1.138J/2.062J/18.376J, WAVE PROPAGATION

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CHAPTER THREE

TWO DIMENSIONAL WAVES

1 Reflection and tranmission of sound at an interface

Reference : L. M. Brekhovskikh and O. A. Godin: *Acoustics of Layered Media* I. Springer.§.2.2.

The governing equation for sound in a honmogeneous fluid is given by (7.31) and (7.32) in Chapter One. In term of the the veloctiy potential defined by

$$\mathbf{u} = \nabla\phi \tag{1.1}$$

it is

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \nabla^2\phi \tag{1.2}$$

where c denotes the sound speed. Recall that the fluid pressure

$$p = -\rho\partial\phi/\partial t \tag{1.3}$$

also satisfies the same equation.

1.1 Plane wave in Infinite space

Let us first consider a plane sinusoidal wave in three dimensional space

$$\phi(\mathbf{x}, t) = \phi_o e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} = \phi_o e^{i(k\mathbf{n}\cdot\mathbf{x}-\omega t)} \tag{1.4}$$

Here the phase function is

$$\theta(\mathbf{x}, t) = \mathbf{k}\cdot\mathbf{x} - \omega t \tag{1.5}$$

The equation of constant phase $\theta(\mathbf{x}, t) = \theta_o$ describes a moving surface. The wave number vector $\mathbf{k} = k\mathbf{n}$ is defined to be

$$\mathbf{k} = k\mathbf{n} = \nabla\theta \tag{1.6}$$

hence is orthogonal to the surface of constant phase, and represens the direction of wave propagation. The frequency is defined to be

$$\omega = -\frac{\partial\theta}{\partial t} \tag{1.7}$$

Is (1.4) a solution? Let us check (1.2).

$$\nabla\phi = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\phi = i\mathbf{k}\phi$$
$$\nabla^2\phi = \nabla\cdot\nabla\phi = i\mathbf{k}\cdot i\mathbf{k}\phi = -k^2\phi$$
$$\frac{\partial^2\phi}{\partial t^2} = -\omega^2\phi$$

Hence (1.2) is satisfied if

$$\omega = kc. \tag{1.8}$$

Sound in an infinite space is non-dispersive.

1.2 Two-dimensional reflection from a plane interface

REfering to figure **??**, let us cnsider two semi-infinite fluids separated by the plane interface along $z = 0$. The lower fluid is distinguished from the upper fluid by the subscript "1". The densities and sound speeds in the upper and lower fluids are ρ, c and ρ_1, c_1 respectively. Let a plane incident wave arive from $z > 0$ at the incident angle of θ with respect to the z axis, the sound pressure and the velocity potential are

$$p_i = P_0 \exp[ik(x\sin\theta - z\cos\theta)] \tag{1.9}$$

The velocity potential is

$$\phi_i = -\frac{iP_0}{\omega\rho}\exp[ik(x\sin\theta - z\cos\theta] \tag{1.10}$$

The indient wave number vector is

$$\mathbf{k}^i = (k_x^i, k_z^i) = k(\sin\theta, -\cos\theta) \tag{1.11}$$

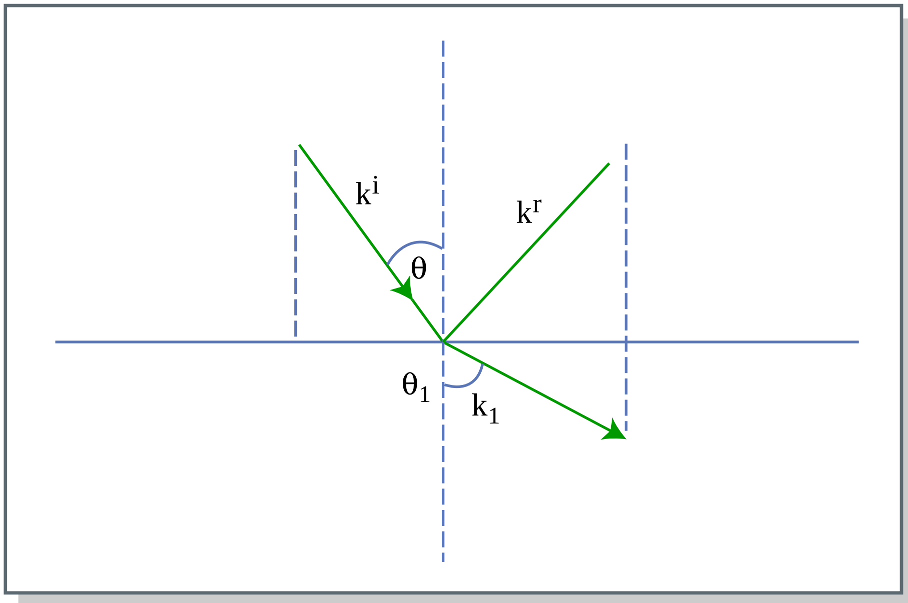


Figure by MIT OCW.

Figure 1: Plane wave incident towards the interface of two fluids.

The motion is confined in the x, z plane.

On the same (incidence) side of the interface we have the reflected wave

$$p_r = R \exp[ik(x \sin \theta + z \cos \theta)] \tag{1.12}$$

where R denotes the reflection coefficient. The wavenumber vector is

$$\mathbf{k}^r = (k^r_x, k^r_z) = k(\sin \theta, \cos \theta) \tag{1.13}$$

The total pressure and potential are

$$p = P_0 \left\{ \exp[ik(x \sin \theta - z \cos \theta)] + R \exp[ik(x \sin \theta + z \cos \theta)] \right\} \tag{1.14}$$

$$\phi = -\frac{iP_0}{\rho\omega} \left\{ \exp[ik(x \sin \theta - z \cos \theta)] + R \exp[ik(x \sin \theta + z \cos \theta)] \right\} \tag{1.15}$$

In the lower medium $z < 0$ the transmitted wave has the pressure

$$p_1 = TP_0 \exp[ik_1(x \sin \theta_1 - z \cos \theta_1)] \tag{1.16}$$

where T is the transmission coefficient, and the potential

$$\phi_1 = -\frac{iP_0}{\rho_1\omega} T \exp[ik_1(x \sin \theta_1 - z \cos \theta_1)] \tag{1.17}$$

Along the interface $z = 0$ we require the continutiy of pressure and normal velocity, i.e.,

$$p = p_1, \quad z = 0 \tag{1.18}$$

and

$$w = w_1, \quad z = 0, \tag{1.19}$$

Applying (1.18), we get

$$P_0 \left\{ e^{ikx\sin\theta} + Re^{ikx\sin\theta} \right\} = TP_0 e^{ik_1 x \sin\theta_1}, \qquad -\infty < x < \infty.$$

Clearly we must have

$$k\sin\theta = k_1 \sin\theta_1 \tag{1.20}$$

or,

$$\frac{\sin\theta}{c} = \frac{\sin\theta_1}{c_1} \tag{1.21}$$

This is just Snell's law. With (1.20), we must have

$$1 + R = T \tag{1.22}$$

Applying (1.19), we have

$$\frac{iP_0}{\rho\omega}\left[-k\cos\theta e^{ik\sin\theta} + Rk\cos\theta e^{ik\sin\theta}\right] = \frac{iP_0}{\rho_1\omega}\left[-k_1\cos\theta_1 T e^{ik_1\sin\theta_1}\right]$$

which implies

$$1 - R = \frac{\rho k_1 \cos\theta_1}{\rho_1 k \cos\theta} T \tag{1.23}$$

Eqs (1.22) and (1.23) can be solved to give

$$T = \frac{2\rho_1 k\cos\theta}{\rho k_1 \cos\theta_1 + \rho_1 k\cos\theta} \tag{1.24}$$

$$R = \frac{\rho_1 k\cos\theta - \rho k_1\cos\theta_1}{\rho_1 k\cos\theta + \rho k_1\cos\theta_1} \tag{1.25}$$

Alternatively, we have

$$T = \frac{2\rho_1 c_1\cos\theta}{\rho c\cos\theta_1 + \rho_1 c_1\cos\theta} \tag{1.26}$$

$$R = \frac{\rho_1 c_1\cos\theta - \rho c\cos\theta_1}{\rho_1 c_1\cos\theta + \rho c\cos\theta_1} \tag{1.27}$$

Let

$$m = \frac{\rho_1}{\rho}, \quad n = \frac{c}{c_1} \tag{1.28}$$

where the ratio of sound speeds n is called the index of refraction. We get after using Snell's law that

$$R = \frac{m\cos\theta - n\cos\theta_1}{m\cos\theta + n\cos\theta_1} = \frac{m\cos\theta - n\sqrt{1-\frac{\sin^2\theta}{n^2}}}{m\cos\theta + n\sqrt{1-\frac{\sin^2\theta}{n^2}}} \tag{1.29}$$

The transmission coefficient is

$$T = 1 + R = \frac{2m\cos\theta}{m\cos\theta + n\sqrt{1-\frac{\sin^2\theta}{n^2}}} \tag{1.30}$$

We now examine the physics.

1. If $n = c/c_1 > 1$, the wave enter from a faster to a slower medium, then $\theta > \theta_1$ and $\sin\theta/n < 1$ always. R is real. In particular, for normal incidence $\theta = \theta_1 = 0$,

$$R = \frac{m-n}{m+n} \tag{1.31}$$

is real. If $m > n > 1$, i.e., $\rho_1/\rho > c/c_1 > 1$, then R is positive. If $n > m > 1$ then R is negative. As θ increases, θ_1 also increases. R decreases until at $\theta = \pi/2$, so that

$$R = -\frac{n}{n} = -1 \tag{1.32}$$

Hence R lies on a segment of the real axis as shown in Figur 1.a. or Figure 1.b. Note that there is no reflection $R = 0$ at the special incidence angle $\theta = \theta_B$, called the Brewster angle:

$$\sin\theta_B = \frac{n}{\sqrt{1-m^2}} \tag{1.33}$$

2. If however $n < 1$, i.e., the wave enters from a slower medium to a faster medium, then $\theta_1 > \theta$. for sufficiently small θ, R is real. When θ increases to a critical value δ, defined by

$$\sin\delta = n \tag{1.34}$$

$\sin\theta_1 = 1$ so that θ_1 becomes $\pi/2$, and

$$R = \frac{m\cos\delta}{m\cos\delta} = 1$$

Below this critical angle $(0 < \theta < \delta)$, R is real. In particular, when $\theta = 0$, (1.31) applies. See Figure 2.c for $m > n$ and in 2.d. for $m < n$.

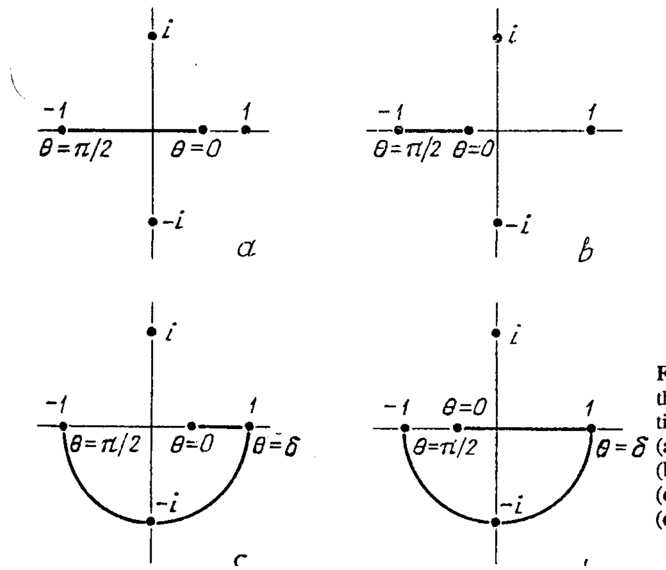


Fig. 2.2a–d. Reflection coefficient in the complex plane for various relations between the media parameters: **(a)** $1 < c/c_1 < \varrho_1/\varrho$; **(b)** $1 < c/c_1$, $\varrho_1/\varrho < c/c_1$; **(c)** $c/c_1 < 1$, $c/c_1 < \varrho_1/\varrho$; **(d)** $\varrho_1/\varrho < c/c_1 < 1$

Figure 2: Complex reflection coefficient. From Brekhovskikh and Godin §.2.2.

When $\theta > \delta$, the square roots above become imaginary. We must then take

$$\cos\theta_1 = \sqrt{1 - \frac{\sin^2\theta}{n^2}} = i\sqrt{\frac{\sin^2\theta}{n^2} - 1} \tag{1.35}$$

The reflection coefficient is now complex

$$R = \frac{m\cos\theta - in\sqrt{\frac{\sin^2\theta}{n^2} - 1}}{m\cos\theta + in\sqrt{\frac{\sin^2\theta}{n^2} - 1}} \tag{1.36}$$

with $|R| = 1$, implying complete reflection. As a check the transmitted wave is now given by

$$p_t = T\exp\left[k_1\left(ix\sin\theta_1 + z\sqrt{\sin^2\theta/n^2 - 1}\right)\right] \tag{1.37}$$

so the amplitude attenuates exponentially in z as $z \to -\infty$. Thus the wave train cannot penetrate much below the interface. The dependence of R on various parameters is best displayed in the complex plane $R = \Re R + i\Im R$. It is clear from (1.36) that $\Im R < 0$ so that R falls on the half circle in the lower half of the complex plane as shown in Figures 2.c and 2.d.

2 Equations for elastic waves

Refs:

Graff: *Wave Motion in Elastic Solids*

Aki & Richards *Quantitative Seismology,* V. 1.

