I-campus project
School-wide Program on Fluid Mechanics
Modules on Waves in fluids
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CHAPTER SEVEN
INTERNAL WAVES IN A STRATIFIED FLUID

1 Introduction.

The atmosphere and ocean are continuously stratified due to change in temperature, composition and pressure. These changes in the ocean and atmosphere can lead to significant variations of density of the fluid in the vertical direction. As an example, fresh water from rivers can rest on top of sea water, and due to the small diffusivity, the density contrast remains for a long time. The density stratification allows oscillation of the fluid to happen. The restoring force that produces the oscillation is the buoyancy force. The wave phenomena associated with these oscillations are called internal waves and are discussed in this chapter.

2 Governing Equations for Incompressible Density-stratified Fluid.

We are going to derive the system of equations governing wave motion of an incompressible fluid with continuous density stratification. Cartesian coordinates x, y and z will be used, with z measured vertically upward. The velocity components in the directions of increasing x, y and z will be denoted as u, v and w. The fluid particle has to satisfy the continuity equation

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{2.1}$$

and the momentum equations

$$\rho\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x}, \tag{2.2}$$
$$\rho\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y}, \tag{2.3}$$
$$\rho\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} - g\rho, \tag{2.4}$$

where ρ and p are, respectively, the fluid density and pressure. The fluid is taken to be such that the density depends only on entropy and on composition, i.e., ρ depends only on the potential temperature θ and on the concentrations of constituents, e.g., the salinity s or humidity q. Then for fixed θ and q (or s), ρ is *independent of pressure*:

$$\rho = \rho(\theta, q). \tag{2.5}$$

The motion that takes place is assumed to be isentropic and without change of phase, so that θ and q are constant for a material element. Therefore

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial \theta}\frac{D\theta}{Dt} + \frac{\partial \rho}{\partial q}\frac{Dq}{Dt} = 0. \tag{2.6}$$

In other words, ρ is constant for a material element because θ and q are, and ρ depends only on θ and q. Such a fluid is said to be *incompressible*, and because of (2.6) the continuity equation (2.1) becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{2.7}$$

For an incompressible fluid, the density ρ satisfies the density equation

$$\frac{1}{\rho}\frac{D\rho}{Dt} = 0. \tag{2.8}$$

Assuming that the velocities are small, we can linearize the momentum equations to obtain

$$\rho\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x}, \tag{2.9}$$

$$\rho\frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y}, \tag{2.10}$$

$$\rho\frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} - g\rho. \tag{2.11}$$

Next, we consider that the wave motion results from the perturbation of a state of equilibrium, which is the state of rest. So the distribution of density and pressure is the hydrostatic equilibrium distribution given by

$$\frac{\partial \bar{p}}{\partial z} = -g\bar{\rho}. \tag{2.12}$$

When the motion develops, the pressure and density changes to

$$p = \bar{p}(z) + p', \tag{2.13}$$

$$\rho = \bar{\rho}(z) + \rho', \tag{2.14}$$

where p' and ρ' are, respectively, the pressure and density perturbations of the "background" state in which the density $\bar{\rho}$ and the pressure $\bar{p}$ are in hydrostatic balance. The density equation now assume the form

$$\frac{\partial \rho'}{\partial t} + u\frac{\partial \rho'}{\partial x} + v\frac{\partial \rho'}{\partial y} + w\frac{\partial \bar{\rho}}{\partial z} + w\frac{\partial \rho'}{\partial z} = 0. \tag{2.15}$$

The nonlinear terms $u(\partial\rho'/\partial x)$, $v(\partial\rho'/\partial y)$ and $w(\partial\rho'/\partial z)$ are negligible for small amplitude motion, so the equation (2.15) simplifies to

$$\frac{\partial \rho'}{\partial t} + w\frac{\partial \bar{\rho}}{\partial z} = 0, \tag{2.16}$$

which states that the density perturbation at a point is generated by a vertical advection of the background density distribution. The continuity equation (2.7) for incompressible fluid stays the same, but the momentum equations (2.9) to (2.11) assume the form

$$\bar{\rho}\frac{\partial u}{\partial t} = -\frac{\partial p'}{\partial x}, \tag{2.17}$$

$$\bar{\rho}\frac{\partial v}{\partial t} = -\frac{\partial p'}{\partial y}, \tag{2.18}$$

$$\bar{\rho}\frac{\partial w}{\partial t} = -\frac{\partial p'}{\partial z} - g\rho'. \tag{2.19}$$

We would like to reduce the systems of equations (2.7), (2.16) and (2.17) to (2.19) to a single partial differential equation. This can be achieved as follows. First, we take the time derivative of the continuity equation to obtain

$$\frac{\partial^2 u}{\partial t \partial x} + \frac{\partial^2 v}{\partial t \partial y} + \frac{\partial^2 w}{\partial t \partial z} = 0. \tag{2.20}$$

Second, we take the x, y and t derivatives, respectively, of the equations (2.17) to (2.19), and we obtain

$$\bar{\rho}\frac{\partial^2 u}{\partial x \partial t} = -\frac{\partial^2 p'}{\partial x^2}, \tag{2.21}$$

$$\bar{\rho}\frac{\partial^2 v}{\partial y \partial t} = -\frac{\partial^2 p'}{\partial y^2}, \tag{2.22}$$

$$\bar{\rho}\frac{\partial^2 w}{\partial t^2} = -\frac{\partial^2 p'}{\partial t \partial z} - g\frac{\partial \rho'}{\partial t}. \tag{2.23}$$

If we substitute equations (2.21) and (2.22) into equation (2.20), we obtain

$$-\frac{1}{\bar{\rho}}\left(\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2}\right) + \frac{\partial^2 w}{\partial t \partial z} = 0. \tag{2.24}$$

We can eliminate ρ' from (2.23) by using equation (2.16) to obtain

$$\bar{\rho}\frac{\partial^2 w}{\partial t^2} = -\frac{\partial^2 p'}{\partial t \partial z} + g\frac{\partial \bar{\rho}}{\partial z}w. \tag{2.25}$$

Third, we apply the operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ to equation (2.25) to obtain

$$\bar{\rho}\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = -\frac{\partial^2}{\partial t \partial z}\left(\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2}\right) + g\frac{\partial \bar{\rho}}{\partial z}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right). \tag{2.26}$$

Next, we use equation (2.24) to eliminate p' from equation (2.26), which gives the following partial differential equation for w:

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{1}{\bar{\rho}}\frac{\partial}{\partial z}\left[\bar{\rho}\frac{\partial w}{\partial z}\right]\right)+N^2\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)=0, \tag{2.27}$$

where we define

$$N^2(z)=-\frac{g}{\rho}\frac{\partial\bar{\rho}}{\partial z}, \tag{2.28}$$

which has the units of frequency (rad/sec) and is called the Brunt-Väisälä frequency or buoyancy frequency. If we assume that w varies with z much more rapidly than $\bar{\rho}(z)$, then

$$\frac{1}{\bar{\rho}}\frac{\partial}{\partial z}\left(\bar{\rho}\frac{\partial}{\partial z}\right)w\sim\frac{\partial^2 w}{\partial z^2}, \tag{2.29}$$

and (2.27) can be approximated by the equation

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+N^2\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)=0. \tag{2.30}$$

The assumption above is equivalent to the Boussinesq approximation, which applies when the motion has vertical scale small compared with the scale of the background density. It consists in taking the density to be constant in computing rates of change of momentum from accelerations, but taking full account of the density variations when they give rise to buoyancy forces, i.e., when there is a multiplying factor g in the vertical component of the momentum equations. The Boussinesq approximation leads to equation (2.30) for the vertical velocity w.

3 The Buoyancy Frequency (Brunt-Väisälä frequency).

Consider a calm stratified fluid with a static density distribution $\bar{\rho}(z)$ which decreases with height z. If a fluid parcel is moved from the level z upward to $z+\zeta$, it is surrounded by lighter fluid of density $\bar{\rho}(z+\zeta)$. The upward buoyancy force per unit volume is

$$g\left[\bar{\rho}(z+\zeta)-\bar{\rho}(z)\right] \approx g\frac{d\bar{\rho}}{dz}\zeta, \tag{3.31}$$

and it is negative. Applying Newton's law to the fluid parcel of unit volume, we have

$$\bar{\rho}\frac{\partial^2\zeta}{\partial t^2} = g\frac{d\bar{\rho}}{dz}\zeta \tag{3.32}$$

or

$$\frac{\partial^2\zeta}{\partial t^2} + N^2\zeta = 0, \tag{3.33}$$

where

$$N^2(z) = -\frac{g}{\bar{\rho}}\frac{d\bar{\rho}}{dz}, \tag{3.34}$$

which is called the buoyancy frequency or the Brunt Väsälä frequency. This elementary consideration shows that once a fluid is displaced from its equilibrium position, gravity and density gradient provide restoring force to enable oscillations.

4 Internal Gravity Waves in Unbounded Stratified Fluid.

Consider the case in which the buoyancy (Brunt-Väisälä) frequency N is constant throughout the fluid. Traveling wave solutions of (2.30) can be found of the form

$$w = w_0\cos(kx+ly+mz-\omega t), \tag{4.35}$$

where w_0 is the vertical velocity amplitude and $\overrightarrow{k} = (k,l,m)$ is the wavenumber of the disturbance, and ω is the frequency. In order for (4.35) to satisfy the governing equation (2.30) for the vertical perturbation velocity, ω and $\overrightarrow{k}$ must be related by the dispersion relation

$$\omega^2 = \frac{(k^2+l^2)N^2}{k^2+l^2+m^2}. \tag{4.36}$$

Thus internal waves can have any frequency between zero and a maximum value of N. The dispersion relation for internal waves is of quite a different character compared to that for surface waves. In particular, the frequency of surface waves depends only on the magnitude $|\overrightarrow{k}|$ of the wavenumber, whereas the frequency of internal waves is independent of the magnitude of the wavenumber and depends only on the angle ϕ that the wavenumber vector makes with the horizontal. To illustrate this, we consider the spherical system of coordinates in the wavenumber space, namely,

$$k = |\overrightarrow{k}| \cos(\phi)\cos(\theta) \tag{4.37}$$

$$l = |\overrightarrow{k}| \cos(\phi)\sin(\theta) \tag{4.38}$$

$$m = |\overrightarrow{k}| \sin(\phi) \tag{4.39}$$

The coordinate system in the wavenumber space is given in the figure 1.

The dispersion relation given by equation (4.36) reduces to

$$\omega^2 = N\cos(\phi). \tag{4.40}$$

Now we can write expressions for the quantities p', ρ', u and v. From equation (2.20) we can write

$$-\frac{1}{\rho_0}\left(\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2}\right) = \frac{\partial^2 w}{\partial t \partial z} = \omega m w_0 \cos(kx+ly+mz-\omega t),$$

which implies that the perturbation pressure p' is given by

$$p' = -\frac{\omega m w_0 \rho_0}{(k^2+l^2)^{1/2}}\cos(kx+ly+mz-\omega t). \tag{4.41}$$

From equation (2.16) we have the perturbation density ρ' given by

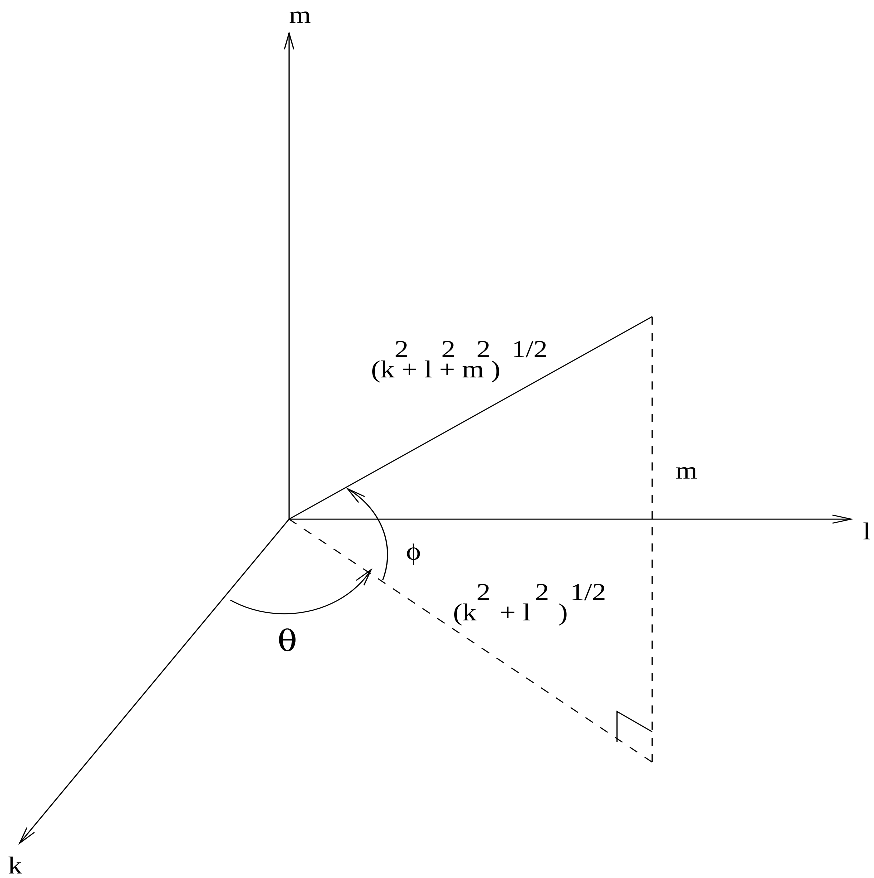

Figure 1: Coordinate system in the wavenumber space.

