u = c\left(x^2 + 10\right)$$
$$v = 2cyz$$
$$w = c\left(-xy + z^2\right)$$

where $c=10^{-4}$. Determine the state of strain on an element positioned at *(0, 2, 1)*.

Problem 2-6 Solution:

Substitute the displacement into the expression of strain components, we have

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = c \cdot 2x = 10^{-4} \cdot 2 \cdot 0 = 0$$
$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = 2cz = 2 \cdot 10^{-4} \cdot 1 = 2 \times 10^{-4}$$
$$\varepsilon_{zz} = \frac{\partial w}{\partial z} = 2cz = 2 \cdot 10^{-4} \cdot 1 = 2 \times 10^{-4}$$
$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}\left(0+0\right) = 0$$
$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \frac{1}{2}\left(2cy - cx\right) = \frac{1}{2}\left(2 \cdot 2 - 0\right) = 2 \times 10^{-4}$$
$$\varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \frac{1}{2}\left(0 - cy\right) = -1 \times 10^{-4}$$

At (0, 2, 1), the state of strain is

$$\underline{\underline{\varepsilon}} = \begin{bmatrix} 0 & 0 & -1\times 10^{-4} \\ 0 & 2\times 10^{-4} & 2\times 10^{-4} \\ -1\times 10^{-4} & 2\times 10^{-4} & 2\times 10^{-4} \end{bmatrix}$$

Problem 2-7: The distribution of stress in an aluminum machine component is given by:

$$\sigma_x = d(cy + 2z^2), \qquad \tau_{xy} = 3dz^2$$
$$\sigma_y = d(cx + cz), \qquad \tau_{yz} = dx^2$$
$$\sigma_z = d(3cx + cy), \qquad \tau_{xz} = 2dy^2$$

where c= 1 mm and d=1MPa/mm^2. Calculate the state of strain of a point positioned at *(1, 2, 4)* mm. Use *E*=70 GPa and *ν*=0.3.

Problem 2-7 Solution:

Substitute the given value of c, d, E, γ into stress component expressions, we have

$$\sigma_x = 34MPa \qquad \tau_{yx} = 48MPa$$
$$\sigma_y = 5MPa \qquad \tau_{yz} = 1MPa$$
$$\sigma_z = 34MPa \qquad \tau_{xz} = 8MPa$$

Use constitutive equations

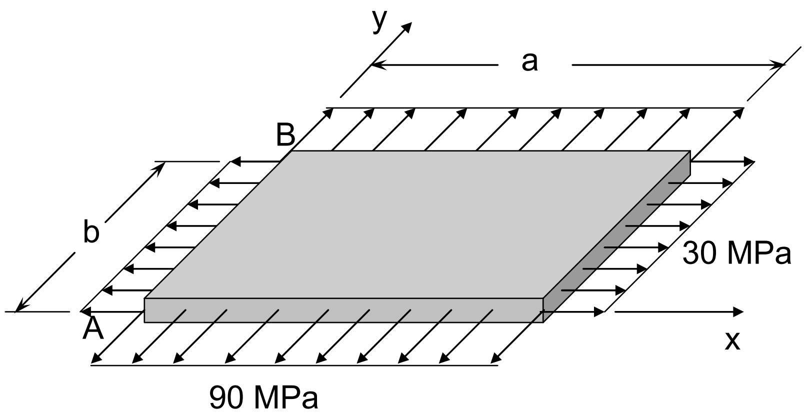
$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right] \qquad \gamma_{xy} = \frac{1}{G}\tau_{xy}$$
$$\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{zz}\right)\right] \qquad \gamma_{yz} = \frac{1}{G}\tau_{yz}$$
$$\varepsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{yy} + \sigma_{xx}\right)\right] \qquad \gamma_{xz} = \frac{1}{G}\tau_{xz}$$

And $G = \dfrac{E}{2(1+\nu)}$

We have the strain components

$$\boxed{\begin{aligned} \varepsilon_{xx} &= 4.42\times10^{-4} \\ \varepsilon_{yy} &= -9.57\times10^{-5} \\ \varepsilon_{zz} &= -9.57\times10^{-5} \end{aligned}} \qquad \boxed{\begin{aligned} \gamma_{xy} &= 8.91\times10^{-4} \\ \gamma_{yz} &= 1.86\times10^{-5} \\ \gamma_{xz} &= 1.49\times10^{-4} \end{aligned}}$$

Problem 2-8: An aluminum alloy plate (E=70 GPa, ν =1/3) of dimensions a=300 mm, b=400 mm, and thickness t=10 mm is subjected to biaxial stresses as shown below. Calculate the change in (a) the length AB; (b) the volume of the plate.