Achenbach. *Wave Propagation in Elastic Solids*

Let the displacement vector at a point x_j and time t be denoted by $u_i(x_j, t)$, then Newton's law applied to an material element of unit volume reads

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \tau_{ij}}{\partial x_j} \tag{2.1}$$

where τ_{ij} is the stress tensor. We have neglected body force such as gravity. For a homogeneous and isotropic elastic solid, we have the following relation between stress and strain

$$\tau_{ij} = \lambda e_{kk} \delta_{ij} + 2\mu e_{ij} \tag{2.2}$$

where λ and μ are Lamé constants and

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{2.3}$$

is the strain tensor. Eq. (2.2) can be inverted to give

$$e_{ij} = \frac{1+\nu}{E}\tau_{ij} - \frac{\nu}{E}\tau_{kk}\delta_{ij} \tag{2.4}$$

where

$$E = \frac{\mu(3\lambda + \mu)}{\lambda + \mu} \tag{2.5}$$

is Young's modulus and

$$\nu = \frac{\lambda}{2(\lambda + \mu)}. \tag{2.6}$$

Poisson's ratio.

Substituting (2.2) and (2.3) into (2.1) we get

$$\begin{aligned}
\frac{\partial \tau_{ij}}{\partial x_j} &= \lambda \frac{\partial e_{kk}}{\partial x_j}\delta_{ij} + \mu \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \\
&= \lambda \frac{\partial e_{kk}}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \mu \frac{\partial^2 u_j}{\partial x_i \partial x_j} \\
&= (\lambda + \mu)\frac{\partial^2 u_j}{\partial x_i x_j} + \mu \nabla^2 u_i
\end{aligned}$$

In vector form (2.1) becomes

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \mu \nabla^2 \mathbf{u} \tag{2.7}$$

Taking the divergence of (2.1) and denoting the dilatation by

$$\Delta \equiv e_{kk} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \tag{2.8}$$

we get the equation governing the dilatation alone

$$\rho \frac{\partial^2 \Delta}{\partial t^2} = (\lambda + \mu)\nabla \cdot \nabla \Delta + \mu \nabla^2 \Delta = (\lambda + 2\mu)\nabla^2 \Delta \tag{2.9}$$

or,

$$\frac{\partial^2 \Delta}{\partial t^2} = c_L^2 \nabla^2 \Delta \tag{2.10}$$

where

$$c_L = \sqrt{\frac{\lambda + 2\mu}{\rho}} \tag{2.11}$$

Thus the dilatation propagates as a wave at the speed c_L. To be explained shortly, this is a longitudinal waves, hence the subscript L. On the other hand, taking the curl of (2.7) and denoting by $\vec{\omega}$ the rotation vector:

$$\vec{\omega} = \nabla \times \mathbf{u} \tag{2.12}$$

we then get the governing equation for the rotation alone

$$\frac{\partial^2 \vec{\omega}}{\partial t^2} = c_T^2 \nabla^2 \vec{\omega} \tag{2.13}$$

where

$$c_T = \sqrt{\frac{\mu}{\rho}} \tag{2.14}$$

Thus the rotation propagates as a wave at the slower speed c_T. The subscript T indicates that this is a transverse wave, to be shown later.

The ratio of two wave speeds is

$$\frac{c_L}{c_T} = \sqrt{\frac{\lambda + \mu}{\mu}} > 1. \tag{2.15}$$

Since

$$\frac{\mu}{\lambda} = \frac{1}{2\nu} - 1 \tag{2.16}$$

it follows that the speed ratio depends only on Poisson's ratio

$$\frac{c_L}{c_T} = \sqrt{\frac{2-2\nu}{1-2\nu}} \tag{2.17}$$

There is a general theorem due to Helmholtz that any vector can be expressed as the sum of an irrotational vector and a solenoidal vector i.e.,

$$\mathbf{u} = \nabla\phi + \nabla \times \mathbf{H} \tag{2.18}$$

subject to the constraint that

$$\nabla \cdot \mathbf{H} = 0 \tag{2.19}$$

The scalar ϕ and the vector $\mathbf{H}$ are called the displacement potentials. Substituting this into (2.7), we get

$$\rho\frac{\partial^2}{\partial t^2}[\nabla\phi + \nabla \times \mathbf{H}] = \mu\nabla^2[\nabla\phi + \nabla \times \mathbf{H}] + (\lambda+\mu)\nabla\nabla\cdot[\nabla\phi + \nabla \times \mathbf{H}]$$

Since $\nabla \cdot \nabla\phi = \nabla^2\phi$, and $\nabla \cdot \nabla \times \mathbf{H} = 0$ we get

$$\nabla\left[(\lambda+2\mu)\nabla^2\phi - \rho\frac{\partial^2\phi}{\partial t^2}\right] + \nabla \times \left[\mu\nabla^2\mathbf{H} - \rho\frac{\partial^2\mathbf{H}}{\partial t^2}\right] = 0 \tag{2.20}$$

Clearly the above equation is satisied if

$$(\lambda+2\mu)\nabla^2\phi - \rho\frac{\partial^2\phi}{\partial t^2} = 0 \tag{2.21}$$

and

$$\mu\nabla^2\mathbf{H} - \rho\frac{\partial^2\mathbf{H}}{\partial t^2} = 0 \tag{2.22}$$

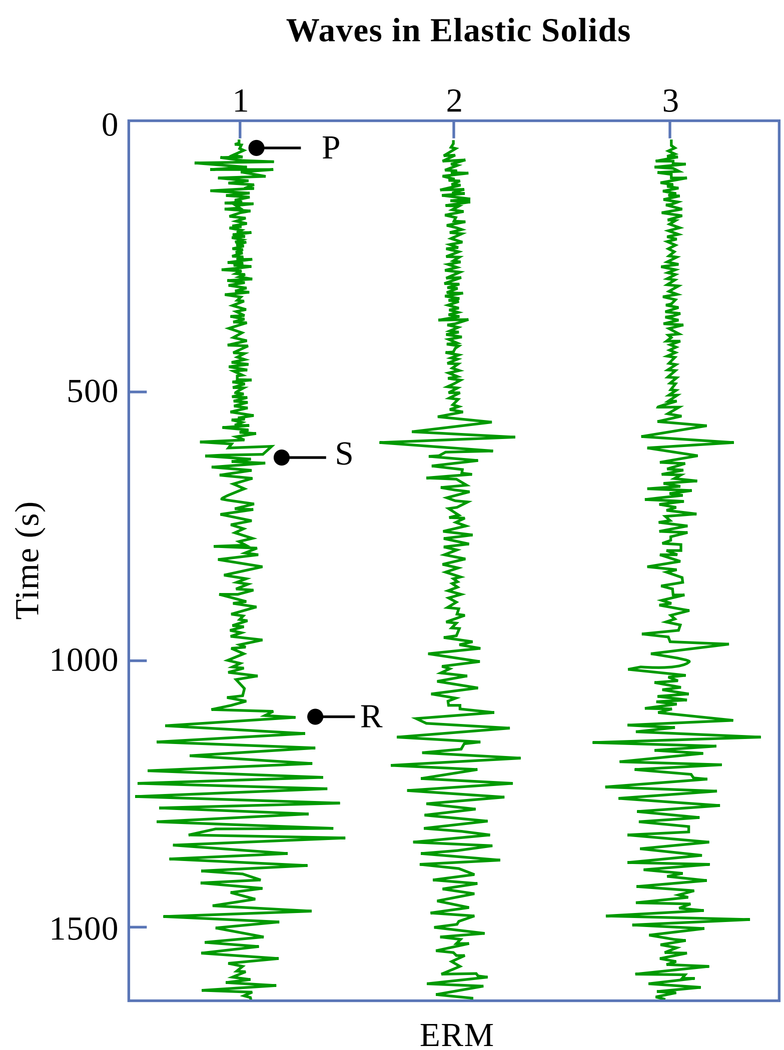
Although the governing equations are simplified, the two potentials are usually coupled by boundary conditions, unless the physical domain is infinite.

A typical seismic record is shown in Figure 2.

3 Free waves in infinite space

The dilatational wave equation admits a plane sinusoidal wave solution:

$$\phi(\mathbf{x},t) = \phi_o e^{ik(\mathbf{n}\cdot\mathbf{x}-c_L t)} \tag{3.1}$$



A typical seismogram during an earthquake, showing the arrival of the P(primary), S(secondary) and R(Rayleigh) waves. Note that the Rayleigh waves have the largest amplitude.

Figure by MIT OCW.

Figure 3: A typical seismic record. P: Longitudinal wave; S: Transverse wave; R : Rayleigh surface wave. From Billingham & King. *Wave Motion*, Cambridge U Press.

Here the phase function is

$$\theta(\mathbf{x}, t) = k(\mathbf{n} \cdot \mathbf{x} - c_L t) \tag{3.2}$$

which describes a moving surface. The wave number vector $\mathbf{k} = k\mathbf{n}$ is defined to be

$$\mathbf{k} = k\mathbf{n} = \nabla\theta \tag{3.3}$$

hence is orthogonal to the surface of constant phase, and represents the direction of wave propagation. The frequency is

$$\omega = kc_T = -\frac{\partial\theta}{\partial t} \tag{3.4}$$

A general solution is

$$\phi = \phi(\mathbf{n} \cdot \mathbf{x} - c_L t) \tag{3.5}$$

Similarly the the following sinusoidal wave is a solution to the shear wave equation;

$$\mathbf{H} = \mathbf{H}_o e^{ik(\mathbf{n}\cdot\mathbf{x} - c_T t)} \tag{3.6}$$

A general solution is

$$\mathbf{H} = \mathbf{H}(\mathbf{n} \cdot \mathbf{x} - c_T t) \tag{3.7}$$

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Note:

We can also write (3.5) and (3.9) as

$$\phi = \phi(t - \frac{\mathbf{n} \cdot \mathbf{x}}{c_L}) \tag{3.8}$$

and

$$\mathbf{H} = \mathbf{H}(t - \frac{\mathbf{n} \cdot \mathbf{x}}{c_T}) \tag{3.9}$$

where

$$\mathbf{s}_L = \frac{\mathbf{n}}{c_L}, \quad \mathbf{s}_T = \frac{\mathbf{n}}{c_T} \tag{3.10}$$

are called the slowness vectors of longitudinal and transverse waves respectively.

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In a dilatational wave the displacement vector is parallel to the wave number vector:

$$\mathbf{u}_L = \nabla\phi = \phi'\mathbf{n} \tag{3.11}$$

from (3.5), where ϕ' is the ordinary derivative of ϕ with repect to its argument. Hence the dilatational wave is a *longitudinal* (compression) wave. On the other hand in a rotational wave the displacement vector is perpendicular to the wave number vector,

$$\begin{aligned}
\mathbf{u}_T &= \nabla \times \mathbf{H} = \mathbf{e}_x \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) + \mathbf{e}_y \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) + \mathbf{e}_z \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \\
&= \mathbf{e}_x \left(H_z' n_y - H_y' n_z \right) + \mathbf{e}_y \left(H_x' n_z - H_z' n_x \right) + \mathbf{e}_z \left(H_y' n_x - H_x' n_y \right) \\
&= \mathbf{n} \times \mathbf{H}' \qquad (3.12)
\end{aligned}$$

from (3.7). Hence a rotational wave is a *transverse* (shear) wave.

4 Elastic waves in a plane

Refs. Graff, Achenbach,

Aki and Richards : *Quantitative Seismology, v.1*

Let us examine waves propagating in the vertical plane of x, y. All physical quantities are assumed to be uniform in the direction of z, hence $\partial/\partial z = 0$, then

$$u_x = \frac{\partial \phi}{\partial x} + \frac{\partial H_z}{\partial y}, \quad u_y = \frac{\partial \phi}{\partial y} - \frac{\partial H_z}{\partial x}, \quad u_z = -\frac{\partial H_x}{\partial y} + \frac{\partial H_y}{\partial x} \qquad (4.13)$$

and

$$\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} = 0 \qquad (4.14)$$

where

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{1}{c_L^2} \frac{\partial^2 \phi}{\partial t^2}, \qquad (4.15)$$

$$\frac{\partial^2 H_p}{\partial x^2} + \frac{\partial^2 H_p}{\partial y^2} = \frac{1}{c_T^2} \frac{\partial^2 H_p}{\partial t^2}, \quad p = x, y, z \qquad (4.16)$$

Note that u_z is also governed by (4.16).

Note that the in-plane displacements u_x, u_y depend only on ϕ and H_z, and not on H_x, H_y. Out-of-plane motion u_z depends on H_x, H_y but not on H_z. Hence the in-plane displacement components u_x, u_y are independent of the out-of-plane component u_z. The in-plane displacements (u_x, u_y) are associated with dilatation and in-plane shear, represented respectively by ϕ and H_z, which will be refered to as the P wave and the SV wave. The out-of-plane displacement u_z is associated with H_x and H_y, and will be refered to as the SH wave.