$$\rho' = -\left(\frac{N^2}{\omega g}\right)\rho_0 w_0 \sin(kx + ly + mz - \omega t). \tag{4.42}$$

The horizontal velocity components can be found from equations (2.17) and (2.18), which give

$$(u, v) = -(k, l)(k^2 + l^2)^{-1} m w_0 \cos(kx + ly + mz - \omega t) \tag{4.43}$$

$$= (k, l)(\omega\rho_0)^{-1} p'. \tag{4.44}$$

The above relations between pressure and velocity fluctuations can be useful for deducing wave properties from observations at a fixed point. For instance, if the horizontal velocity components and perturbation pressure of a progressive wave are measured, the horizontal component of the wavenumber vector can be deduced from (4.44).

A sketch showing the properties of a plane progressive internal wave in the vertical plane that contains the wavenumber vector is presented in figure 2. The particle motion is along wave crests, and there is no pressure gradient in this direction. The restoring force on a particle is therefore due solely to the component $g\cos\phi$ of gravity in the direction of motion. The restoring force is also proportional to the component of density change in this direction, which is $\cos\phi\frac{d\rho}{dz}$ per unit displacement.

Consider now the succession of solutions as ϕ progressively increases from zero to $\pi/2$. When $\phi = 0$, a vertical line of particles moves together like a rigid rod undergoing longitudinal vibrations. When the line of particles is displaced from its equilibrium, buoyancy restoring forces come into play just as if the line of particles were on a spring, resulting in oscillations of frequency N. The solution for increasing values of ϕ correspond to lines of particles moving together at angle ϕ to the vertical. The restoring force per unit displacement ($\cos\phi d\rho'/dz$) is less than the case where $\phi = 0$, so the frequency of vibration is less. As ϕ tends to $\pi/2$, the frequency of vibration tends to zero. The case $\phi = \pi/2$ is not an internal wave, but it represents an important form of motion that is often observed. For instance, it is quite common on airplane journeys to see thick layers of cloud that are remarkably flat and extensive. Each cloud layer is moving in its own horizontal plane, but different layers are moving relative to each other.

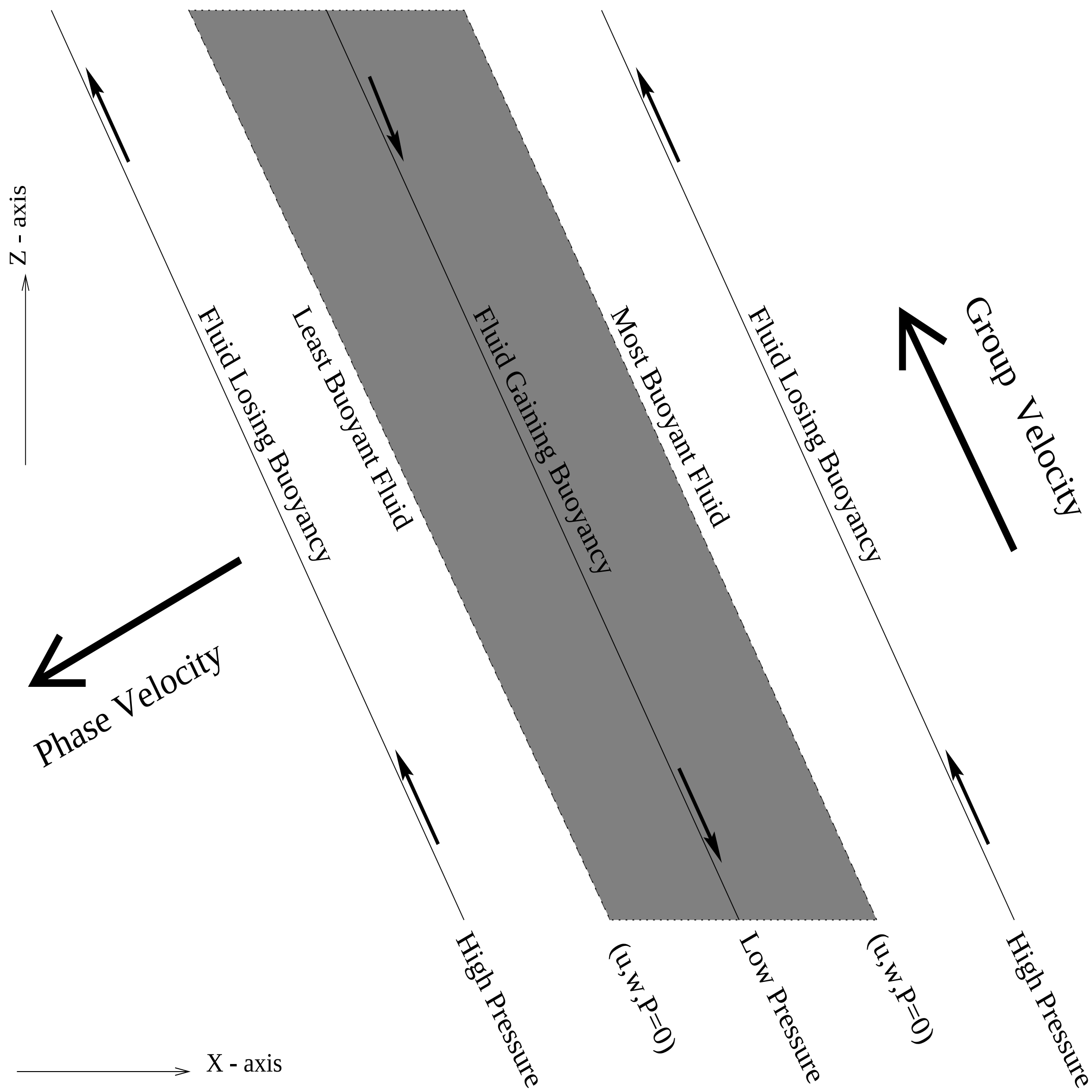

Figure 2: The instantaneous distribution of velocity, pressure, and buoyancy perturbations in an internal gravity wave. This is a view in the x, z plane. The phase of the wave is constant along the slanting, dashed, and solid lines. Velocity and pressure perturbations have extrema along the solid lines; buoyancy perturbations are zero along the solid lines. Buoyancy perturbations have extrema, and velocity and pressure perturbations are zero along dashed lines. Small arrows indicate the perturbation velocities, which are always parallel to the lines of constant phase. Large heavy arrows indicate the direction of phase propagation and group velocity.

