

Lecture 12

Fundamental Concepts in Structural Plasticity

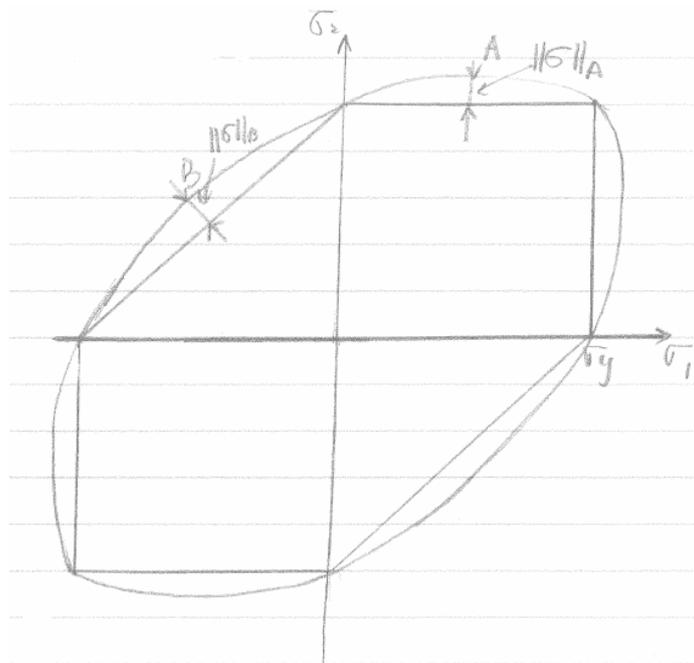
Problem 12-1: Stress yield condition

Consider the plane stress yield condition in the principal coordinate system σ_1, σ_2

- Calculate the maximum difference $\|\sigma\|$ between the Von-Mises and Tresca yield condition
- Show the locations on the plane stress graph where the maximum difference occurs

Problem 12-1 Solution:

Through observation, the maximum difference $\|\sigma\|$ occurs at either A or B, as indicated in the below figure



Mises yield condition can be expressed as

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = \sigma_y^2$$

The difference $\|\sigma\|$ at A

At Tresca yield surface

$$\sigma_{2_A}^{Tresca} = \sigma_y$$

At Mises Yield surface

$$\sigma_1 = \frac{1}{2} \sigma_y$$

Substitute $\sigma_1 = \frac{1}{2} \sigma_y$ into Mises yield condition, we have

$$\begin{aligned} \left(\frac{1}{2} \sigma_y\right)^2 - \frac{1}{2} \sigma_y \sigma_2 + \sigma_2^2 &= \sigma_y^2 \\ \Rightarrow \sigma_{2_A}^{Mises} &= 1.15 \sigma_y \end{aligned}$$

The difference $\|\sigma\|$ at A is

$$\begin{aligned} \|\sigma\|_A &= \sigma_{2_A}^{Mises} - \sigma_{2_A}^{Tresca} = 1.15 \sigma_y - \sigma_y = 0.15 \sigma_y \\ \boxed{\|\sigma\|_A &= 0.15 \sigma_y} \end{aligned}$$

The difference $\|\sigma\|$ at B

Tresca yield surface in the second quadrant is

$$\sigma_1^2 + \sigma_2^2 = \frac{1}{2} \sigma_y^2$$

The distance from Tresca yield surface B to the origin is

$$d_B^{Tresca} = \sqrt{\sigma_1^2 + \sigma_2^2} = \frac{\sqrt{2}}{2} \sigma_y$$

Mises yield surface in the second quadrant is

$$\sigma_1^2 - \sigma_1 \sigma_2 + \sigma_2^2 = \sigma_y^2$$

Also, at Mises yield surface

$$\sigma_1 + \sigma_2 = 0$$

Combining the above two equations, we have the coordinate of Mises yield surface B

$$\sigma_1 = \frac{1}{\sqrt{3}}\sigma_y, \quad \sigma_2 = -\frac{1}{\sqrt{3}}\sigma_y$$