From Hooke's law the stress components can be written

$$\begin{aligned}\tau_{xx} &= \lambda\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}\right)+2\mu\frac{\partial u_x}{\partial x}=(\lambda+2\mu)\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}\right)-2\mu\frac{\partial u_y}{\partial y}\\ &= (\lambda+2\mu)\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)-2\mu\left(\frac{\partial^2\phi}{\partial y^2}-\frac{\partial^2 H_z}{\partial y\partial x}\right) \qquad (4.17)\end{aligned}$$

$$\begin{aligned}\tau_{yy} &= \lambda\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}\right)+2\mu\frac{\partial u_y}{\partial y}=(\lambda+2\mu)\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}\right)-2\mu\frac{\partial u_x}{\partial x}\\ &= (\lambda+2\mu)\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)-2\mu\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2 H_z}{\partial x\partial y}\right) \qquad (4.18)\end{aligned}$$

$$\tau_{zz}=\frac{\lambda}{2(\lambda+\mu)}(\tau_{xx}+\tau_{yy})=\nu(\tau_{xx}+\tau_{yy})=\lambda\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right) \qquad (4.19)$$

$$\tau_{xy}=\mu\left(\frac{\partial u_y}{\partial x}+\frac{\partial u_x}{\partial y}\right)=\mu\left(2\frac{\partial^2\phi}{\partial x\partial y}-\frac{\partial^2 H_z}{\partial x^2}+\frac{\partial^2 H_z}{\partial y^2}\right) \qquad (4.20)$$

$$\tau_{yz}=\mu\frac{\partial u_z}{\partial y}=\mu\left(-\frac{\partial^2 H_x}{\partial y^2}+\frac{\partial^2 H_y}{\partial y\partial x}\right) \qquad (4.21)$$

$$\tau_{xz}=\mu\frac{\partial u_z}{\partial x}=\mu\left(-\frac{\partial^2 H_x}{\partial\partial x\partial y}+\frac{\partial^2 H_y}{\partial x^2}\right) \qquad (4.22)$$

Different physical situations arise for different boundary conditions. We shall consider first the half plane problem bounded by the plane $y=0$.

5 Reflection of elastic waves from a plane boundary

Consider the half space $y>0$, $-\infty<x<\infty$. Several types of boundary conditions can be prescribed on the plane boundary : (i) dynamic: the stress components only (the traction condition); (ii) kinematic: the displacement components only, or (iii). a combination of stress components and displacement components. Most difficult are (iv) the mixed conditions in which stresses are given over part of the boundary and displacements over the other.

We consider the simplest case where the plane $y=0$ is completely free of external stresses,

$$\tau_{yy}=\tau_{xy}=0, \qquad (5.23)$$

and

$$\tau_{yz}=0 \qquad (5.24)$$

It is clear that (5.23) affects the P and SV waves only, while (5.24) affects the SH wave only. Therefore we have two uncoupled problems each of which can be treated separately.

5.1 P and SV waves

Consider the case where only P and SV waves are present, then $H_x = H_y = 0$. Let all waves have wavenumber vectors inclined in the positve x direction:

$$\phi = f(y)e^{i\xi x - i\omega t}, \quad H_z = h_z(y)e^{i\zeta x - i\omega t} \tag{5.25}$$

It follows from (4.15) and (4.16) that

$$\frac{d^2 f}{dy^2} + \alpha^2 f = 0, \quad \frac{d^2 h_z}{dy^2} + \beta^2 h_z = 0, \tag{5.26}$$

with

$$\alpha = \sqrt{\frac{\omega^2}{c_L^2} - \xi^2} = \sqrt{k_L^2 - \xi^2}, \quad \beta = \sqrt{\frac{\omega^2}{c_T^2} - \zeta^2} = \sqrt{k_T^2 - \zeta^2} \tag{5.27}$$

We first take the square roots to be real; the general solution to (5.26) are sinusoids, hence,

$$\phi = A_P e^{i(\xi x - \alpha y - \omega t)} + B_P e^{i(\xi x + \alpha y - \omega t)}, \quad H_z = A_S e^{i(\zeta x - \beta y - \omega t)} + B_S e^{i(\zeta x + \beta y - \omega t)} \tag{5.28}$$

On the right-hand sides the first terms are the incident waves and the second are the reflected waves. If the incident amplitudes A_P, A_S and are given, what are the properties of the reflected waves B_P, B_S? The wave number components can be written in the polar form:

$$(\xi, \alpha) = k_L(\sin\theta_L, \cos\theta_L), \quad (\zeta, \beta) = k_T(\sin\theta_T, \cos\theta_T) \tag{5.29}$$

where (k_L, k_T) are the wavenumbers, the (θ_L, θ_T) the directions of the P wave and SV wave, respectively. In terms of these we rewrite (5.28)

$$\phi = A_P e^{ik_L(\sin\theta_L x - \cos\theta_L y - \omega t)} + B_P e^{ik_L(\sin\theta_L x + \cos\theta_L y - \omega t)} \tag{5.30}$$

$$H_z = A_S e^{ik_T(\sin\theta_T x - \cos\theta_L y - \omega t)} + B_S e^{ik_T(\sin\theta_T x + \cos\theta_T y - \omega t)} \tag{5.31}$$

In order to satisfy (5.23) ($\tau_{yy} = \tau_{xy} = 0$) on $y = 0$ for all x, we must insist:

$$k_L \sin\theta_L = k_T \sin\theta_T, \quad (\xi = \zeta) \tag{5.32}$$

This is in the form of Snell's law:

$$\frac{\sin\theta_L}{c_L} = \frac{\sin\theta_T}{c_T} \tag{5.33}$$

implying

$$\frac{\sin\theta_L}{\sin\theta_T} = \frac{c_L}{c_T} = \sqrt{\frac{\lambda+2\mu}{\mu}} = \frac{k_T}{k_L} \equiv \kappa \tag{5.34}$$

When (5.23) are applied on $y = 0$ the exponential factors cancel, and we get two algebraic conditions for the two unknown amplitudes of the reflected waves (B_P, B_S) :

$$k_L^2(2\sin^2\theta_L - \kappa^2)(A_P + B_P) - k_T^2 \sin 2\theta_T(A_S - B_S) = 0 \tag{5.35}$$

$$k_L^2 \sin 2\theta_L(A_P - B_P) - k_T^2 \cos\theta_T(A_S + B_S) = 0. \tag{5.36}$$

Using (5.34), we get

$$2\sin^2\theta_L - \kappa^2 = \kappa^2(2\sin^2\theta_T - 1) = -\kappa^2\cos 2\theta_T$$

The two equations can be solved and the solution expressed in matrix form:

$$\left\{\begin{array}{c} B_P \\ B_S \end{array}\right\} = \left[\begin{array}{cc} S_{PP} & S_{SP} \\ S_{PS} & S_{SS} \end{array}\right] \left\{\begin{array}{c} A_P \\ A_S \end{array}\right\} \tag{5.37}$$

where

$$\mathbf{S} = \left[\begin{array}{cc} S_{PP} & S_{SP} \\ S_{PS} & S_{SS} \end{array}\right] \tag{5.38}$$

denotes the scattering matrrix. Thus S_{PS} represents the reflected S-wave due to incident P wave of unit amplitude, etc. It is straightforward to verify that

$$S_{PP} = \frac{\sin 2\theta_L \sin 2\theta_T - \kappa^2\cos^2 2\theta_T}{\sin 2\theta_L \sin 2\theta_T + \kappa^2\cos^2 2\theta_T} \tag{5.39}$$

$$S_{SP} = \frac{-2\kappa^2 \sin 2\theta_T \cos 2\theta_T}{\sin 2\theta_L \sin 2\theta_T + \kappa^2\cos^2 2\theta_T} \tag{5.40}$$

$$S_{PS} = \frac{2\sin 2\theta_L \cos 2\theta_T}{\sin 2\theta_L \sin 2\theta_T + \kappa^2\cos^2 2\theta_T} \tag{5.41}$$

$$S_{SS} = \frac{\sin 2\theta_L \sin 2\theta_T - \kappa^2\cos^2 2\theta_T}{\sin 2\theta_L \sin 2\theta_T + \kappa^2\cos^2 2\theta_T} \tag{5.42}$$

In view of (5.33) and

$$\kappa = \frac{c_L}{c_T} = \sqrt{\frac{2-2\nu}{1-2\nu}} \tag{5.43}$$

The scattering matrix is a function of Poisson's ratio and the angle of incidence.

(i) P- wave Incidence : In this case θ_L is the incidence angle. Consider the special case when the only incident wave is a P wave. Then $A_P \neq 0$ and $A_S = 0$ and only S_{PP} and S_{SP} are relevant. . Note first that $\theta_L > \theta_T$ in general . For normal incidence, $\theta_L = 0$, hence $\theta_T = 0$. We find

$$S_{PP} = -1, \quad S_{PS} = 0 \tag{5.44}$$

there is no SV wave. The refelcted wave is a P wave. On the other hand if

$$\sin 2\theta_L \sin 2\theta_T - \kappa^2 \cos^2 2\theta_T = 0 \tag{5.45}$$

then $S_{PP} = 0$, hence $B_P = 0$ but $B_S \neq 0$; only SV wave is reflected. This is the case of mode conversion, whereby an incident P waves changes to a SV wave after reflection. The amplitude of the reflected SV wave is

$$\frac{B_S}{A_P} = S_{PS} = \frac{\tan 2\theta_T}{\kappa^2} \tag{5.46}$$

(ii) SV wave Incidence : Let $A_P = 0$ but $A_S \neq 0$. In this case θ_T is the incidence angle. Then only S_{SP} and S_{SS} are relevant. For normal incidence, $\theta_L = \theta_T = 0$, $S_{SS} = -1$, and $S_{SP} = 0$; no P wave is reflected. Mode conversion ($B_P \neq 0, B_S = 0$) also happens when (5.45) is satisfied. Since $\theta_L > \theta_T$, there is a critical incidence angle θ_T beyond which the P wave cannot be reflected back into the solid and propagates only along the x axis. At the critical angle

$$\sin\theta_L = 1, \ (\theta_L = \pi/2), \quad \text{or} \quad \sin\theta_T = 1/\kappa \tag{5.47}$$

by Snell's law. Thus for $\nu = 1/3$, $\kappa = 2$ and the critical incidence angle is $\theta_T = 30°$.

Beyond the critical angle of incidence, the P waves decay exponentially away from the free surface. The amplitude of the SV wave is linear in y which is unphysical, suggesting the limitation of unbounded space assumption.

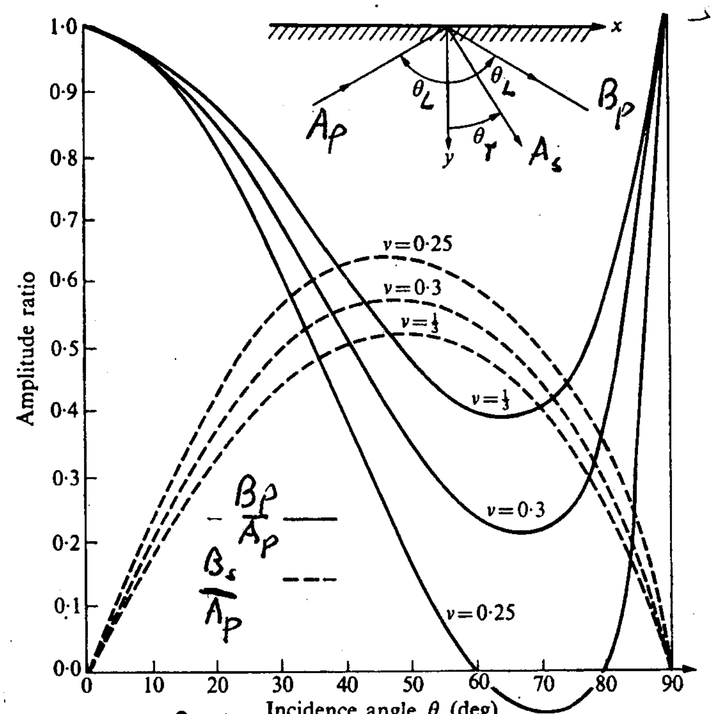

FIG. 6.3. Amplitude ratios B_P/A_P, B_S/A_P for incident P waves, for various Poisson's ratios, with a ray representation of the reflection also shown.

From Graff

Figure 4: Amplitude ratios for incident P waves for various Possion's ratios. From Graff: *Waves in Elastic Solids*. Symbols should be converted according to : $A_1 \to A_P, A_2 \to B_P, B_1 \to A_S, B_2 \to B_S$.

320 Waves in semi-infinite media

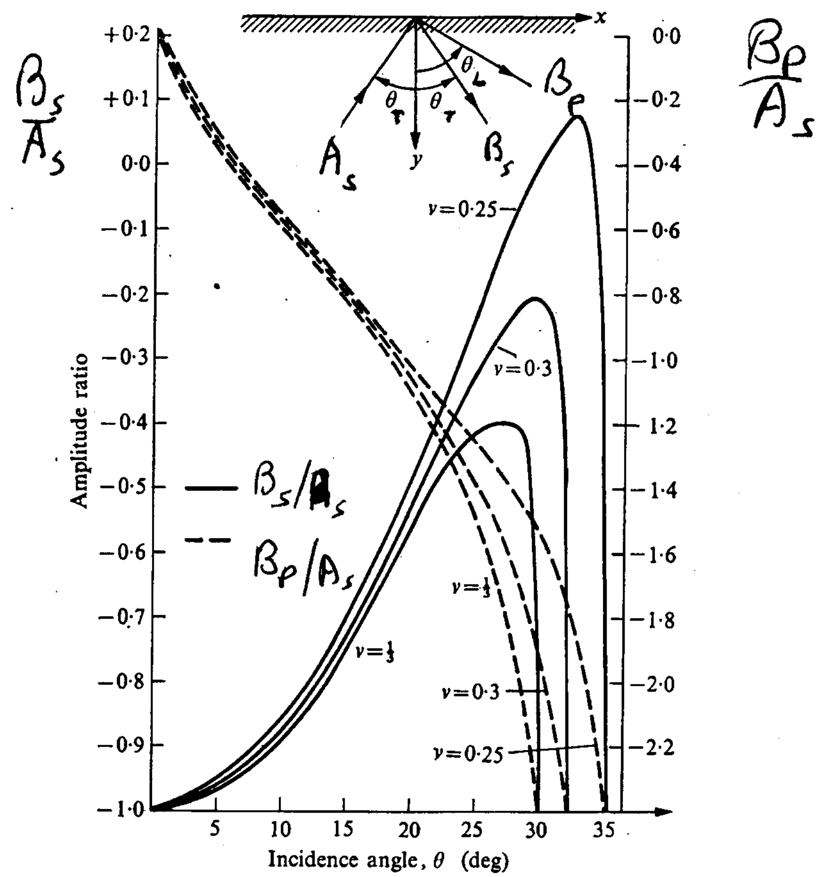

FIG. 6.4. Reflected wave amplitude ratios . , for incident SV waves and various Poisson's ratios, with the ray representation of the reflection also shown.