4.1 Dispersion Effects.

In practice, internal gravity waves never have the form of the exact plane wave given by equation (4.35), so it is necessary to consider superposition of such waves. As a consequence, dispersion effects become evident, since waves with different frequencies have different phase and group velocities as we are going to show in this section. For internal waves, surfaces of constant frequency in the wavenumber space are the cones $\phi =$ constant. The phase velocity is parallel to the wavenumber vector and it lies on a cone of constant phase. Its magnitude is

$$\frac{\omega}{|\overrightarrow{k}|} = \left(\frac{N}{|\overrightarrow{k}|}\right) \cos\phi. \tag{4.45}$$

The *group velocity* C_g is the gradient of the frequency ω in the wavenumber space and therefore is normal to the surface of constant frequency ω. It follows that the group velocity is at right angles to the wavenumber vector. When the group velocity has an upward component, therefore, the phase velocity has a downward component, and vice versa. The group velocity vector is

$$C_g = \frac{N}{|\overrightarrow{k}|} \sin\phi(\sin\phi\cos\theta, \sin\phi\sin\theta, -\cos\phi). \tag{4.46}$$

Therefore, the magnitude of the group velocity is $(\frac{N}{|\overrightarrow{k}|})\sin\phi$, and its direction is at an angle ϕ to the vertical, as illustrated in the figure 3.

To illustrate the effects of dispersion, we consider the case of two dimensional motions. We consider only the coordinates x and z. In this case, the wavenumber is the vector (k, m). We consider an initially localized wave packet. Due to dispersion effects, the wave packet spreads and moves according to the group velocity vector C_g, which now simplifies to

$$C_g = \frac{N}{|\overrightarrow{k}|} \sin\phi(\sin\phi, -\cos\phi). \tag{4.47}$$

The phase velocity is perpendicular to the group velocity vector, so the wave crests (lines of constant phase) move perpendicularly to the direction of propagation of the

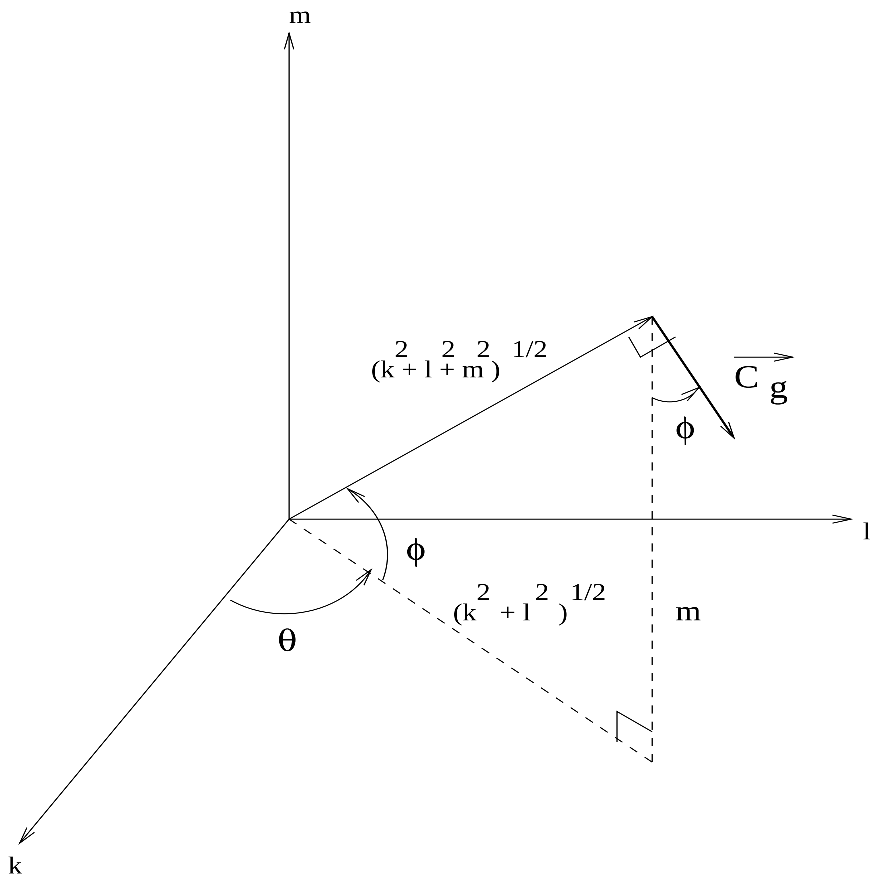

Figure 3: Wavenumber vector and group velocity vector.

wave packet. The phase velocity is given by the equation (4.45), where the wavenumber vector $\overrightarrow{k}$ makes an angle ϕ with the horizontal direction (see figure 1, but now set $\theta = 0$).