The distance from Tresca yield surface B to the origin is

$$d_B^{Mises} = \sqrt{\sigma_1^2 + \sigma_2^2} = \sqrt{\frac{2}{3}}\sigma_y$$

The difference $\|\sigma\|$ at B is

$$\|\sigma\|_B = d_B^{Mises} - d_B^{Tresca} = \frac{\sqrt{2}}{2}\sigma_y - \sqrt{\frac{2}{3}}\sigma_y = 0.11\sigma_y$$

$$\boxed{\|\sigma\|_B = 0.11\sigma_y}$$

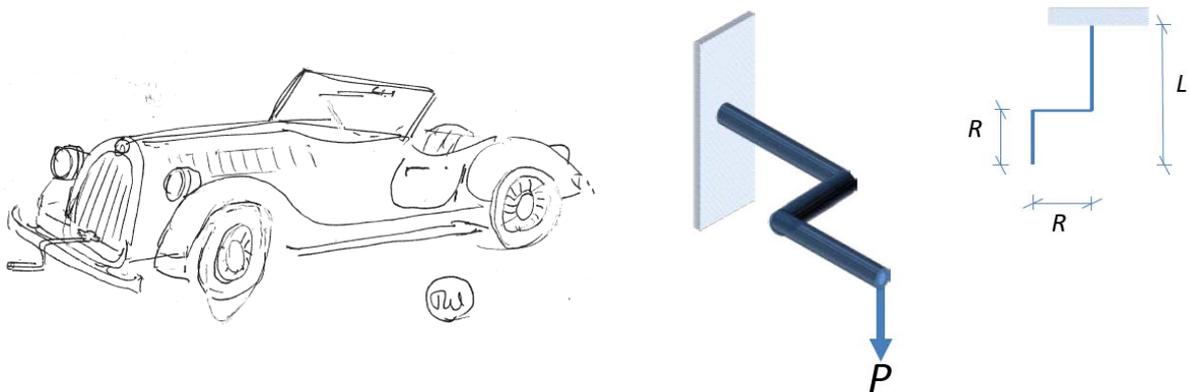
Compare $\|\sigma\|_A$ and $\|\sigma\|_B$, the maximum difference $\|\sigma\|$ occurs at A, where

$$\boxed{\|\sigma\|_{\max} = 0.15\sigma_y}$$

Problem 12-2:

In the early twenties, passenger cars did not have electric starters. The driver had to use a crank to start the engine. The crank is a solid rod of radius r and the geometry of the crank is shown below. Define the equivalent stress by $\bar{\sigma} = \sqrt{\sigma^2 + 3\tau^2}$ where σ sigma is the stress produced by bending and τ is the shear stress due to torsion.

- Find the relationship between the maximum equivalent stress in the crank and the magnitude of the crank load P . (Use the principle of superposition)
- Derive a formula for the elastic deflection under the load P in the direction of the load P .



Problem 12-2 Solution:

- Determine the magnitude of the load P causing first yield in the mostly stressed point on the crank

We will use the Mises yield condition which simplified to

$$\sigma_o^2 = \sigma^2 + 3\tau^2$$

We calculate σ from the moment, and τ from the torsion

$$\sigma = \frac{Mr}{I} \text{ where } I = \frac{\pi r^4}{4}$$

$$\tau = \frac{Tr}{J} \text{ where } J = \frac{\pi r^4}{2}$$

Because we are asked to find P causing first yield at the maximum stressed point, we need to find σ_{\max} and τ_{\max}

From physics we can see that the M_{\max} and T_{\max} will occur at the support

$$M_{\max} = Pl \Rightarrow \sigma_{\max} = \frac{M_{\max} r}{I} = \frac{Plr}{\pi r^4/4}$$

$$T_{\max} = PR \Rightarrow \tau = \frac{T_{\max} r}{J} = \frac{PRr}{\pi r^4/2}$$

$$\boxed{\begin{aligned} \sigma_{\max} &= \frac{4 Pl}{\pi r^3} \\ \tau_{\max} &= \frac{2 PR}{\pi r^3} \end{aligned}}$$