From Graff $\frac{B_P}{A_S}, \frac{B_S}{A_S}$

Figure 5: Reflected wave amplitude ratios for incident SV waves for various Possion's ratios. From Graff: *Waves in Elastic Solids*. Symbols should be converted according to : $A_1 \to A_P, A_2 \to B_P, B_1 \to A_S, B_2 \to B_S$.

5.2 SH wave

Because of (5.2)

$$\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} = 0$$

we can introduce a stream function ψ so that

$$H_x = -\frac{\partial \psi}{\partial y}, \quad H_y = \frac{\partial \psi}{\partial x} \tag{5.48}$$

where

$$\nabla^2 \psi = \frac{1}{c_T^2}\frac{\partial^2 \psi}{\partial t^2} \tag{5.49}$$

Clearly the out-of-plane dispacement is

$$u_z = -\frac{\partial H_x}{\partial y} + \frac{\partial H_y}{\partial x} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla^2 \psi \tag{5.50}$$

and

$$\tau_{yz} = \mu \frac{\partial}{\partial y}\nabla^2 \psi = \frac{\mu}{c_T^2}\frac{\partial}{\partial y}\frac{\partial^2 \psi}{\partial t^2} \tag{5.51}$$

The zero-stress boundary condition implies

$$\frac{\partial \psi}{\partial y} = 0 \tag{5.52}$$

Thus the problem for ψ is analogous to one for sound waves reflected by a solid plane. Again for monochromatic incident waves, the solution is easily shown to be

$$\psi = \left(Ae^{-i\beta y} - Ae^{i\beta y}\right) e^{i\alpha x - i\omega t} \tag{5.53}$$

where

$$\alpha^2 + \beta^2 = k_T^2 \tag{5.54}$$

We remark that when the boundary is any cylindrical surface with axis parallel to the z axis, the the stress-free condition reads

$$\tau_{zn} = 0, \quad \text{on} \quad B. \tag{5.55}$$

where n is the unit outward normal to B. Since in the pure SH wave problem

$$\tau_{zn} = \mu \frac{\partial u_z}{\partial n} = \frac{\partial}{\partial n}\nabla^2 \psi = \frac{\mu}{c_T^2}\frac{\partial}{\partial n}\frac{\partial^2 \psi}{\partial t^2}$$

Condition (5.55) implies

$$\frac{\partial \psi}{\partial n} = 0, \quad \text{on} \quad B. \tag{5.56}$$

Thus the analogy to acoustic scattering by a hard object is true irrespective of the geomntry of the scatterer.

6 Rayleigh surface waves

Refs. Graff, Achenbach, Fung

In a homogeneous elastic half plane, in addition to P, SV and SH waves, another wave which is trapped along the surface of a half plane can also be present. Because most of the action is near the surface, this *surface wave* is of special importance to seismic effects on the ground surface.

Let us start from the governing equations again

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{1}{c_L^2}\frac{\partial^2 \phi}{\partial t^2}, \tag{6.1}$$

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} = \frac{1}{c_T^2}\frac{\partial^2 H_z}{\partial t^2} \tag{6.2}$$

We now seek waves propagating along the x direction

$$\phi = \Re\left(f(y)e^{i\xi x - i\omega t}\right), \quad H_z = \Re\left(h(y)e^{i\xi x - i\omega t}\right) \tag{6.3}$$

Then $f(y), h(y)$ must satisfy

$$\frac{d^2 f}{dy^2} + \left(\omega^2/c_L^2 - \xi^2\right) f = 0, \qquad \frac{d^2 h}{dy^2} + \left(\omega^2/c_T^2 - \xi^2\right) h = 0, \tag{6.4}$$

To have surface waves we insist that

$$\overline{\alpha} = \sqrt{\xi^2 - \omega^2/c_L^2}, \quad \overline{\beta} = \sqrt{\xi^2 - \omega^2/c_T^2} \tag{6.5}$$

be real and postive. Keeping only the solutions which are bounded for $y \sim \infty$, we get

$$\phi = Ae^{-\overline{\alpha}y}e^{i(\xi x - \omega t)}, \quad H_z = Be^{-\overline{\beta}y}e^{i(\xi x - \omega t)}. \tag{6.6}$$

The expressions for the displacements and stresses can be found straightforwardly.

$$u_x = \left(i\xi Ae^{-\overline{\alpha}y} - \overline{\beta}Be^{-\overline{\beta}y}\right)e^{i(\xi x-\omega t)}, \tag{6.7}$$

$$u_y = -\left(\overline{\alpha}Ae^{-\overline{\alpha}y} + i\xi Be^{-\overline{\beta}y}\right)e^{i(\xi x-\omega t)}, \tag{6.8}$$

$$\tau_{xx} = \mu\left\{\left(\overline{\beta}^2 - \xi^2 - 2\overline{\alpha}^2\right)Ae^{-\overline{\alpha}y} - 2i\overline{\beta}\xi Be^{-\overline{\beta}y}\right\}e^{i(\xi x-\omega t)}, \tag{6.9}$$

$$\tau_{yy} = \mu\left\{\left(\overline{\beta}^2 + \xi^2\right)Ae^{-\overline{\alpha}y} + 2i\overline{\beta}\xi Be^{-\overline{\beta}y}\right\}e^{i(\xi x-\omega t)}, \tag{6.10}$$

$$\tau_{xy} = \mu\left\{-2i\overline{\alpha}\xi Ae^{-\overline{\alpha}y} + \left(\xi^2 + \overline{\beta}^2\right)Be^{-\overline{\beta}y}\right\}e^{i(\xi x-\omega t)} \tag{6.11}$$

On the free surface the traction-free conditions $\tau_{yy} = \tau_{xy} = 0$ require that

$$\left(\overline{\beta}^2 + \xi^2\right) A + 2i\overline{\beta}\xi B = 0, \tag{6.12}$$

$$-2i\overline{\alpha}\xi A + \left(\overline{\beta}^2 + \xi^2\right) B = 0. \tag{6.13}$$

For nontrivial solutions of A, B the coefficient determinant must vanish,

$$\left(\overline{\beta}^2 + \xi^2\right)^2 - 4\overline{\alpha}\overline{\beta}\xi^2 = 0, \tag{6.14}$$

or

$$\left[2\xi^2 - \frac{\omega^2}{c_T^2}\right]^2 - 4\xi^2\sqrt{\xi^2 - \frac{\omega^2}{c_L^2}}\sqrt{\xi^2 - \frac{\omega^2}{c_T^2}} = 0 \tag{6.15}$$

which is the dispersion relation between frequency ω and wavenumber ξ. From either (6.12) or (6.13) we get the amplitude ratio:

$$\frac{A}{B} = -\frac{2i\overline{\beta}\xi}{\overline{\beta}^2 + \xi^2} = \frac{\overline{\beta}^2 + \xi^2}{2i\overline{\alpha}\xi}, \tag{6.16}$$

In terms of the wave velocity $c = \omega/\xi$, (6.15) becomes

$$\left(2 - \frac{c^2}{c_T^2}\right)^2 = 4\left(1 - \frac{c^2}{c_L^2}\right)^{\frac{1}{2}}\left(1 - \frac{c^2}{c_T^2}\right)^{\frac{1}{2}}. \tag{6.17}$$

or, upon squaring both sides, finally

$$\frac{c^2}{c_T^2}\left\{\left(\frac{c}{c_T}\right)^6 - 8\left(\frac{c}{c_T}\right)^4 + \left(24 - \frac{16}{\kappa^2}\right)\left(\frac{c}{c_T}\right)^2 - 16\left(1 - \frac{1}{\kappa^2}\right)\right\} = 0. \tag{6.18}$$

where

$$k = \frac{c_L}{c_T} = \sqrt{\frac{\lambda + 2\mu}{\mu}} = \sqrt{\frac{2 - 2\nu}{1 - 2\nu}}$$

The first solution $c = \omega = 0$ is at best a static problem. In fact $\overline{\alpha} = \overline{\beta} = \xi$ and $A = -iB$, so that $u_x = u_y \equiv 0$ which is of no interest.

We need only consider the cubic equation for c^2. Note that the roots of the cubic equation depend only on Poisson's ratio, through $\kappa^2 = 2(1-\nu)/(1-2\nu)$. There can be three real roots for c or ω, or one real root and two complex-conjugate roots. We rule out the latter because the complex roots imply either temporal damping or instability; neither of which is a propagating wave. When all three roots are real we must pick the one so that both $\overline{\alpha}$ and $\overline{\beta}$ are real. We shall denote the speed of Rayleigh wave by c_R.

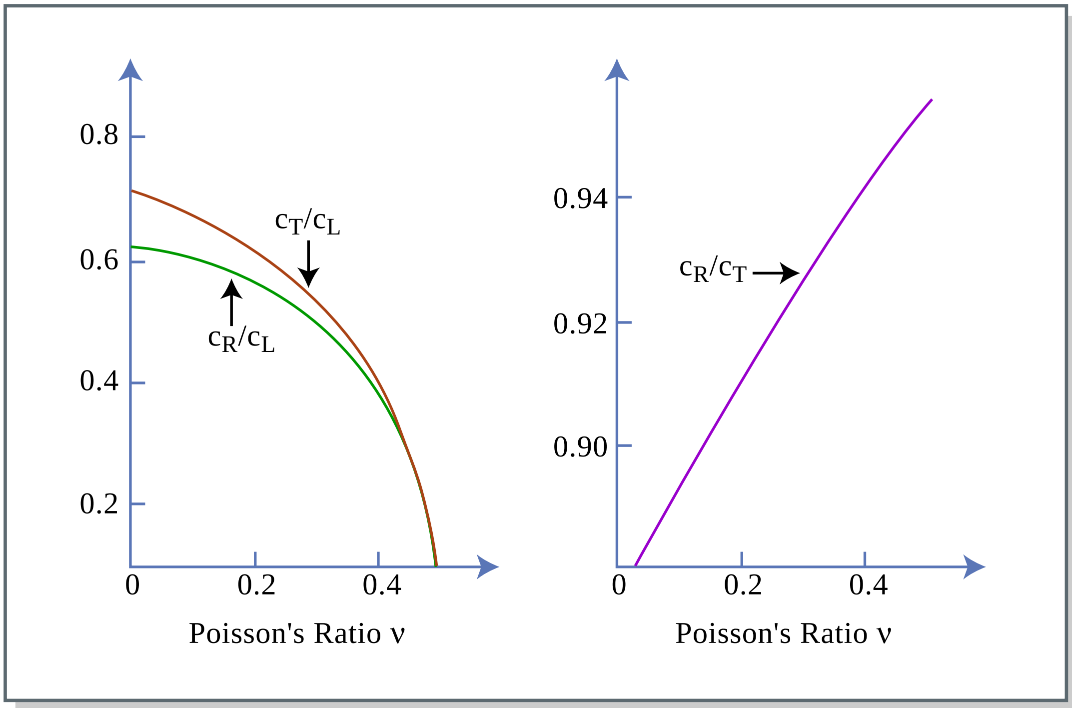

Figure by MIT OCW.

Figure 6: The velocity of Rayleigh surface waves c_R. From Fung *Foundations of Solid Mechanics*.

For $c = 0$, the factor in curley brackets is

$$\{.\} = -16\left(1 - \frac{c_T^2}{c_L^2}\right) < 0$$

For $c = c_T$ the same factor is equal to unity and hence positive. There must be a solution for c such that $0 < c < c_T$. Furthermore, we cannot have roots in the range $c/c_T > 1$. If so,

$$\overline{\beta}^2 = \xi^2\left(1 - \frac{c^2}{c_T^2}\right) < 0$$

which is not a surface wave. Thus the surface wave, if it exists, is slower than the shear wave.

Numerical studies for the entire range of Poisson's ratio $(0 < \nu < 0.5)$ have shown that there are one real and two complex conjugate roots if $\nu > 0.263\ldots$ and three real roots if $\nu < 0.263\ldots$. But there is only one real root that gives the surface wave velocity c_R. A graph of c_R for all values of Poisson's ratio, due to Knopoff , is shown in Fig. 6. A curve-fitted expression for the Rayleigh wave velocity is

$$c_R/c_T = (0 \cdot 87 + 1 \cdot 12\nu)/(1 + \nu). \tag{6.19}$$

For rocks, $\lambda = \mu$ and $\nu = \frac{1}{4}$, the roots are

$$(c/c_T)^2 = 4, 2 + 2/\sqrt{3}, 2 - 2/\sqrt{3}. \tag{6.20}$$

The only acceptable root for Rayleigh wave speed c_R is

$$(c_R/c_T)^2 = (2 - 2/\sqrt{3})^{\frac{1}{2}} = 0 \cdot 9194 \tag{6.21}$$

or

$$c_R = 0.9588 c_T. \tag{6.22}$$

The particle displacement of a particle on the free surface is, from (6.7) and (6.8)

$$u_x = iA\left(\xi - \frac{\overline{\beta}^2 + \xi^2}{2\xi}\right) e^{i(\xi x - \omega t)} \tag{6.23}$$

$$u_y = A\left(-\overline{\alpha} + \frac{\overline{\beta}^2 + \xi^2}{2\xi}\right) e^{i(\xi x - \omega t)} \tag{6.24}$$

Note that

$$a = A\left[\xi - \frac{\overline{\beta}^2 + \xi^2}{2\xi}\right] = A\left[\xi + \frac{k_T^2}{2\xi}\right] > 0$$

$$b = A\left[-\overline{\alpha} + \frac{\overline{\beta}^2 + \xi^2}{2\xi}\right] = A\left[\frac{(\overline{\alpha} - \overline{\beta})^2 + k_L^2}{2\overline{\beta}}\right] > 0$$

hence

$$u_x = a\sin(\omega t - \xi x), \quad u_y = b\cos(\omega t - \xi x)$$

and

$$\frac{u_x^2}{a^2} + \frac{u_y^2}{b^2} = 1 \tag{6.25}$$

The particle trajectory is an ellipse. In complex form we have

$$\frac{u_x}{a} + i\frac{u_y}{b} = \exp\{i(\omega t - \xi x - \pi/2)\} \tag{6.26}$$

Hence as t increases,a particle at $(x, 0)$ traces the ellipse in the counter-clockwise direction. See figure (6).