To illustrate the effects of dispersion, we consider three different animations of a localized wave packet for the density perturbation ρ'. The perturbation density ρ' is related to the vertical velocity w by the equation

$$\frac{\partial \rho'}{\partial t} = \frac{\rho_0 N^2}{g} w, \tag{4.48}$$

and the governing equation for the vertical velocity w is given by the equation (2.30). To obtain the evolution in time of an initially localized wave packet for the perturbation density, we apply a two-dimensional Fourier transform to equations (2.30) and (4.48). The two-dimensional Fourier transform pair considered is

$$\hat{u}(k,m) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dz \left\{ \exp(-ikx - imz) u(x,z) \right\} \tag{4.49}$$

and

$$u(x,z) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dm \left\{ \exp(-ikx - imz) \hat{u}(k,m) \right\}. \tag{4.50}$$

The Fourier transform of the equation (2.30) is given by the equation

$$\frac{\partial^2 \hat{w}}{\partial t^2} + \frac{N^2 k^2}{k^2 + m^2} \hat{w} = 0, \tag{4.51}$$

which has solution of the form

$$\hat{w}(k,m,t) = A(k,m) \exp(i\omega t) + B(k,m) \exp(-i\omega t), \tag{4.52}$$

where ω is given by the dispersion relation

$$\omega = \frac{Nk}{\sqrt{k^2 + m^2}}. \tag{4.53}$$

The Fourier transform of the equation (4.48) is given by the equation

$$\frac{\partial \hat{\rho}'}{\partial t} = \frac{\rho_0 N^2}{g}\hat{w}. \tag{4.54}$$

From equations (4.51) and (4.54) we have that

$$\hat{\rho}'(k,m,t) = \frac{\rho_0 N^2}{g\omega(k,m)} \left\{-iA(k,m)\exp(i\omega t) + iB(k,m)\exp(i\omega t)\right\}, \tag{4.55}$$

where the constants A and B are determined from the Fourier transform of the initial conditions for ρ', given by the equations

$$\rho'(x,z,0) = f(x,z), \tag{4.56}$$

$$\frac{\partial \rho'}{\partial t}(x,z,0) = 0, \tag{4.57}$$

which implies that the constants $A(k,m)$ and $B(k,m)$ are given by the equations

$$A(k,m) = \frac{ig\omega}{2\rho_0 N^2}\hat{f}, \tag{4.58}$$

$$A(k,m) = -\frac{ig\omega}{2\rho_0 N^2}\hat{f}. \tag{4.59}$$

The perturbation density $\rho'(x,z,t)$ is finally given by the equation

$$\rho'(x,z,t) = \frac{1}{8\pi g}\int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dm \left\{\hat{f}(k,m)\exp(i\omega(k,m)t) + \hat{f}(k,m)\exp(-i\omega(k,m)t)\right\}\exp(-ikx - imz). \tag{4.60}$$

The function $f(x,z)$ and its Fourier transform $\hat{f}(k,m)$ are given by the equations

$$f(x,z) = \frac{1}{2\pi}\exp\left(-\frac{1}{2}x^2\sigma^2 - \frac{1}{2}z^2\tau^2\right)\cos(\tilde{k}x + \tilde{m}z), \tag{4.61}$$

$$\hat{f}(k,m) = \frac{1}{2\sigma\tau}\left\{\exp\left(-\frac{1}{2}\frac{(k-\tilde{k})^2}{\sigma^2} - \frac{1}{2}\frac{(m-\tilde{m})^2}{\tau^2}\right) + \exp\left(-\frac{1}{2}\frac{(k+\tilde{k})^2}{\sigma^2} - \frac{1}{2}\frac{(m+\tilde{m})^2}{\tau^2}\right)\right\}. \tag{4.62}$$

In the animation which follows, we show the results from the numerical evaluation of the inverse Fourier transform in equation (4.60) for a sequence of values of the variable t with $\hat{f}$ given by equation (4.62).

The first animation has as initial condition a Gaussian wave packet with $\sigma = 1/4$, $\tau = 1/4$ and $\tilde{k} = \tilde{m} = \frac{\pi}{2}$. This initial wave packet has a circular shape and splits in two parts as time increases. This two parts propagate in opposite directions from each other. Since the x and z components of the main wavenumber are equal and positive and the wave packet has the same modulation along the x and z directions ($\sigma = \tau$), the two parts of the initial wave packet travels towards the middle of the second and fourth quadrants, as we see in the animation. For the wave packet in the second (fourth) quadrant the group velocity vector points away from the origin towards the middle of the second (fourth) quadrant, so the phase velocity, which is orthogonal to the group velocity, is oriented in the anti-clockwise (clockwise) sense, as we can see from the crests movement in the animation. When the two splited parts of the initial wave packet are still close, we see some constructive and destructive interference. To see this animation, click here.

The second animation has as initial condition a Gaussian wave packet with $\sigma = 1/2$, $\tau = 1/100$ and $\tilde{k} = \tilde{m} = \frac{\pi}{2}$. This initial wave packet has a shape of an elongated ellipse in the x direction. In the movie frame, this initial wave packet looks almost without variation in the x direction. The wave packet splits in two parts as time increases. This two parts propagate in opposite directions from each other, in a way similar to the previous example. The interference effect between the two splited wave packet for earlier times is more intense than what was observed in the previous example, as we can see in the animation. To see it, click here.

The third animation has as initial condition a Gaussian wave packet with $\sigma = 1/2$, $\tau = 1/20$ and $\tilde{k} = \tilde{m} = \frac{\pi}{2}$. This initial wave packet has a shape of an elongated ellipse in the x direction. The movie frame shows the whole wave packet, which splits in two parts as time increases. This two parts propagate in opposite directions from each other, but the group velocity vector has a slightly smaller component in the x direction. This is due to the difference of the modulation of the wave packet in the x and z directions, as we can see in the animation. To see this animation, click here.

4.2 Saint Andrew's Cross.

Here we discuss the wave pattern for internal waves produced by a localized source on a sinusoidal oscillation, like an oscillating cylinder for example, in a fluid with constant density gradient (the buoyancy frequency is constant). For sinusoidal internal waves, the wave energy flux $\overrightarrow{I} = p'\overrightarrow{u}$ (the perturbation pressure p' is given by equation (4.41) and the components of the velocity vector are given by equations (4.44) and (4.35)) averaged over a period is given by the equation

$$\overrightarrow{I} = \frac{1}{2}\frac{w_0^2 N m \rho_0}{k^2 + l^2}\left\{\sin\phi\cos\theta, \sin\phi\sin\theta, -\cos\phi\right\}, \tag{4.63}$$

which is parallel to the group velocity, according to equation (4.46). Therefore, for internal waves the energy propagates in the direction of the group velocity, which is parallel to the surfaces of constant phase. This fact means that internal waves generated by a localized source could never have the familiar appearance of concentric circular crests centered on the source, as we see, for example, for gravity surface waves. Instead, the crests and other surfaces of constant phase stretch radially outward from the source because wave energy travels with the group velocity, which is parallel to surfaces of constant phase.