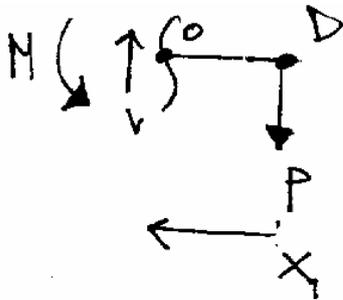
$$\begin{aligned} \sigma_o^2 &= \sigma_{\max}^2 + 3\tau_{\max}^2 \\ &= \left(\frac{4 Pl}{\pi r^3}\right)^2 + 3\left(\frac{2 PR}{\pi r^3}\right)^2 \end{aligned}$$

$$\boxed{\sigma_o^2 = \frac{4}{\pi^2 r^6} P^2 (4l^2 + 3R^2)}$$

$$\boxed{P = \frac{\pi \sigma_o r^2}{2} \sqrt{4\left(\frac{l}{r}\right)^2 + 3\left(\frac{R}{r}\right)^2}}$$

b) Calculate the elastic deflection in the direction of the load P

Beam CD

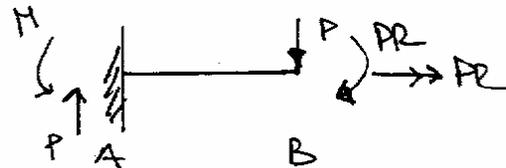


$$T = 0$$

$$\sum M_o = M - Px_1 = 0$$

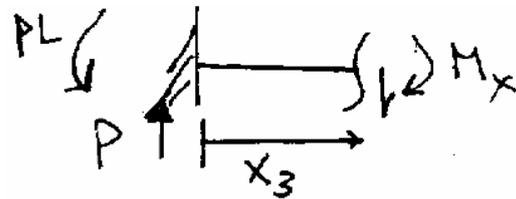
$$\boxed{M = Px_1 \text{ at } 0 < x_1 < R}$$