Problem 2-8 Solution:

The state of stress is given

$$\sigma_{xx} = 30\text{MPa}$$
$$\sigma_{yy} = 90\text{MPa}$$
$$\sigma_{zz} = 0\text{MPa}$$

Use constitutive equations to calculate strains ε_{xx} and ε_{yy}

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right] = \frac{1}{70\times10^9}\left[30 - \frac{1}{3}\times 90 - 0\right]\times10^6 = 0$$
$$\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{zz}\right)\right] = \frac{1}{70\times10^9}\left[90 - \frac{1}{3}\times 30 - 0\right]\times10^6 = 0.00114$$
$$\varepsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right] = \frac{1}{70\times10^9}\left[0 - \frac{1}{3}\left(30 + 90\right)\right] = -0.00057$$

Given strain ε_{yy}, calculate the change in length in y

$$\varepsilon_{yy} = \frac{\Delta_y}{L_o}$$
$$\Delta_y = L_o\varepsilon_{yy}$$
$$\boxed{\Delta_y = 0.46mm}$$

Using solutions in problem 2-2, calculate the change in volume

$$\varepsilon_v = \frac{\Delta V}{V_o}$$

$$\Rightarrow \Delta V = V_0 \varepsilon_v = V_0 \left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} \right) = a \times b \times t \times \left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} \right)$$

$$\boxed{\Delta V = 685.72 mm^3}$$

<u>Problem 2-9:</u> Consider the general definition of the strain tensor in the 3D continuum. The three Euler-Bernoulli hypotheses of the elementary beam theory state:

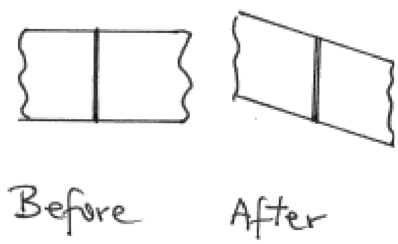
1. Plane remain plane
2. Normal remain normal
3. Transverse deflections are only a function of the length coordinate x.

Proof that under the above assumptions the state in the beam is uni-axial, meaning that the only surviving component of the strain is in the length, x-direction. The proof is sketched in the lecture notes, but we want you to redo the step by step derivation.

<u>Problem 2-9 Solution:</u>

1. Plane remains Plane

This Hypothesis is satisfied when the u-component of the displacement vector is a linear function of z

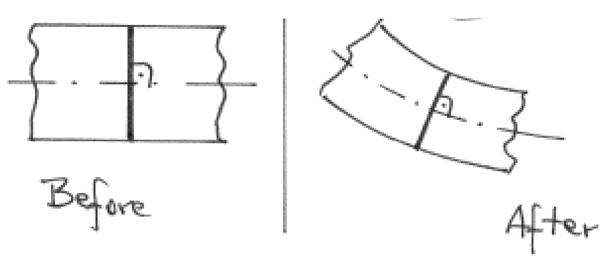


$$u_z = u^{\circ} - \theta z \quad (1)$$

where u° is the displacement of the beam axis

2. Normal remains normal

This hypothesis is satisfied when the rotation of the deformed cross-section θ is equal to the local slope of the middle axis



$$\theta = \frac{dw}{dx} \quad (2)$$

Combing (1) and (2), we have

$$u(x,z) = u^{\circ} - \frac{dw}{dx} z \quad (3)$$

3. Deflections are only a function of x

$$w = w(x) \quad (4)$$

Also, because of planar deformation

$$v \equiv 0 \quad (5)$$

Evaluate all components of strain tensor

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}$$

From equation(5): $v = 0 \quad \Rightarrow \boxed{\varepsilon_{yy} = \frac{\partial v}{\partial y} = 0}$

From equation(4):$w = w(x) \quad \Rightarrow \boxed{\varepsilon_{zz} = \frac{\partial w(x)}{\partial z} = 0}$

From equation(4):$w = w(x)$ and equation(5): $v = 0 \quad \Rightarrow \boxed{\varepsilon_{yz} = \frac{1}{2}(\frac{\partial v}{\partial z} + \frac{\partial w(x)}{\partial y}) = \frac{1}{2}(0+0) = 0}$

From equation(3): $u = u^{\circ} - \frac{dw}{dx} z \quad \Rightarrow \boxed{\varepsilon_{zx} = \frac{1}{2}(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}) = \frac{1}{2}(-\frac{\partial w}{\partial x} + \frac{\partial w}{\partial x}) = 0}$

From equation(3):$u = u(x,z)$ and equation(5): $v = 0 \quad \Rightarrow \boxed{\varepsilon_{xy} = \frac{1}{2}(\frac{\partial u(x,z)}{\partial y} + \frac{\partial v}{\partial x}) = \frac{1}{2}(0+0) = 0}$

Only ε_{xx} survives, therefore the state of strain of a classical beam theory is uniaxial

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