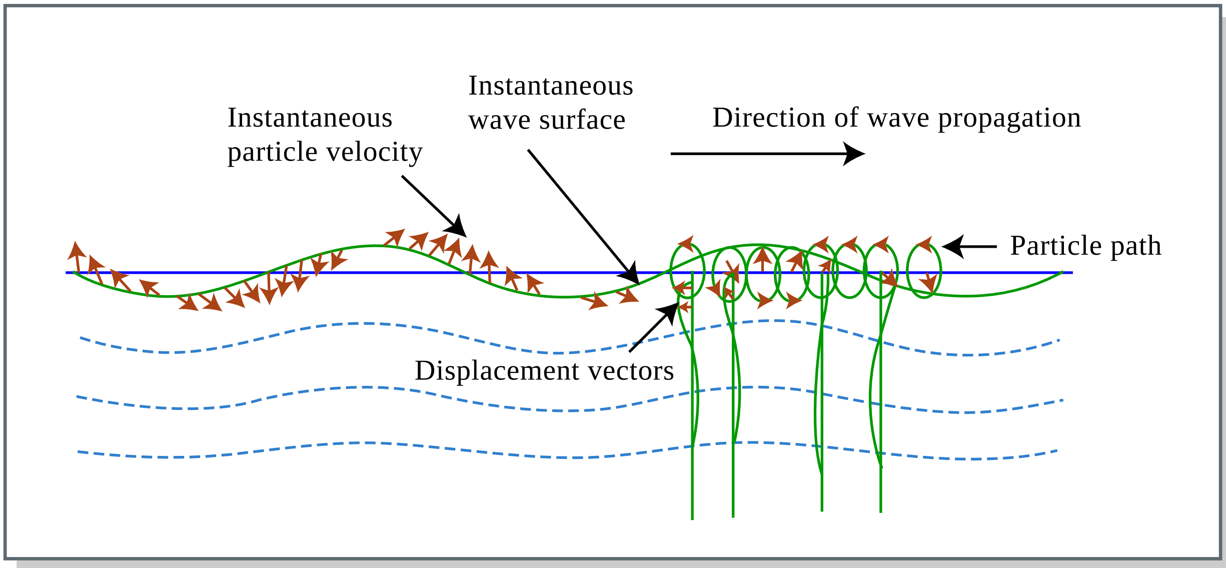

Figure by MIT OCW.

Figure 7: Displacement of particles on the ground surface in Rayleigh surface wave From Fung *Foundations of Solid Mechanics*.

7 Scattering of monochromatic SH waves by a cavity

7.1 The boundary-value problem

We consider the scattering of two-dimensional SH waves of single frequency. The time-dependent potential can be wirtten as

$$\psi(x, y, t) = \Re\left[\phi(x, y)e^{-i\omega t}\right] \tag{7.1}$$

where the potential ϕ is governed by the Helmholtz equation

$$\nabla^2\phi + k^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi^2}{\partial y^2} + k^2\phi = 0, \quad k = \frac{\omega}{c_T} \tag{7.2}$$

To be specific consider the scatterer to be a finite cavity of some general geometry. On the stress-free boundary B the shear stress vanishes,

$$\tau_{zn} = -\frac{\mu\omega^2}{c_T^2}\Re\left(\frac{\partial\phi}{\partial n}e^{-i\omega t}\right) = 0 \tag{7.3}$$

hence

$$\frac{\partial\phi}{\partial n} = 0, \quad \text{on} \quad B \tag{7.4}$$

Let the incident wave be a plane wave

$$\phi_I = Ae^{i\mathbf{k}\cdot\mathbf{x}} \tag{7.5}$$

Let the angle of incidence with respect to the positive x axis be θ_o. In polar coordinates the incident wave vector is

$$\mathbf{k} = k(\cos\theta_o, \sin\theta_o), \quad \mathbf{x} = r(\cos\theta, \sin\theta) \tag{7.6}$$

and the potential is

$$\phi_I = A\exp\left[ikr(\cos\theta_o\cos\theta + \sin\theta_o\sin\theta)\right] = Ae^{ikr\cos(\theta-\theta_o)} \tag{7.7}$$

Let the total wave be the sum of the incident and scattered waves

$$\phi = \phi_I + \phi_S \tag{7.8}$$

For the scattered wave the boundary condition on the cavity surface is

$$\frac{\partial\phi_S}{\partial n} = -\frac{\partial\phi_S}{\partial n}, \quad \text{on} \quad B. \tag{7.9}$$

In addition, the scattered wave must satisfy the *radiation condition* at infinity, i.e., it must propagate outward at infinty.

Note that this boundary value problem is identical to the one for sound plane sound wave in air (or water) scattered by a perfectly rigid cylinder.

We treat below the special case of a circular cavity of readius a This is one of the few geomentries that can be solved analytically.

7.2 The circular cavity

In polar coordinates the governing equation reads

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + k^2\phi = 0, \quad r > a. \tag{7.10}$$

Since ϕ_I satisfies the preceding equation, so does ϕ_S.

First, It is shown in Appendix A that the plane wave can be expanded in Fourier-Bessel series :

$$e^{ikr\cos(\theta-\theta_o)} = \sum_{n=0}^{\infty}\epsilon_n i^n J_n(kr)\cos n(\theta-\theta_o) \tag{7.11}$$

where ϵ_n is the Jacobi symbol:

$$\epsilon_0 = 0, \quad \epsilon_n = 2, \; n = 1, 2, 3, \ldots \tag{7.12}$$

Each term in the series (7.11) is called a partial wave.

By the method of separation of variables,

$$\phi_S(r,\theta) = R(r)\Theta(\theta)$$

we find

$$r^2R'' + rR' + (k^2r^2 - n^2)R = 0, \quad \text{and} \quad \Theta'' + n^2\Theta = 0$$

where $n = 0, 1, 2, \dots$ are eigenvalues in order that Θ is periodic in θ with period 2π. For each eigenvalue n the possible solutions are

$$\Theta_n = (\sin n\theta, \cos n\theta),$$

$$R_n = \left(H_n^{(1)}(kr), H_n^{(2)}(kr)\right),$$

where $H_n^{(1)}(kr), H_n^{(2)}(kr)$ are Hankel functions of the first and second kind, related to the Bessel and Weber functions by

$$H_n^{(1)}(kr) = J_n(kr) + iY_n(kr), \quad H_n^{(2)}(kr) = J_n(kr) - iY_n(kr) \tag{7.13}$$

The most general solution to the Helmholtz equation is

$$\phi_S = A\sum_{n=0}^{\infty}(A_n \sin n\theta + B_n \cos n\theta)\left[C_n H_n^{(1)}(kr) + D_n H_n^{(2)}(kr)\right], \tag{7.14}$$

For large radius the asymptotic form of the Hankel functions are

$$H_n^{(1)} \sim \sqrt{\frac{2}{\pi kr}} e^{i(kr - \frac{\pi}{4} - \frac{n\pi}{2})}, \quad H_n^{(2)} \sim \sqrt{\frac{2}{\pi kr}} e^{-i(kr - \frac{\pi}{4} - \frac{n\pi}{2})} \tag{7.15}$$

In conjunction with the time factor $\exp(-i\omega t)$, $H_n^{(1)}$ gives an outgoing wave while $H_n^{(2)}$ gives an incoming wave. To satisfy the radiation condition, we must discard all terms involving $H_n^{(2)}$. From here on we shall abbreviate $H_n^{(1)}$ simply by H_n. The scattered wave is now

$$\phi_S = A\sum_{n=0}^{\infty}(A_n \sin n\theta + B_n \cos n\theta) H_n(kr) \tag{7.16}$$

The expansion coefficients (A_n, B_n) must be chosen to satisfy the boundary condition on the cavity surface[1] Once they are determined, the wave is found everywhere. In

[1] In one of the numerical solution techniques, one divides the physical region by a circle enclosing the cavity. Between the cavity and the circle, finite elements are used. Outside the circle, (7.16) is used. By constructing a suitable variational principle, finite element computation yields the nodal coefficients as well as the expansion coefficients. See (Chen & Mei , 1974).

particular in the far field, we can use the asymptotic formula to get

$$\phi_S \sim A \sum_{n=0}^{\infty} (A_n \sin n\theta + B_n \cos n\theta)\, e^{-in\pi/2} \sqrt{\frac{2}{\pi k r}} e^{ikr - i\pi/4} \tag{7.17}$$

Let us define the dimensionless directivity factor

$$\mathcal{A}(\theta) = \sum_{n=0}^{\infty} (A_n \sin n\theta + B_n \cos n\theta)\, e^{-in\pi/2} \tag{7.18}$$

which indicates the angular variation of the far-field amplitude, then

$$\phi_S \sim A\mathcal{A}(\theta) \sqrt{\frac{2}{\pi k r}} e^{ikr - i\pi/4} \tag{7.19}$$

This expression exhibits clearly the asymptotic behaviour of ϕ_S as an outgoing wave. By differentiation, we readily see that

$$\lim_{kr\to\infty} \sqrt{r} \left(\frac{\partial \phi_S}{\partial r} - \phi_S \right) = 0 \tag{7.20}$$

which is one way of stating the radiation condition for two dimensional SH waves.

Let us complete the solution.

Without loss of generality we can take $\theta_o = 0$. On the surface of the cylindrical cavity $r = a$, we impose

$$\frac{\partial \phi_I}{\partial r} + \frac{\partial \phi_S}{\partial r} = 0, \quad r = a$$

It follows that $A_n = 0$ and

$$\epsilon^n i^n A J_n'(ka) + B_n k H_n'(ka) = 0, \quad n = 0, 1, 2, 3, \ldots n$$

where primes denote differentiation with respect to the argument. Hence

$$B_n = -A\epsilon_n i^n \frac{J_n'(ka)}{H_n'(ka)}$$

The sum of incident and scattered waves is

$$\phi = A \sum_{n=0}^{\infty} \epsilon_n i^n \left[J_n(kr) - \frac{J_n'(ka)}{H_n'(ka)} H_n(kr) \right] \cos n\theta \tag{7.21}$$

and

$$\psi = A e^{-i\omega t} \sum_{n=0}^{\infty} \epsilon_n i^n \left[J_n(kr) - \frac{J_n'(ka)}{H_n'(ka)} H_n(kr) \right] \cos n\theta \tag{7.22}$$

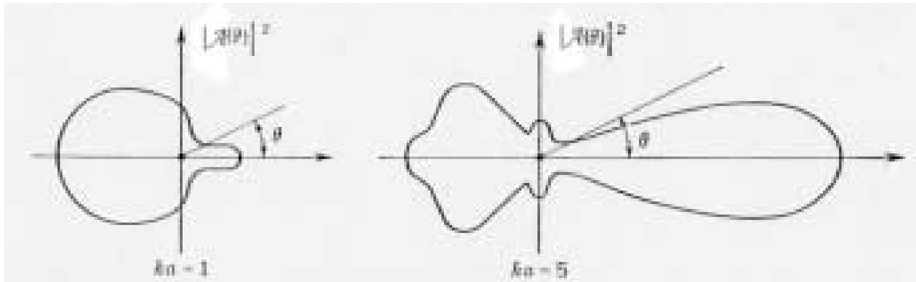

Figure 8: Angular distribution of scattered energy in the far field in cylindrical scattering

The limit of long waves can be approximatedly analyzed by using the expansions for Bessel functions for small argument

$$
\begin{aligned}
J_0(x) \sim 1 - \frac{x^2}{4} + O(x^4), \quad J_n(x) \sim \frac{x^n}{2^n n!}, ; n = 1, 2, 3... \\
Y_0(x) \sim \frac{2}{\pi} \log x, \quad Y_n(x) \sim \frac{2^n (n-1)!}{\pi x^n}, n = 1, 2, 3...
\end{aligned}
\tag{7.23}
$$

Then the scattered wave has the potential

$$
\begin{aligned}
\frac{\phi_S}{A} &\sim -H_0(kr)\frac{J_0'(ka)}{H_0'(ka)} - 2iH_1(kr)\frac{J_1'(ka)}{H_1'(ka)}\cos\theta + O(ka)^3 \\
&= \frac{\pi}{2}(ka)^2\left(-\frac{i}{2}H_0(kr) - H_1(kr)\cos\theta\right) + O(ka)^3
\end{aligned}
\tag{7.24}
$$

The term $H_0(kr)$ coresponds to a oscillating source which sends istropic waves in all directions. The second term is a dipole sending scattered waves mostly in forward and backward directions. For large kr, the angular variation is a lot more complex. The far field pattern for various ka is shown in fig 4.