Beam AB



$$\sum M_A = M - P(L - R) - PR = 0$$

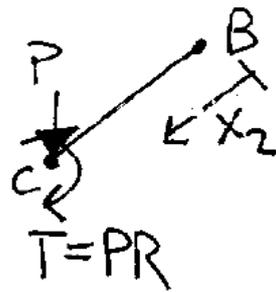
$$M = PL$$



$$-M_x + PL - Px_3 = 0$$

$$\boxed{M_x = PL - Px_3 \text{ at } 0 < x_3 < (L - R)}$$

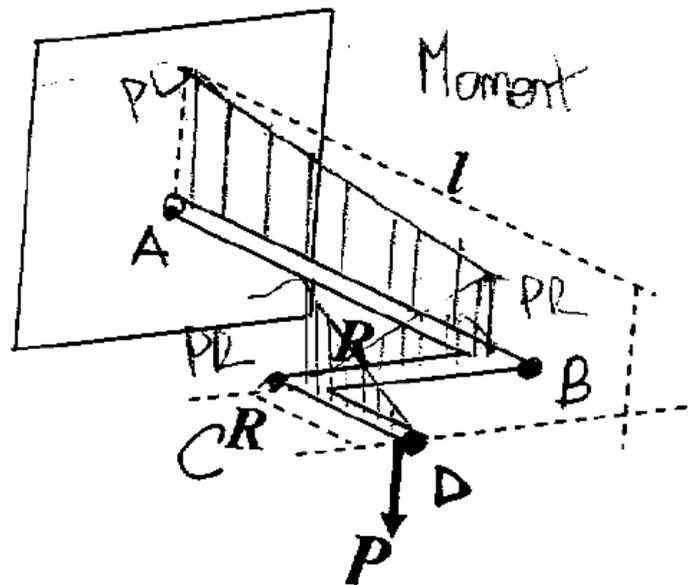
Beam CB



$$M = P(R - x_2) \text{ at } 0 < x_2 < R$$

$$T = PR \text{ at } 0 < x_2 < R$$

In summary, here are the moment distributions



Torsion is constant along
Beams AB and BC: $T = PR$
Beam CD: $T = 0$

$$\begin{aligned}
U_{bending} &= \int \frac{M^2}{2EI} dx \\
&= \int_0^l \frac{(Px_1)^2}{2EI} dx_1 + \int_0^R \frac{(P(R-x_2))^2}{2EI} dx_2 + \int_0^{L-R} \frac{(P(L-x_3))^2}{2EI} dx_3 \\
&= \frac{P^2}{2EI} \left[\frac{x_1^3}{3} \Big|_0^l + \left(R^2x_2 - \frac{2Rx_2^2}{2} + \frac{x_2^3}{3} \right) \Big|_0^R + \left(L^2x_3 - \frac{2Lx_3^2}{2} + \frac{x_3^3}{3} \right) \Big|_0^{L-R} \right] \\
&= \dots (\text{after lengthy algebra}) \\
&= \frac{P^2}{6EI} (R^3 + L^3)
\end{aligned}$$

$$\boxed{U_{bending} = \frac{P^2}{6EI} (R^3 + L^3)}$$

$$\begin{aligned}
U_{torsion} &= \int \frac{T^2}{2GJ} dx \\
&= \int_0^R \frac{(PR)^2}{2GJ} dx_2 + \int_0^{L-R} \frac{(PR)^2}{2GJ} dx_2 \\
&= \frac{(PR)^2}{2GJ} (R + l - R)
\end{aligned}$$

$$\boxed{U_{torsion} = \frac{(PR)^2 L}{2GJ}}$$

Use Castigliano's Theorem to calculate the deflection where the load is applied, in the direction of the load

$$\begin{aligned}
\Delta &= \frac{\partial U}{\partial P} \\
&= \frac{\partial (U_{torsion} + U_{bending})}{\partial P} \\
&= \frac{\partial}{\partial P} \left[\frac{P^2}{6EI} (R^3 + L^3) + \frac{(PR)^2 L}{2GJ} \right] \\
&= P \left[\frac{(R^3 + L^3)}{3EI} + \frac{R^2 L}{GJ} \right]
\end{aligned}$$

Recall that for solid circular cross-section $I = \frac{\pi r^4}{4}$ and $J = \frac{\pi r^4}{2}$

$$\text{Also, } G = \frac{E}{2}(1+\nu)$$

$$\begin{aligned}\Delta &= P \left[\frac{(R^3 + L^3)}{3EI} + \frac{R^2L}{GJ} \right] \\ &= P \left[\frac{R^3 + L^3}{3E \pi r^4/4} + \frac{R^2L(1+\nu)}{E \pi r^4/4} \right] \\ &= \frac{4}{\pi} \frac{P}{Er^2} \left[\frac{R^3 + L^3}{3r^2} + \frac{R^2L(1+\nu)}{r^2} \right] \\ \Delta &= \frac{4}{\pi} \frac{P}{Er^2} \left[\frac{R^3 + L^3}{3r^2} + \frac{R^2L(1+\nu)}{r^2} \right]\end{aligned}$$

Note, if $R = 0$, we have a cantilever beam whose deflection is

$$\Delta = \frac{PL^3}{3EI}$$

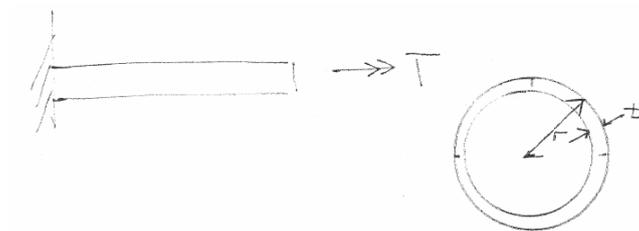
Problem 12-3:

Consider a thin-walled tube of radius r , thickness t and length L . The tube is fully clamped on one end and free on the other. It is twisted at the free end by an axial torque T .