For a source of definite frequency $\omega \leq N$ (less than the buoyancy frequency), those surfaces are all at a definite angle

$$\phi = \cos^{-1}(\omega/N), \tag{4.64}$$

to the *vertical*; therefore, all the wave energy generated in the source region travels at that angle to the *vertical*. Accordingly, it is confined to a double cone with semi-angle ϕ. The direction of the group velocity vector along the double cone is specified by the fact that energy has to radiate out from the source. The direction of propagation of the lines of constant phase is also specified in terms of the direction of the group velocity and by the fact that the phase velocity

$$\overrightarrow{C} = \frac{N\cos\phi}{|\overrightarrow{k}|}\left\{\cos\phi\cos\theta, \cos\phi\sin\theta, \sin\phi\right\} \tag{4.65}$$

is orthogonal to the group velocity, and that

$$\vec{C} + \vec{C}_g = \frac{N}{|\vec{k}|}\{\cos\theta, \sin\theta, 0\}\,. \tag{4.66}$$

Then, given the direction of the group velocity, the orthogonality of the phase and group velocity plus the condition (4.66), the direction of the phase velocity is specified. If the group velocity has a positive vertical component, the phase velocity has a negative vertical component and vice-versa. The two-dimensional case of an oscillating cylinder is illustrated in figure 4.

This unique property of anisotropy has been verified in dramatic experiments by Mowbray and Stevenson. By oscillating a long cylinder at various frequencies vertically in a stratified fluid, equal phase lines are only found along four beams forming "St. Andrew's Cross", see figure 5 for $\omega/N = 0.7$ and $\omega/N = 0.9$. It can be verified that the angles are $\phi = 45$ degrees for $\omega/N = 0.7$, and $\phi = 26$ degrees for $\omega/N = 0.9$, in close accordance with the condition (4.64).

5 Waveguide behavior.

In this section we study free wave propagation in a continuously stratified fluid in the presence of boundaries, like an ocean or an atmosphere. Attention is restricted to the case in which the bottom is flat, but neither the hydrostatic approximation nor long-wave approximation will be made. The equilibrium state that is being perturbed is the one at rest, so density, and hence buoyancy frequency, is a function only of the vertical coordinate z. We start with an ocean, which has an upper boundary. The atmosphere is somewhat different from the ocean since it has no definite upper boundary, so a study of waves in this situation is made later in this section.

5.1 The oceanic waveguide

Since we assume the undisturbed state as the state of rest, fluid properties are constant on horizontal surfaces and, furthermore, the boundaries are horizontal. Solutions of the perturbation equation (2.27) can be found in the form

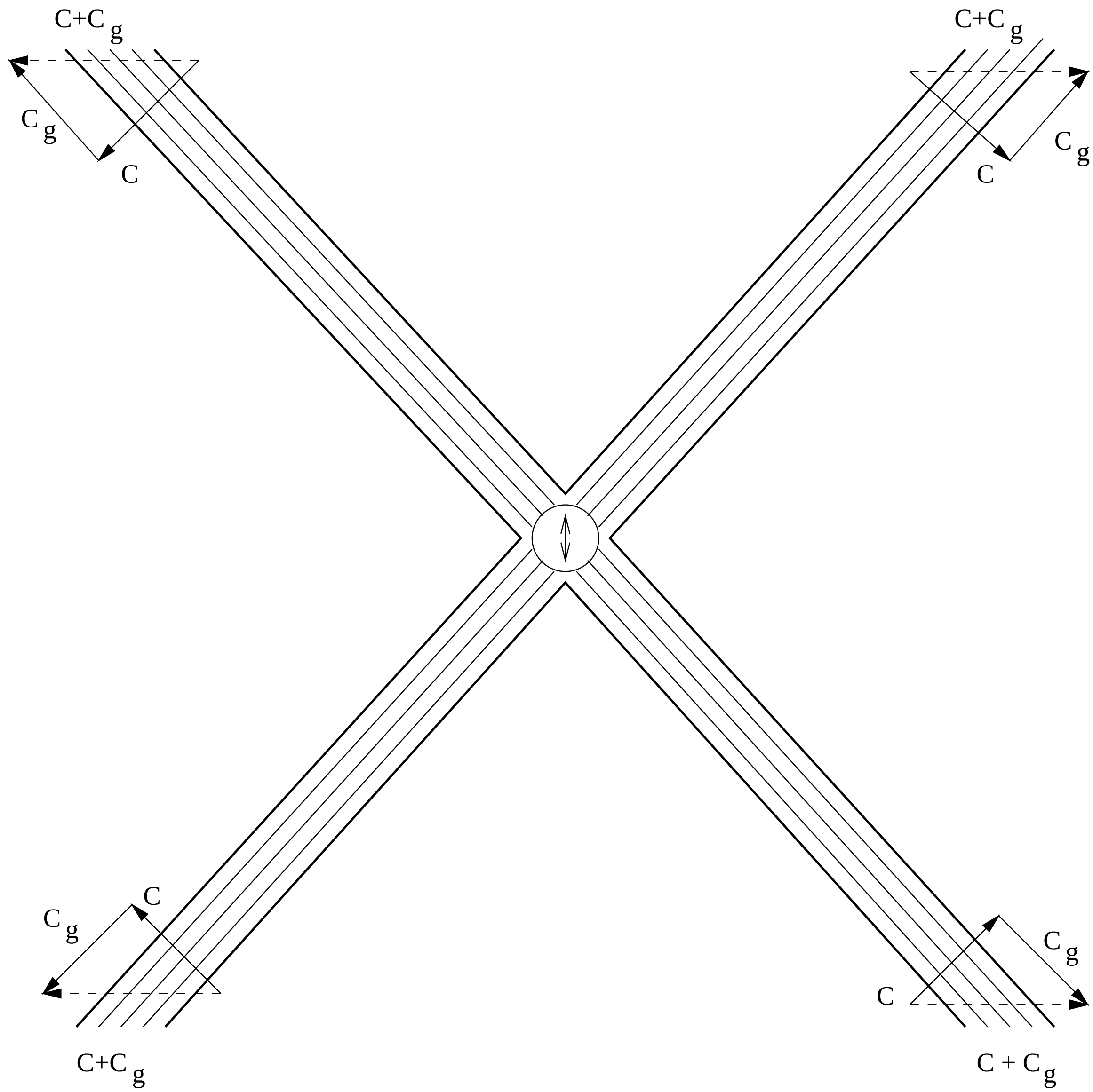

Figure 4: Phase and group velocities.