On the cavity surface surface, the displacement is proportional to $\nabla^2\psi(a,\theta)$ or $\nabla^2\phi(a,\theta)$. The angular variation is plotted for several ka in figure 5.

For numerical simulations, see the website
http://ocw.mit.edu/OcwWeb/Civil-and-Environmental-Engineering/1-138JFall-2004/Simulations/

Remark on energy flux: At any radius r the total rate of energy outflux by the

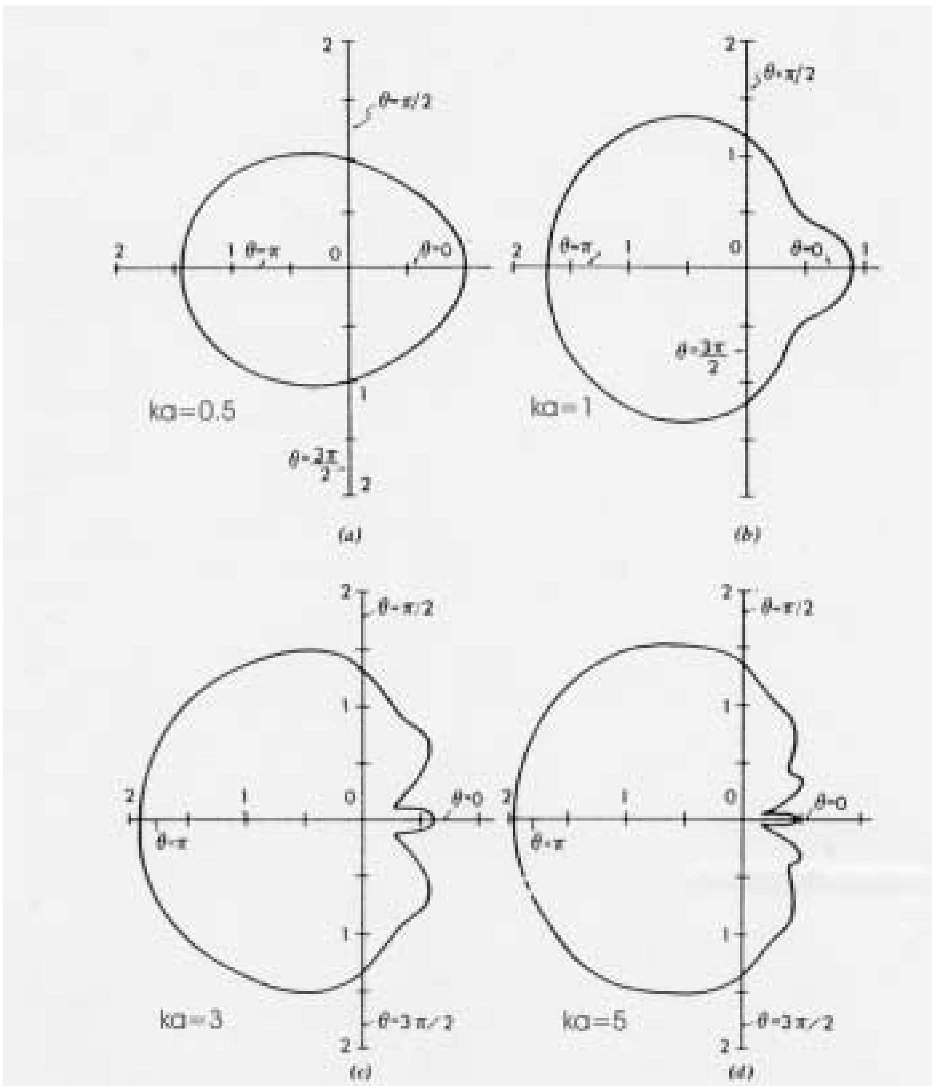

Figure 9: Polar distribution of $\phi(a, \theta)$ on a circular cylinder.

scattered wave is

$$r\int_0^{2\pi} d\theta \overline{\tau_{rz}\frac{\partial u_z}{\partial t}} = \mu r\int_0^{2\pi} d\theta \overline{\Re\left[-\mu k^2 \frac{\partial \phi}{\partial r} e^{-i\omega t}\right] \Re\left[i\omega k^2 \phi e^{-i\omega t}\right]}$$
$$= -\frac{\mu\omega k^4 r}{2}\int_0^{2\pi} d\theta \Re\left[i\phi^* \frac{\partial \phi}{\partial r}\right] = -\frac{\mu\omega k^4 r}{2}\Im\int_0^{2\pi} d\theta \left[\phi^* \frac{\partial \phi}{\partial r}\right] \quad (7.25)$$

where overline indicates time averaging over a wave period $2\pi/\omega$. The integral can be evaluated by using the asymptotic expression.

Remark: In the analogous case of plane acoustics where the sound pressure and radial fluid velocity are respectively,

$$p = -\rho_o \frac{\partial \phi}{\partial t}, \quad \text{and} \quad u_r = \frac{\partial \phi}{\partial r} \quad (7.26)$$

the energy scattering rate is

$$r\int_0^{2\pi} d\theta \overline{pu_r} = \frac{\omega\rho_o r}{2}\Re\int_C d\theta \left(-i\phi^* \frac{\partial \phi}{\partial r}\right) = -\frac{\omega\rho_o r}{2}\Im\int_C d\theta \left(\phi^* \frac{\partial \phi}{\partial r}\right) \quad (7.27)$$

8 The optical theorem

For the same scatterer and the same frequency ω, different angles of incidence θ_j define different scattering problems ϕ_j. In particular at infinty, we have

$$\phi_j \sim A_j \left\{ e^{ikr\cos(\theta-\theta_j)} + \mathcal{A}_j(\theta)\sqrt{\frac{2}{\pi kr}} e^{ikr - i\pi/4} \right\} \quad (8.1)$$

Let us apply Green's formula to ϕ_1 and ϕ_2 over a closed area bounded by a closed contour C,

$$\iint_S \left(\phi_2\nabla^2\phi_1 - \phi_1\nabla^2\phi_2\right) dA = \int_B \left(\phi_2\frac{\partial \phi_1}{\partial n} - \phi_1\frac{\partial \phi_2}{\partial n}\right) ds + \int_C ds \left(\phi_2\frac{\partial \phi_1}{\partial n} - \phi_1\frac{\partial \phi_1}{\partial n}\right) ds$$

where $\mathbf{n}$ refers to the unit normal vector pointing out of S. The surface integral vanishes on account of the Helmholtz equation, while the line integral along the cavity surface vanishes by virture of the boundary condition, hence

$$\int_C ds \left(\phi_2\frac{\partial \phi_1}{\partial n} - \phi_1\frac{\partial \phi_2}{\partial n}\right) ds = 0 \quad (8.2)$$

By similar reasioning, we get

$$\int_C ds \left(\phi_2 \frac{\partial \phi_1^*}{\partial n} - \phi_1^* \frac{\partial \phi_2}{\partial n} \right) ds = 0 \tag{8.3}$$

where ϕ_1^* denotes the complex conjugate of ϕ_1.

Let us choose $\phi_1 = \phi_2 = \phi$ in (8.3), and get

$$\int_C ds \left(\phi \frac{\partial \phi^*}{\partial n} - \phi^* \frac{\partial \phi}{\partial n} \right) ds = 2\Im \left(\int_C ds \, \phi \frac{\partial \phi^*}{\partial n} \right) = 0 \tag{8.4}$$

Physically, across any circle the net rate of energy flux vanishes, i.e., the scattered power must be balanced by the incident power.

Making use of (8.1) we get

$$\begin{aligned}
0 = \Im \int_0^{2\pi} r d\theta & \left[e^{ikr\cos(\theta-\theta_o)} + \sqrt{\frac{2}{\pi kr}} \mathcal{A}_o(\theta) e^{ikr - i\pi/4} \right] \\
& \cdot \left[-ik\cos(\theta-\theta_o) e^{-ikr\cos(\theta-\theta_o)} - ik\sqrt{\frac{2}{\pi kr}} \mathcal{A}_o^*(\theta) e^{-ikr + i\pi/4} \right] \\
& = \Im \int_0^{2\pi} r d\theta \left\{ -ik\cos(\theta-\theta_o) + \frac{2}{\pi kr}(-ik)|\mathcal{A}_o|^2 \right. \\
& + e^{ikr[\cos\theta-\theta_o)-1] + i\pi/4}(-ik)\sqrt{\frac{2}{\pi kr}} \mathcal{A}_o^* \\
& \left. + \, e^{-ikr[\cos\theta-\theta_o)-1] - i\pi/4}(-ik)\cos(\theta-\theta_o)\sqrt{\frac{2}{\pi kr}} \mathcal{A}_o \right\}
\end{aligned}$$

The first term in the integrand gives no contribution to the integral above because of periodicity. Since $\Im(if) = \Im(if^*)$, we get

$$\begin{aligned}
0 = & -\frac{2}{\pi} \int_0^{2\pi} |\mathcal{A}_o(\theta)|^2 d\theta \\
& + \Im \int_0^{2\pi} r d\theta \left\{ \mathcal{A}_o(-ik)\sqrt{\frac{2}{\pi kr}} [1 + \cos(\theta-\theta_o)] e^{i\pi/4} e^{ikr(1-\cos(\theta-\theta_o))} \right\} \\
= & -\frac{2}{\pi} \int_0^{2\pi} |\mathcal{A}_o(\theta)|^2 d\theta \\
& - \Re \left\{ e^{-i\pi/4} \left[r\sqrt{\frac{2}{\pi kr}} \int_0^{2\pi} d\theta \mathcal{A}_o(\theta) [1 + \cos(\theta-\theta_o)] e^{ikr(1-\cos(\theta-\theta_o))} \right] \right\}
\end{aligned}$$

For large kr the remaining integral can be found approximately by the method of stationary phase (see Appendix B), with the result

$$\int_0^{2\pi} d\theta \mathcal{A}_o(\theta) [1 + \cos(\theta-\theta_o)] e^{ikr(1-\cos(\theta-\theta_o))} \sim \mathcal{A}_o(\theta_o) \sqrt{\frac{2\pi}{kr}} e^{i\pi/4} \tag{8.5}$$

We get finally

$$\int_0^{2\pi} |\mathcal{A}_o|^2 d\theta = -2\Re\mathcal{A}_o(\theta_o) \tag{8.6}$$

Thus the total scattered energy in all directions is related to the amplitude of the scattered wave in the forward direction. In atomic physics, where this theorem was originated (by Niels Bohr), measurement of the scattering amplitude in all directions is not easy. This theorem suggests an econmical alternative.

Homework For the same scatterer, consider two scattering problems ϕ_1 and ϕ_2. Show that

$$\mathcal{A}_1(\theta_2 + \pi) = \mathcal{A}_2(\theta_1 + \pi) \tag{8.7}$$

For general elastic waves, see Mei (1978) for similar and other identities in elastodynamics with rigid inclusions. *J . Acoust. Soc. Am.* 64(5), 1514-1522.

9 Diffraction of SH wave by a long crack - the parabolic approximation

References

Morse & Ingard, *Theoretical Acoustics* Series expansions.

Born & Wolf, *Principle of Optics* Fourier Transform and the method of steepest descent.

B. Noble. *The Wiener-Hopf Technique.*

If the obstacle is large, there is always a shadow behind where the incident wave cannot penetrate deeply. The phenomenon of scattering by large obstacles is usually referred to as diffraction.

Diffraction of plane incident SH waves by a long crack is identical to that of a hard screen in acoustics. The exact solution was due to A. Sommerfeld. We shall apply the boundary layer idea and give the approximate solution valid far away from the tip $kr \gg 1$ by the *parabolic approximation*, due to V. Fock.

Refering to figure () let us make a crude division of the entire field into the illuminated zone I , dominated by the incident wave alone, the reflection zone II dominated the sum

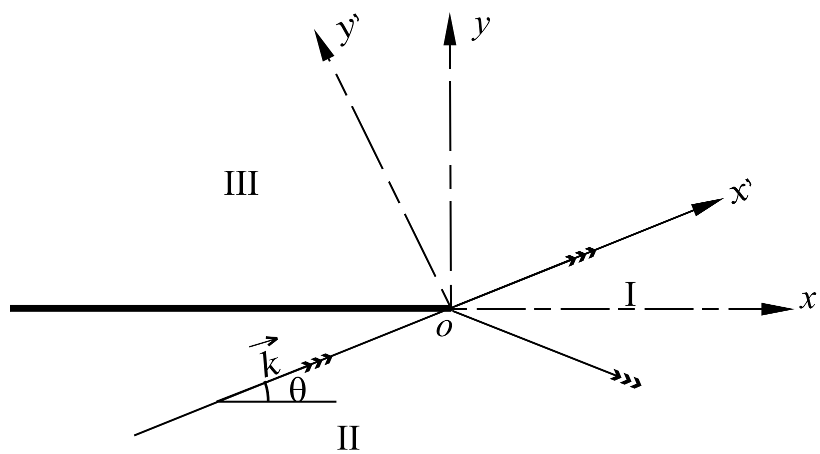

Figure 10: Wave zones near a long crack

of the incident and the reflected wave, and the shadow zone III where there is no wave. The boundaries of these zones are the rays touching the crack tip. According to this crude picture the solution is

$$\phi = \begin{cases} A_o \exp(ik\cos\theta x + ik\sin\theta y), & I \\ A_o[\exp(ik\cos\theta x + ik\sin\theta y) + \exp(ik\cos\theta x - ik\sin\theta y)], & II \\ 0, & III \end{cases} \tag{9.1}$$

Clearly (9.1) is inadquate because the potential cannot be discontinuous across the boundaries. A remedy to provide smooth transitions is needed.