- (a) Derive an expression between torsional moment and the relative end rotation.
- (b) Assuming $L/R=10$ and $R/t=10$, give the expression for the critical torque that will cause the tube stress to reach yield in shear.

Problem 12-3 Solution:

- (a) Express T as a function of ϕ



$$\begin{aligned} T &= \int \tau_{x\theta} r dA \\ &= \int G r^2 \frac{d\phi}{dx} dA \\ &= G \frac{d\phi}{dx} \int r^2 dA \\ &= GJ \frac{d\phi}{dx} \end{aligned}$$

$$\Delta\phi|_{x=0} = 0$$

$$\Delta\phi|_{x=L} = \Delta\phi_{\max}$$

$$\Delta\phi = \frac{T}{GJ} \left(\frac{x}{L} \right)$$

- (b) The distribution of torsional shear stress can be expressed as

$$\tau_{x\theta} = \frac{Tr}{J}$$

Where

$$\begin{aligned}
 J &= \frac{\pi}{2} (r_o^4 - r_i^4) \\
 &= \frac{\pi}{2} (r_o^4 - (r_o - t)^4)
 \end{aligned}$$

Given

$$\frac{r}{t} = 10 \Rightarrow t = \frac{r}{10}$$

$$\begin{aligned}
 J &= \frac{\pi}{2} \left(r_o^4 - \left(r_o - \frac{r_o}{10} \right)^4 \right) \\
 &\boxed{J = 0.071\pi r_o^4}
 \end{aligned}$$

$$\tau_{x\theta} = \frac{Tr}{0.071\pi r_o^4} = \frac{5.8T}{\pi r_o^3}$$

Plane stress yield condition states:

$$\sigma_{xx}^2 + \sigma_{yy}^2 + 3\sigma_{xy}^2 = \sigma_y^2$$

In pure shear

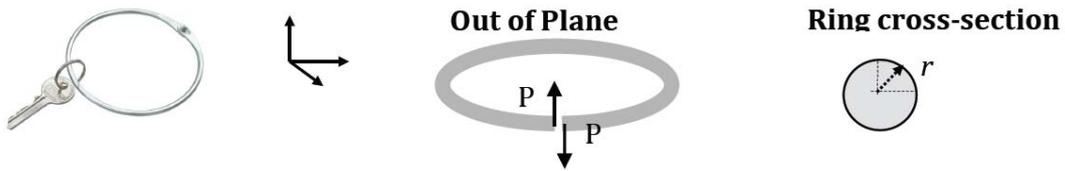
$$\sigma_{xx} = \sigma_{yy} = 0$$

$$\boxed{\sigma_{xy} = \frac{\sigma_y}{\sqrt{3}}}$$

$$\tau_{x\theta} = \frac{5.8T}{\pi r_o^3} = \frac{\sigma_y}{\sqrt{3}}$$

$$\boxed{\sigma_y = 3.2 \frac{T}{r_o^3}}$$

Problem 12-4: Consider the following key ring problem



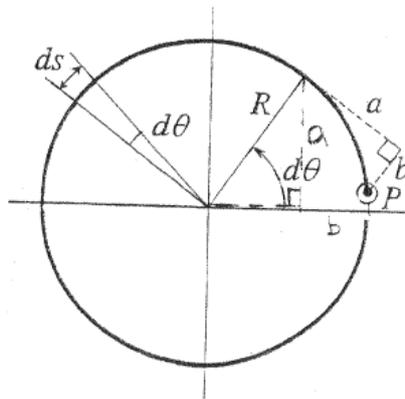
- Derive the out of plane displacement where the force is applied.
- Determine the magnitude and distribution of the bending stress and the shear stress along the ring.
- Find the location of the maximum equivalent plastic strain.
- Determine the critical opening force for which first yield would occur. Consider the plane stress