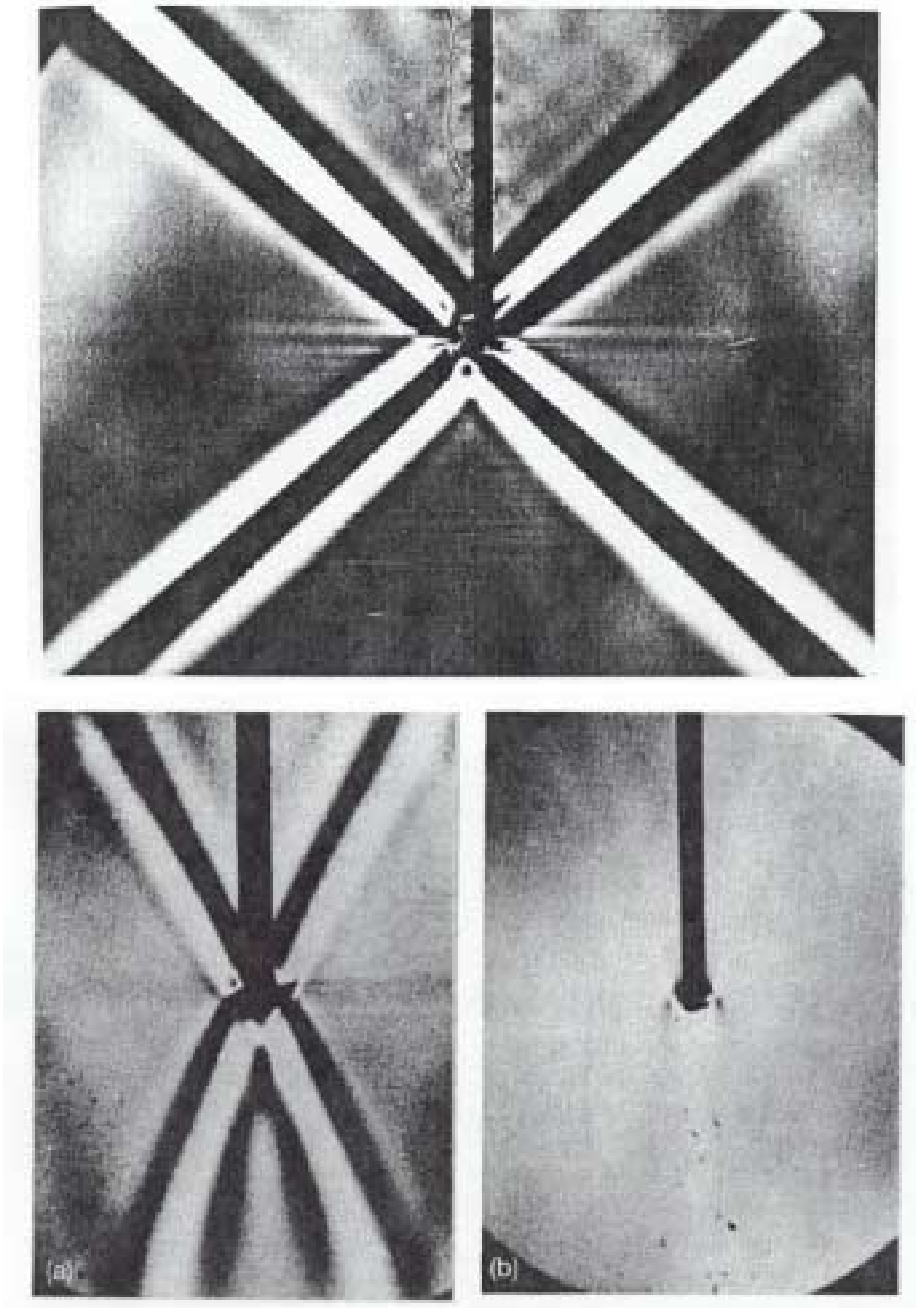

Figure 5: St Andrew's Cross in a stratified fluid. In the top figure $\omega/N = 0.9$ and in the left bottom figure $\omega/N = 0.7$.

$$w(x, y, z, t) = \hat{w}(z) \exp[i(kx + ly - \omega t)] \tag{5.67}$$

The equation for $\hat{w}(z)$ can be found by substitution of equation (5.67) into the governing equation (2.27). We obtain

$$\frac{1}{\bar{\rho}} \frac{\partial}{\partial z} \left[\bar{\rho} \frac{\partial \hat{w}}{\partial z} \right] + \frac{(N^2 - \omega^2)}{\omega^2} (k^2 + l^2) \hat{w}(z) = 0 \tag{5.68}$$

The boundary conditions for this equation are the bottom condition of no flux across it, given by the equation

$$\hat{w}(z) = 0 \text{ at } z = -H, \tag{5.69}$$

and at the free-surface we have the linearized condition

$$\frac{\partial p'}{\partial t} = \bar{\rho} g w(z) \text{ at } z = 0, \tag{5.70}$$

where p' is the perturbation pressure. From this equation we can obtain a free-surface boundary condition for $\hat{w}(z)$. We apply the operator $\frac{\partial}{\partial t}$ to the equation (2.24), and then we substitute equation (5.70) into the resulting equation. As a result, we obtain the equation

$$\frac{\partial^3 w}{\partial t^2 \partial z} = g \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) \text{ at } z = 0 \tag{5.71}$$

Now, if we substitute equation (5.67) into the equation (5.71), we obtain the free-surface boundary condition for $\hat{w}(z)$, which follows

$$\frac{\partial^2 \hat{w}}{\partial z^2} + \frac{(N^2 - \omega^2)}{\omega^2} (k^2 + l^2) \hat{w}(z) = 0 \text{ at } z = 0. \tag{5.72}$$

To simplify the governing equation for $\hat{w}(z)$, we make the Boussinesq approximation, such that equation (5.68) simplifies to

$$\frac{\partial^2 \hat{w}}{\partial z^2} + \frac{(N^2 - \omega^2)}{\omega^2}(k^2 + l^2)\hat{w}(z) = 0, \tag{5.73}$$

with boundary conditions given by equations (5.72) and (5.69). The two boundary (bottom and free-surface) have the effect of confining the wave energy to a region of finite extent, so the ocean can be considered as a *waveguide* that causes the energy to propagate horizontally.

A useful piece of imaginary is to picture internal waves propagating obliquely through the ocean, reflections at the upper and lower boundaries ensuing no loss of energy from the wave guide, whereas horizontal propagation is uninhibited.