Consider the shadow boundary Ox'. Let us introduce a new cartesian coordinate system so that x' axis is along, while the y' axis is normal to, the shadow boundary. The relations between (x, y) and (x', y') are

$$x' = x\cos\theta + y\sin\theta, \quad y' = y\cos\theta - x\sin\theta \tag{9.2}$$

Thus the incident wave is simply

$$\phi_I = A_o e^{ikx'} \tag{9.3}$$

Following the chain rule of differentiation,

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial\phi}{\partial y'}\frac{\partial y'}{\partial x} = \cos\theta\frac{\partial\phi}{\partial x'} - \sin\theta\frac{\partial\phi}{\partial y'}$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial \phi}{\partial y'}\frac{\partial y'}{\partial y} = \sin\theta \frac{\partial \phi}{\partial x'} + \cos\theta \frac{\partial \phi}{\partial y'}$$

we can show straightforwardly that

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \phi}{\partial x'^2} + \frac{\partial^2 \phi}{\partial y'^2}$$

so that the Helmholtz equation is unchanged in form in the x', y' system.

We try to fit a boundary layer along the x' axis and expect the potential to be almost like a plane wave

$$\phi(x,', y') = A(x', y')e^{ikx'} \tag{9.4}$$

, but the amplitude is slowly modulated in both x' and y' directions. Substituting (9.4 into the Helmholtz equation, we get

$$e^{ikx'}\left\{\frac{\partial^2 A}{\partial x'^2} + 2ik\frac{\partial A}{\partial x'} - k^2 A + \frac{\partial^2 A}{\partial y'^2} + k^2 A\right\} = 0 \tag{9.5}$$

Expecting that the characteristic scale L_x of A along x' is much longer than a wavelength, $kL_x \gg 1$, we have

$$\frac{\partial A}{\partial x'} \ll kA, \quad \text{hence} \quad 2ik\frac{\partial A}{\partial x'} \gg \frac{\partial^2 A}{\partial x'^2}$$

We get as the first approximation the Schródinger equation[2]

$$2ik\frac{\partial A}{\partial x'} + \frac{\partial^2 A}{\partial y'^2} \approx 0 \tag{9.7}$$

In this transition zone where the remaining terms are of comparable importance, hence the length scales must be related by

$$\frac{k}{x'} \sim \frac{1}{y'^2}, \quad \text{implying} \quad ky' \sim \sqrt{kx'}$$

Thus the transition zone is the interior of a parabola.

Equation (9.7) is of the parabolic type. The boundary conditions are

$$A(x, \infty) = 0 \tag{9.8}$$

[2]In one-dimensional quantum mechanics the wave function in a potential-free field is governed by the Schrödinger equation

$$ih\frac{\partial \psi}{\partial t} + \frac{1}{2M}\frac{\partial^2 \psi}{\partial x^2} = 0 \tag{9.6}$$

$$A(x, -\infty) = A_o \tag{9.9}$$

The initial condition is

$$A(0, y') = \begin{cases} 0, & y' > 0, \\ A_0, & y' < 0 \end{cases} \tag{9.10}$$

he initial-boundary value for A has no intrinsic length scales except x', y' themselves. Therefore the condition $kL_x \gg 1$ means $kx' \gg 1$ i.e., far away from the tip. This problem is somwhat analogous to the problem of one-dimensional heat diffusion across a boundary. A convenient way of solution is the method of similarity.

Assume the solution

$$A = A_o f(\gamma) \tag{9.11}$$

where

$$\gamma = \frac{-ky'}{\sqrt{\pi k x'}} \tag{9.12}$$

is the similarity variable. We find upon subsitution that f satisfies the ordinary differential equation

$$f'' - i\pi\gamma f' = 0 \tag{9.13}$$

subject to the boundary conditions that

$$f \to 0, \quad \gamma \to -\infty; \quad f \to 1, \quad \gamma \to \infty. \tag{9.14}$$

Rewriting (9.13) as

$$\frac{f''}{f'} = i\pi\gamma$$

we get

$$\log f' = i\pi\gamma/2 + \text{constant.}$$

One more integration gives

$$f = C \int_{-\infty}^{\gamma} \exp\left(\frac{i\pi u^2}{2}\right) du$$

Since

$$\int_0^{\infty} \exp\left(\frac{i\pi u^2}{2}\right) du = \frac{e^{i\pi/4}}{\sqrt{2}}$$

we get

$$C = \frac{e^{-i\pi/4}}{\sqrt{2}}$$

and

$$f = \frac{A}{A_o} = \frac{e^{-i\pi/4}}{\sqrt{2}} \int_{-\infty}^{\gamma} \exp\left(\frac{i\pi u^2}{2}\right) du = \frac{e^{-i\pi/4}}{\sqrt{2}} \left\{ \frac{e^{i\pi/4}}{\sqrt{2}} + \int_0^{\gamma} \exp\left(\frac{i\pi u^2}{2}\right) du \right\} \quad (9.15)$$

Defining the cosine and sine Fresnel integrals by

$$C(\gamma) = \int_0^{\gamma} \cos\left(\frac{\pi v^2}{2}\right) dv, \quad S(\gamma) = \int_0^{\gamma} \sin\left(\frac{\pi v^2}{2}\right) dv \quad (9.16)$$

we can then write

$$\frac{e^{-i\pi/4}}{\sqrt{2}} \left\{ \left[\frac{1}{2} + C(\gamma)\right] + i \left[\frac{1}{2} + S(\gamma)\right] \right\} \quad (9.17)$$

In the complex plane the plot of $C(\gamma) + iS(\gamma)$ vs. γ is the famous Cornu's spiral, shown in figure (9).

The wave intensity is given by

$$\frac{|A|^2}{A_o^2} = \frac{1}{2} \left\{ \left[\frac{1}{2} + C(\gamma)\right]^2 + \left[\frac{1}{2} + S(\gamma)\right]^2 \right\} \quad (9.18)$$

Since $C, S \rightarrow -1/2$ as $\gamma \rightarrow -\infty$, the wave intensity diminshes to zero gradually into the shadow. However, $C, S \rightarrow 1/2$ as $\gamma \rightarrow \infty$ in an oscillatory manner. The wave intensity oscillates while approaching to unity asymptotically. In optics this shows up as alternately light and dark diffraction bands.

In more complex propagation problems, the parabolic approximation can simplify the numerical task in that an elliptic boundary value problem involving an infinite domain is reduced to an initial boundary value problem. One can use Crank-Nicholson scheme to march in "time", i.e., x'.

Homework: Find by the parabolic approximation the transition solution along the edge of the reflection zone.

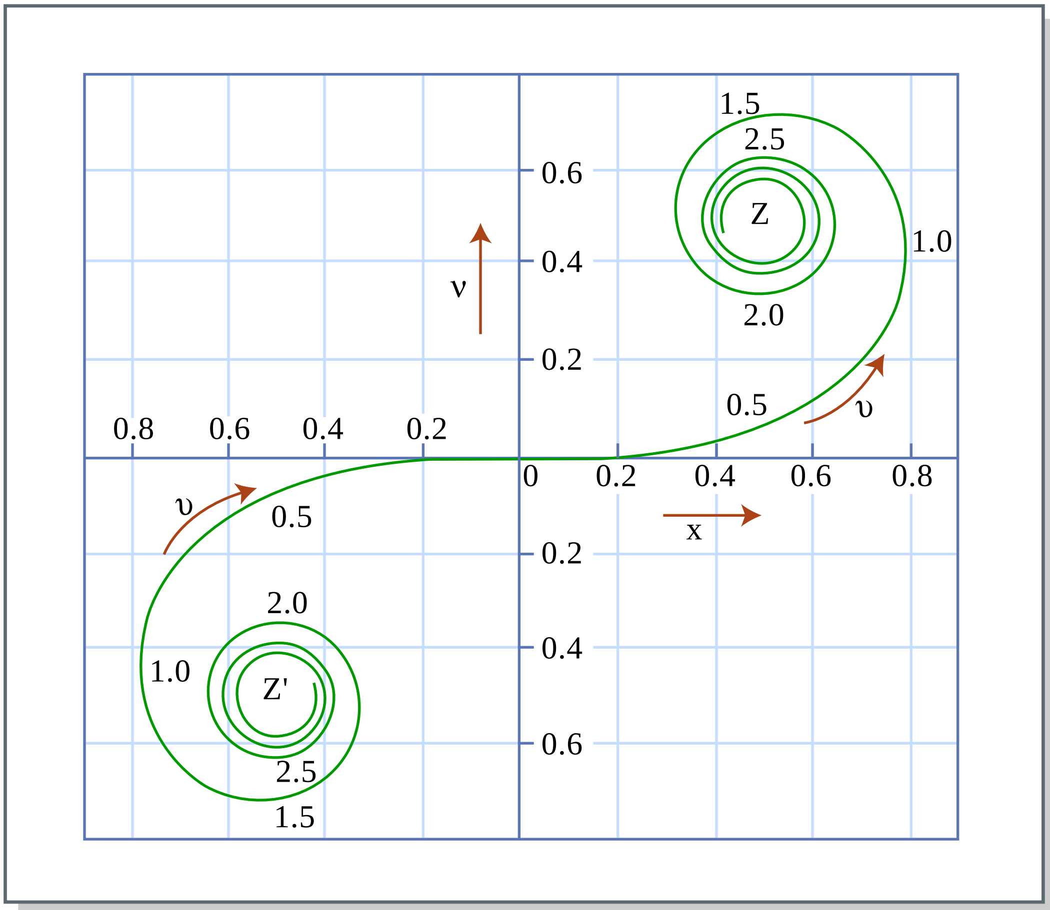

Figure by MIT OCW.

Figure 11: Cornu's spiral, a plot of the Fresnel integrals in the comiplex plane of $C(\gamma)+iS(\gamma)$. Abscissa: $C(\gamma)$. Ordinate : $iS(\gamma)$.

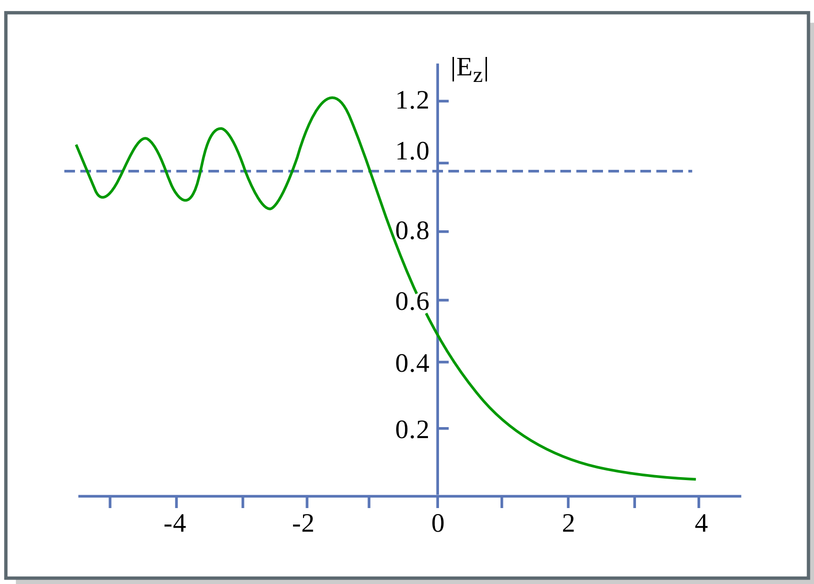

Figure by MIT OCW.

Figure 12: Diffraction of a normally incident plane sound wave on a rigid half screen. Ordinate: $f(\gamma) = A/A_o$. Abscissa: $-\gamma = ky'/\sqrt{\pi k x'}$.

10 Exact theory of Wedge Diffraction

Refs. J. J. Stoker 1957 , *Water waves.* pp 120-125

Born & Wolf, 1950 *Principles of Optics*

Noble, 1963 *The Wiener-Hopf Technique*

The diffraction of plane incident waves by a semi-infinite barrier is a celebrated problem in classical physics and was first solved by A. Sommerfeld. Several analytical treatments are available, including the mathematically very elegant technique of Wiener & Hopf is available . In the last section we gave an approximate theory by parabolic approximation. Here we present an exact theory for the more general case of a wedge, by the more elementary method of series expansion. Though the result is in an infinite series and not in closed form, quantitative information can be calculated quite readily by a computer.

The basic ideas were described for water waves by Stoker (1957) for a wedge. Extensive numerical results were reported by Dr. H.S. Chen (Army Corps of Engineers, Tech Rept: CERC 878-16, 1987). Based on Stoker's analysis more extensive numerical computations have been carried out by Dr. G.D. Li. To facilitate the understanding of the physics, these results are presented in animated form in our subject website (see Simulations):

http://ocw.mit.edu/OcwWeb/Civil-and-Environmental-Engineering/1-138JFall-2004/

We shall describe the problem for water waves; the analysis and results can be adapted for cracks, sound,... etc.

Refering to Figure 10, we consider a vertical wedge of arbitrary apex angle in a sea of constant depth h. Let the tip of the wedge be the z axis orgin and the x axis coincide with one wall. The still water surface is the x, y plane. In the cylindrical polar coordinate system (r, θ, z) with $(x = r\cos\theta, y = r\sin\theta), z$, the walls are given by $\theta = 0$ and $\nu\pi$ with $1 < \nu < 2$. A train of monochromatic waves is incident from infinity at the angle α with respect the x axis.