Problem 12-4 Solution:

- Derive the out of plane displacement where the force is applied.
We can use Castigliano's Theorem to calculate the displacement where the force is applied. In addition, to calculating the strain energy contribution from the moment, we must also account for the contribution from torsion

$$U = \int \frac{1}{2} \frac{M^2}{EI} dx + \int \frac{1}{2} \frac{T^2}{GJ} dx$$

- Calculate M, T



From the geometry above, we can see that

$$M = Pa$$

$$T = Pb$$

where

$$a = R \sin \theta$$

$$b = R - R \cos \theta$$

So

$$M = PR \sin \theta$$

$$T = PR(1 - \cos \theta)$$

2. Calculate strain energy

$$U = \int \frac{1}{2} \frac{M^2}{EI} dx + \int \frac{1}{2} \frac{T^2}{GJ} dx$$

$$= \int_0^\pi \frac{1}{2} \frac{(PR \sin \theta)^2}{EI} R d\theta + \int_0^\pi \frac{1}{2} \frac{[PR(1 - \cos \theta)]^2}{GJ} R d\theta$$

$$= \frac{(PR)^2 R}{EI} \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4} \right]_0^\pi + \frac{(PR)^2 R}{GJ} \left[\theta - 2 \sin \theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_0^\pi$$

$$= \frac{(PR)^2 R}{EI} \left[\frac{\pi}{2} \right] + \frac{(PR)^2 R}{GJ} \left[\pi + \frac{\pi}{2} \right]$$

$$U = \frac{\pi P^2 R^3}{2} \left[\frac{1}{EI} + \frac{3}{GJ} \right]$$

3. Apply Castigliano Theorem, the out of plane displacement where the force is applied is

$$\Delta = \frac{\partial U}{\partial P} = \pi P R^3 \left[\frac{1}{EI} + \frac{3}{GJ} \right]$$

b) Determine the magnitude + distribution of bending stress + shearing

$$\sigma = \frac{Mz}{I} = \frac{PR \sin \theta \cdot r}{\pi r^4 / 4} = \frac{4}{\pi r^3} PR \sin \theta$$

$$\tau = \frac{Tr}{J} = \frac{PR(1 - \cos \theta) \cdot r}{\pi r^4 / 2} = \frac{2}{\pi r^3} PR(1 - \cos \theta)$$

c) Find the location of maximum plastic strain

Recall that relation of plastic strain and plastic stress is

$$\sigma = A \varepsilon^n$$

The maximum plastic strain occurs at location of the maximum Mises equivalent stress,

1. Calculate the Mises equivalent stress

The stress on the outer surface is planar, so we will use the plane stress condition

$$\sigma_{xx}^2 - \sigma_{xx} \sigma_{yy} + \sigma_{yy}^2 + 3\sigma_{xy}^2 = \sigma_o^2$$

Where σ_o is Mises equivalent stress

In our case, $\sigma_{yy} = 0$

So

$$\begin{aligned}\sigma_o^2 &= \sigma^2 + 3\tau^2 \\ &= \left(\frac{4}{\pi r^3} PR \sin \theta\right)^2 + 3\left(\frac{2}{\pi r^3} PR(1 - \cos \theta)\right)^2\end{aligned}$$

2. Calculate the Maximum Mises equivalent stress $\sigma_{o_{\max}}$

$\sigma_{o_{\max}}$ will occur when we have $\sigma_{o_{\max}}^2$. We can find the maximum by taking the derivative, setting it equal to 0, find θ where $\sigma_{o_{\max}}^2$ occurs and then go back to get $\sigma_{o_{\max}}$

$$\begin{aligned}\frac{d\sigma_o^2}{d\theta} &= \left(\frac{4}{\pi r^3} PR\right)^2 2 \sin \theta \cos \theta + 3\left(\frac{2}{\pi r^3} PR\right)^2 2(1 - \cos \theta) \sin \theta \\ &= 8\left(\frac{PR}{\pi r^3}\right)^2 [\sin \theta (\cos \theta + 3)]\end{aligned}$$