Next, we obtain the general solution of equation (5.73) under the boundary conditions (5.72) and (5.69). We first consider the case where $\omega^2 > N^2$. For this case the general solution has the form

$$\hat{w}(z) = \frac{\cosh[m(z + H)]}{\cosh(mH)} \text{ with } m^2 = \frac{(\omega^2 - N^2)}{\omega^2}(k^2 + l^2, \tag{5.74}$$

which already satisfies the bottom boundary condition. The free-surface boundary condition (5.72) gives the dispersion relation

$$m \tanh(mH) = \frac{g}{\omega^2}(k^2 + l^2), \tag{5.75}$$

which is similar to the dispersion relation for surface waves. Actually, the solution (5.74) is not an internal wave, but a surface gravity wave. To have internal waves, we need that $\omega^2 \leq N^2$. This is the next case to consider. We consider the general solution of equation (5.73), which is given by the equation

$$\hat{w}(z) = \sin[m(z + H)] \text{ with } m^2 = \frac{(\omega^2 - N^2)}{\omega^2}(k^2 + l^2), \tag{5.76}$$

which already satisfies the bottom boundary condition. If we substitute equation (5.76) into the free-surface boundary condition (5.72), we obtain the dispersion relation

$$N^2 - \omega^2 = gm \tanh(mH). \tag{5.77}$$

For a given value of the frequency ω, this dispersion relation gives a countable set of values for the modulus of the horizontal component $(k^2 + l^2)$ of the wavenumber, or for a given value of the modulus of the horizontal component of the wavenumber, we have a countable set of possible value for the frequency ω. For ω smaller or of the same order of the buoyancy frequency N, the rigid lid approximation can be made, i. e., the left hand side of equation (5.72) is small compared with the right hand side, so equation (5.72) reduces to

$$\hat{w}(z) = 0 \text{ at } z = 0. \tag{5.78}$$

This boundary condition gives a dispersion relation of the form

$$\sin(mH) = 0 \tag{5.79}$$

or

$$\omega^2 = \frac{(k^2 + l^2)N^2H^2}{n^2\pi^2 + (k^2 + l^2)H^2}, n = 1, 2, 3, \dots, \tag{5.80}$$

which is close to the result given by the dispersion relation given by the free-surface boundary condition (5.77). The value of m for the case with a free-surface is slightly larger than the case with the rigid lid approximation.

If the ocean is perturbed with a spatial structure of one of the modes (a specific value of m for a given ω), then the subsequent behavior in time is described by equation (5.67), i. e., there is an oscillation with a particular frequency. Such a situation, however, is unlikely, so it is necessary to represent the initial structure in space as a *superposition* of modes (for a given ω, we have a countable set of values for $k^2 + l^2$). Then each of these will behave in time as found above, and so the solution can be constructed at all times by taking the appropriate superposition of modes.

5.2 Free Waves in a semi-infinite region.

The atmosphere does not have a definite upper boundary as does the ocean, so solutions of equation (5.73) will now be considered for the case of a semi-infinite domain $z > 0$. In this case there are two types of solutions, the first being typified by the case $N = constant$. The only solutions of equation (5.73) that satisfy the condition at the ground $z = 0$ and remain bounded at infinity are sinusoidal, i. e.,

$$\hat{w}(z) = \sin(mz), \tag{5.81}$$

where m has the same expression as the one given in equation (5.76). There is now no restriction on m, so according to the functional relation between m and ω given in equation (5.76), the frequency ω can have any value in the range $0 \leq \omega < N$, i. e., there is a continuous spectrum of solutions. Superposition of such solutions can be used to solve initial-value problems, and have the from of Fourier integrals.

When N varies with z, there is another type of solution possible, namely, one that satisfies the condition at the ground yet decays as $z \to \infty$. These are waveguide modes, and there are, in general, only a finite number possible. A simple example is provided by the case in which a region of depth H of uniform large buoyancy frequency N_1 underlies a semi-infinite region of uniform small buoyancy frequency N_2. The layer with buoyancy frequency N_1 has depth H and lies at $0 < z < H$ and the semi-infinite layer with buoyancy frequency N_2 lies at $z > H$. For $0 < \omega < N_2$, the solution in both layers has the form given by equation (5.81) with $m = m_1$ in the first layer and $m = m_2$ in the second layer. The wave frequency is constant across the interface of the two layers, which gives the relation

$$\frac{N_1^2}{m_1^2 + k^2 + l^2} = \frac{N_2^2}{m_2^2 + k^2 + l^2} \tag{5.82}$$

between the vertical wavenumbers m_1 and m_2. For this case, the spectrum is continuous and ω can assume any value between 0 and N_2. This is not true for the case when $N_2 < \omega < N_1$, when the frequency ω can assume only a finite set of values in the range $N_2 < \omega < N_1$. In this case, the solution of equation (5.73) for the first layer is given by the equation

$$\hat{w}(z) = \sin(m_1 z) \text{ where } m_1^2 = \frac{(N_1^2 - \omega^2)}{\omega^2}(k^2 + l^2), \tag{5.83}$$

and in the second layer we have solution given by the equation

$$\hat{w}(z) = \exp(-\gamma z) \text{ where } \gamma^2 = \frac{(\omega^2 - N_2^2)}{\omega^2}(k^2 + l^2). \tag{5.84}$$

At the intersection $z = H$ between the two layers, the perturbation pressure p' and the vertical velocity w should be continuous. Alternatively, this condition can be expressed in terms of the ratio

$$Z = \frac{p'}{\rho_0 w}, \tag{5.85}$$

which must be the same on both sides of the boundary. It is convenient to refer to Z as the "impedance". The condition that the impedance in both sides of the layers interface should be the same gives the possible values for ω (eigenvalues). This condition is expressed by the equation

$$\cot^2(m_1 H) = \frac{\omega^2 - N_2^2}{N_1^2 - \omega^2}. \tag{5.86}$$

The spectrum in terms of the wave frequency has a continuous part plus a discrete part, solution of equation (5.86). The modes $\hat{w}$ for $0 < \omega < N_2$ are of sinusoidal shape in both layers, and for $N_2 < \omega < N_1$ the modes $\hat{w}(z)$ are sinusoidal in the first layer and decay exponentially in the second layer. Thus, to deduce how the perturbation will change with time from some initial state, it is necessary to represent this state as a superposition both of discrete waveguide modes and the continuous spectrum of sinusoidal modes. The relative amplitude of the different modes depends on the initial state.