Set $\frac{d\sigma_o^2}{d\theta} = 0$, we have

$$\begin{aligned}\sin \theta (\cos \theta + 3) &= 0 \\ \Rightarrow \boxed{\theta = 0, \pi}\end{aligned}$$

Minimum Mises equivalent stress occurs at $\theta = 0$, where $\sigma_o = 0$

Maximum Mises equivalent stress occurs at $\theta = \pi$, where $\sigma_o = \frac{4\sqrt{3}}{\pi r^3} PR$

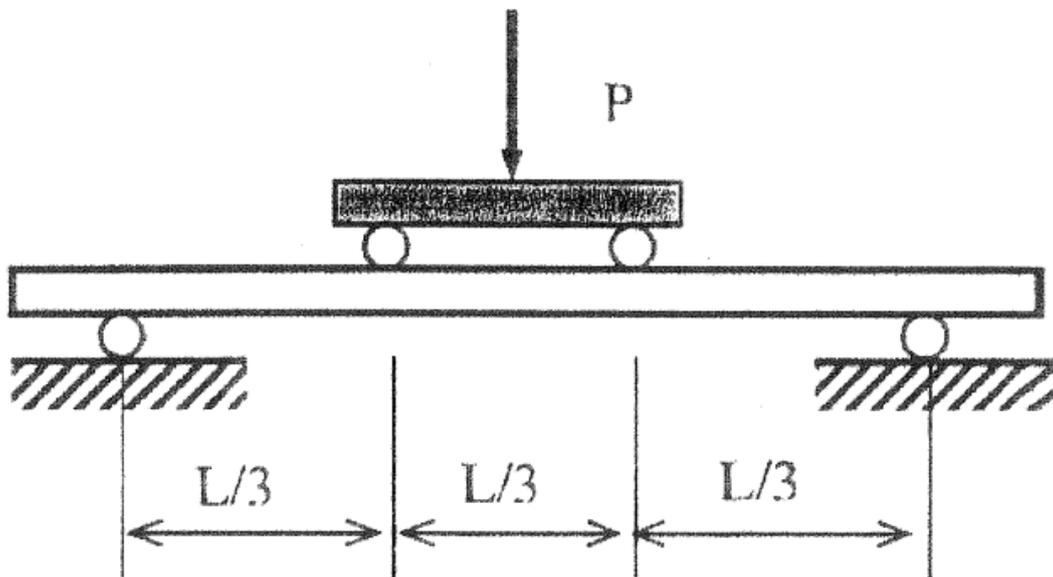
d) The critical opening force for which first yield would occur when $\sigma_o = \sigma_y$, that is

$$\sigma_y = \sigma_o = \frac{4\sqrt{3}}{\pi r^3} P_{cr} R$$

So we have

$$\boxed{P_{cr} = \frac{\pi r^3}{4\sqrt{3}R} \sigma_y}$$

Problem 12-5: Plasticity



Consider the four-point bending of a beam of length L . The beam is loaded by two rollers parted by a distance of $L/3$. The material of the beam is rigid, perfectly plastic. Determine the load capacity of the beam under two different end conditions.

- Write an expression for fully plastic bending moment of a beam of rectangular cross-section $b \times h$.
- Ends of the beam are simply supported
- Ends of the beam are clamped

Problem 12-5 Solution:

a) $M_o = \frac{\sigma_y h^2 b}{4}$

b) Moment distribution

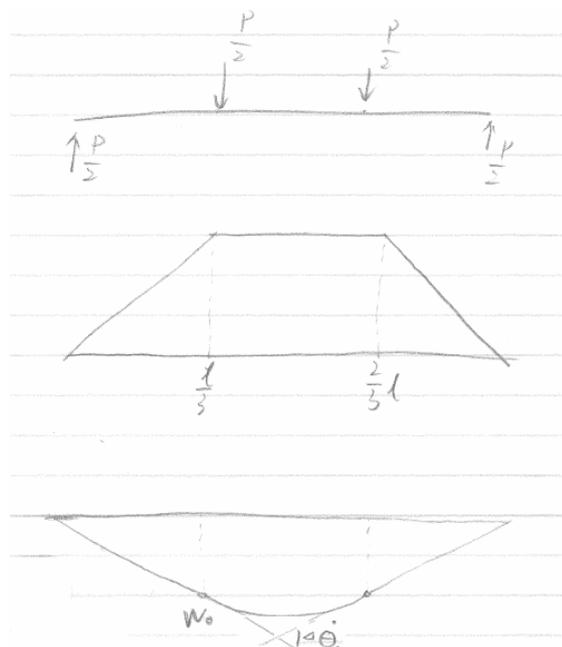
$$M = \frac{Px}{2} \quad 0 < x < \frac{l}{3}$$

$$M = \frac{Pl}{6} \quad \frac{l}{3} < x < \frac{2l}{3}$$

$$M = \frac{P}{2}(l-x) \quad \frac{2l}{3} < x < l$$

We can assume that within $\frac{l}{3} < x < \frac{2l}{3}$

$$M = M_o = \frac{\sigma_y h^2 b}{4}$$



The rate of change of internal energy is

$$\dot{U} = \int_{l/3}^{2l/3} M\dot{K}dx = \int_{l/3}^{2l/3} M_o \frac{d\dot{\theta}}{dx} dx = M_o \dot{\theta} \Big|_{x=l/3}^{x=2l/3} = M_o \Delta\dot{\theta}_o$$

Where geometrically $\Delta\dot{\theta}_o = 2 \frac{\dot{w}_o}{l/3} = 6 \frac{\dot{w}_o}{l}$

Rate of work balance

$$M_o \cdot \Delta\dot{\theta}_o = 2 \cdot \frac{P_c}{2} \cdot \dot{w}_o$$

$$M_o \cdot 6 \frac{\dot{w}_o}{l} = P_c \cdot \dot{w}_o$$

$$\boxed{P_c = \frac{6M_o}{l}}$$

c) If the ends are clamped, we have two plastic hinges at the supports

$$\Delta\dot{\theta}_1 = \Delta\dot{\theta}_2 = \frac{\dot{w}_o}{l/3} = 3 \frac{\dot{w}_o}{l}$$

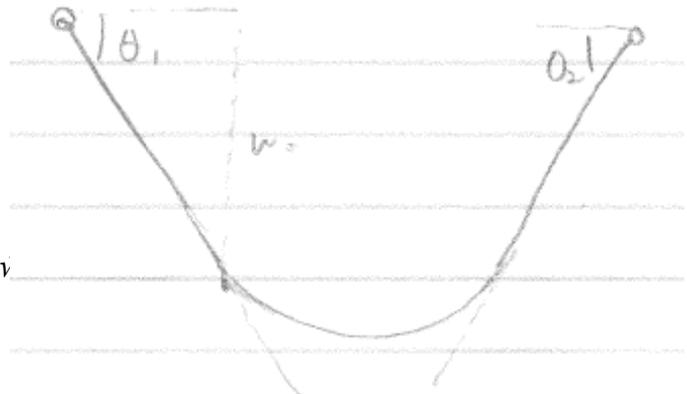
Rate of work balance

$$\sum M_o^i \cdot (\Delta\dot{\theta}_o)^i = 2 \cdot \frac{P_c}{2} \cdot \dot{w}_o$$

$$M_o \cdot \Delta\dot{\theta}_o + M_o \cdot \Delta\dot{\theta}_1 + M_o \cdot \Delta\dot{\theta}_2 = P_c \cdot \dot{w}_o$$

$$M_o \cdot 12 \frac{\dot{w}_o}{l} = P_c \cdot \dot{w}_o$$

$$\boxed{P_c = \frac{12M_o}{l}}$$



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