In the water region defined by $0 < \theta < \nu\pi$ and $0 \geq z \geq -h$, the velocity potential $\Phi(r, \theta, z, t)$ must satisfy the Laplace equation,

$$\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial z^2} = 0 \tag{10.1}$$

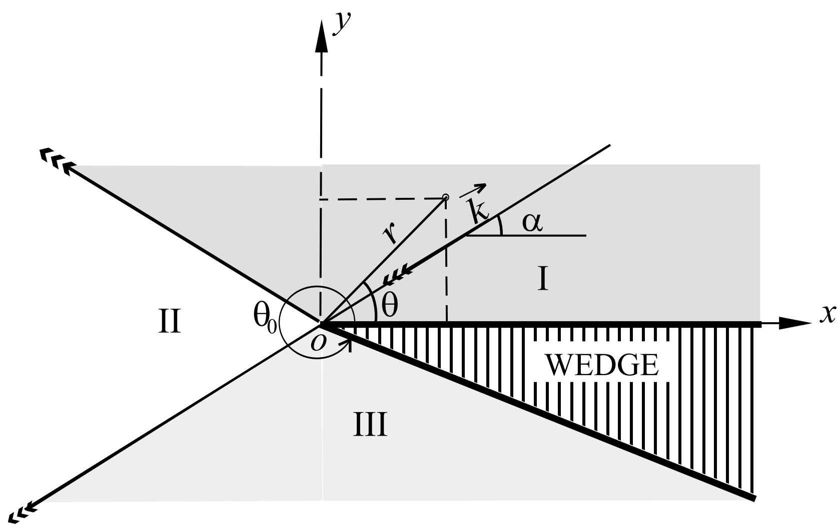

Figure 13: Coordinate system and subregions.

and subject to the linearized free surface boundary conditions

$$\frac{\partial \Phi}{\partial t} = -g\zeta \tag{10.2}$$

$$\frac{\partial \zeta}{\partial t} = \frac{\partial \Phi}{\partial z} \tag{10.3}$$

whicdh can be combined to give

$$\frac{\partial^2 \Phi}{\partial t^2} + g\frac{\partial \Phi}{\partial z} = 0, \quad z = 0. \tag{10.4}$$

Along the impermeable bottom and walls, the no flux boundary conditions are

$$\frac{\partial \Phi}{\partial z} = 0 \qquad on \;\; z = -h \tag{10.5}$$

$$\frac{\partial \Phi}{\partial \theta} = 0 \qquad at \;\; \theta = 0 \;\; and \;\; \nu\pi \tag{10.6}$$

The incident wave train is given by

$$\Phi_i = \frac{-igA_0}{\omega}\frac{\cosh k(z+h)}{\cosh kh}\phi(r,\theta)e^{-ikr\cos(\theta-\alpha)-i\omega t} \tag{10.7}$$

where k is the real wavenumber satisfying the dispersion relation

$$\omega^2 = gk \tanh kh, \tag{10.8}$$

and $\pi + \alpha$ is the angle of incidence measured from the x axis. A_0 is the incident wave amplitude.

Because of the vertical side-walls, the three dimensional problem can be reduced to a two dimensional one by letting

$$\Phi(r, \theta, z, t) = A_0 \frac{\cosh k(z+h)}{\cosh kh} \phi(r, \theta) e^{-i\omega t} \tag{10.9}$$

where $\phi(r, \theta, t)$ is the horizontal pattern of the velocity potential normlized for an incident wave of unit amplitude.

Substituting Equation (10.9) into the Laplace equation and using both the kinematic and dynamic boundary conditions on the free surface, the Laplace equation is then reduced to the Helmholtz equation

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + k^2 \phi = 0, \quad \nu\pi \geq \theta \geq 0. \tag{10.10}$$

with the following boundary conditions on the rigid walls of the wedge:

$$\frac{\partial \phi}{\partial \theta} = 0 \qquad \textit{at} \;\; \theta = 0 \;\; \textit{and} \;\; \theta_0 \tag{10.11}$$

The free surface displacement ζ from the mean water level z=0 can be represented by

$$\zeta(r, \theta, t) = -\frac{1}{g}\frac{\partial \Phi}{\partial t} = A_0 \eta(r, \theta) e^{-i\omega t} \tag{10.12}$$

Note that η is dimensionless.

Refering to Figure 10, the entire water region can be divided into three zones according to the crude picture of geometrical optics. I: the zone of incident and reflected plane waves, II : the zone of incident plane wave and III ; the shadow with no plane wave. This crude picture is discontinuous at the border lines separating the zones. Siknce the physical solution must be smooth every where we must find the transitions. Let us use the ideas of boundary layers.

Let the total potential be expressed in a compact form by

$$\phi = \phi_o(r, \theta) + \phi_s(r, \theta), \quad \text{for all} \;\; 0 < \theta < \nu\pi \tag{10.13}$$

where ϕ_o consists of only the plane waves,

$$\phi_o(r, \theta) = \begin{cases} \phi_i + \phi_r & \pi - \alpha > \theta > 0, \;\; \text{in I;} \\ \phi_i & \pi + \alpha > \theta > \pi - \alpha, \;\; \text{in II;} \\ 0 & \theta_0 > \theta > \pi + \alpha, \;\; \text{in III.} \end{cases} \tag{10.14}$$

Here ϕ_i denotes the incident wave

$$\phi_i = e^{-ikr\cos(\theta-\alpha)} \tag{10.15}$$

where α denotes the angle of incidence, and ϕ_r the reflected wave

$$\phi_r = e^{-ikr\cos(\theta+\alpha)} \tag{10.16}$$

The correction is the diffracted wave ϕ_s which must satisfy the radiation condition and behaves as an outgoing wave at infinity, i.e.,

$$\lim_{r\to 0}\sqrt{r}(\frac{\partial\phi_s}{\partial r} - ik\phi_s) = 0 \tag{10.17}$$

or

$$\phi_s \sim \frac{\mathcal{A}(\theta)e^{ikr}}{\sqrt{kr}} \qquad at \;\; r\to\infty \tag{10.18}$$

10.1 Solution by Fourier series (or, finite Fourier Transform)

Let us solve the scattered wave formally by Fourier series

$$\phi(r,\theta) = \frac{1}{\nu\pi}\bar{\phi}_0(r) + \frac{2}{\nu\pi}\sum_{n=1}^{\infty}\bar{\phi}_n(r)\cos\frac{n\theta}{\nu} \tag{10.19}$$

then, the Fourier coefficients are:

$$\bar{\phi}_n(kr) = \int_0^{\nu\pi}\phi(kr,\theta)\cos\frac{n\theta}{\nu}d\theta \tag{10.20}$$

From (10.10), each Fourier coefficient satisfies

$$r^2\frac{\partial^2\bar{\phi}_n}{\partial r^2} + r\frac{\partial\bar{\phi}_n}{\partial r} + \left[(kr)^2 - \left(\frac{n}{\nu}\right)^2\right]\bar{\phi}_n = 0 \tag{10.21}$$

The general solution finite at the origin is

$$\bar{\phi}_n(kr) = a_n J_{n/\nu}(kr) \tag{10.22}$$

where the coefficient's $a_n, n = 0,1,2,3,...$ are to be determined.

The Fourier coefficient of (10.13) reads

$$\bar{\phi}_n(kr) = a_n J_{n/\nu}(kr) = \int_0^{\nu\pi}\phi_s\cos\frac{n\theta}{\nu}d\theta + \int_0^{\nu\pi}\phi_o\cos\frac{n\theta}{\nu}d\theta \tag{10.23}$$

or

$$\bar{\phi}_{s,n} = a_n J_{n/\nu}(kr) - \bar{\phi}_{o,n} \tag{10.24}$$

Applying the operator $\lim_{r\to\infty}\sqrt{r}(\partial/\partial r - ik)$ to both sides of (10.23), and using the Sommerfeld radiation condition (10.17), we have

$$\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial}{\partial r} - ik\right)\left[a_n J_{n/\nu}(kr) - \int_0^{\nu\pi} \phi_o \cos\frac{n\theta}{\nu} d\theta\right] = 0 \tag{10.25}$$

We now perform some asymptotic analysis for large kr to evaluate a_n.

The first part can be treated explicitly for large kr, since

$$J_{n/\nu}(kr) \sim \sqrt{\frac{2}{\pi kr}}\cos\left(kr - \frac{n\pi}{2\nu} - \frac{\pi}{4}\right) \tag{10.26}$$

It follows that

$$\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial}{\partial r} - ik\right) J_{n/\nu}(kr) = \sqrt{\frac{2k}{\pi}} e^{-i\left(kr - \frac{n\pi}{2\nu} + \frac{\pi}{4}\right)} \tag{10.27}$$

For the second part, we substitute ϕ_o from (10.15) and (10.16) to rewrite the integral as

$$\begin{aligned}\int_0^{\nu\pi} \phi_o \cos\frac{n\theta}{\nu} d\theta \quad &= \overbrace{\int_0^{\pi-\alpha} e^{-ikr\cos(\theta-\alpha)} \cos\frac{n\theta}{\nu} d\theta}^{1} + \overbrace{\int_0^{\pi-\alpha} e^{-ikr\cos(\theta+\alpha)} \cos\frac{n\theta}{\nu} d\theta}^{2} \\ &+ \overbrace{\int_{\pi-\alpha}^{\pi+\alpha} e^{-ikr\cos(\theta-\alpha)} \cos\frac{n\theta}{\nu} d\theta}^{3}\end{aligned} \tag{10.28}$$

Each of the integrals above can be evaluated for large kr by the method of stationary phase (*again*). Details are given in the next subsection. only the results are cited below.

The first integral is appoximately

$$I_1(\theta) = \cos\left(\frac{n\alpha}{\nu}\right) e^{-ikr+\frac{i\pi}{4}} \left[\frac{2\pi}{kr}\right]^{\frac{1}{2}} + O\left(\frac{1}{kr}\right) \tag{10.29}$$

from which

$$\begin{aligned}&\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial}{\partial r} - ik\right)\int_0^{\pi-\alpha} e^{-ikr\cos(\theta-\alpha)} \cos\frac{n\theta}{\nu} d\theta \\ &= \lim_{r\to\infty}\sqrt{r}\left(\frac{\partial}{\partial r} - ik\right)\left\{\cos\left(\frac{n\alpha}{\nu}\right) e^{-ikr+\frac{i\pi}{4}} \left[\frac{2\pi}{kr}\right]^{\frac{1}{2}}\right\} \\ &= 2\sqrt{2\pi k}\cos\left(\frac{n\alpha}{\nu}\right) e^{-ikr-\frac{i\pi}{4}}\end{aligned} \tag{10.30}$$

where we have used $i = e^{i\pi/2}$. By similar analysis the second integral is found to be

$$I_2(\theta) \approx \frac{1}{2} \cos\left(\frac{n(\pi-\alpha)}{\nu}\right) e^{ikr-\frac{i\pi}{4}} \left[\frac{2\pi}{kr}\right]^{\frac{1}{2}} \tag{10.31}$$

It follows that

$$\begin{aligned} &\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial}{\partial r} - ik\right) \int_0^{\pi-\alpha} e^{-ikr\cos(\theta+\alpha)} \cos\frac{n\theta}{\nu} d\theta \\ &= \lim_{r\to\infty} \sqrt{r}\left(\frac{\partial}{\partial r} - ik\right) \left\{\frac{1}{2}\cos(\frac{n(\pi-\alpha)}{\nu}) e^{ikr-\frac{i\pi}{4}} \left[\frac{2\pi}{kr}\right]^{\frac{1}{2}}\right\} \\ &= 0 \end{aligned} \tag{10.32}$$

Finally the third integral is approximately

$$I_3(\theta) \approx \frac{1}{2} \cos(\frac{n(\pi+\alpha)}{\nu}) e^{ikr-\frac{i\pi}{4}} \left[\frac{2\pi}{kr}\right]^{\frac{1}{2}} \tag{10.33}$$

hence

$$\begin{aligned} &\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial}{\partial r} - ik\right) \int_0^{\pi+\alpha} e^{-ikr\cos(\theta+\alpha)} \cos\frac{n\theta}{\nu} d\theta \\ &= \lim_{r\to\infty} \sqrt{r}\left(\frac{\partial}{\partial r} - ik\right) \left\{\frac{1}{2}\cos(\frac{n(\pi+\alpha)}{\nu}) e^{ikr-\frac{i\pi}{4}} \left[\frac{2\pi}{kr}\right]^{\frac{1}{2}}\right\} \\ &= 0 \end{aligned} \tag{10.34}$$

In summary, only the first integral associated with the incident wave furnishes a nonvanishing contribution to the expansion coefficients, i.e.,

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial}{\partial r} - ik\right) \int_0^{\nu\pi} \phi_o \cos\frac{n\theta}{\nu} d\theta \sim 2\sqrt{2\pi k} \cos\frac{n\alpha}{\nu} e^{-i\left(kr+\frac{\pi}{4}\right)} \tag{10.35}$$

With this result we get by substituting (10.27) and (10.35) into (10.25), the coefficients a_n are found

$$a_n = 2\pi \cos\frac{n\alpha}{\nu} e^{-i\frac{n\pi}{2\nu}} \tag{10.36}$$

By the inverse transform, (10.19), we get the exact solution,

$$\phi(r,\theta) = \frac{2}{\nu}\left[J_0(kr) + 2\sum_{n=1}^{\infty} e^{-i\frac{n\pi}{2\nu}} J_{n/\nu}(kr) \cos\frac{n\alpha}{\nu} \cos\frac{n\theta}{\nu}\right] \tag{10.37}$$

Numerical computations by MATLAB is straightforward. Aside from its own practical interest, this solution is useful for checking strictly numerical methods for diffraction problem for other geometries.

10.2 Asymptoic approximation of Integrals

For the first integral I_1, we take the phase to be $f_1(\theta) = k\cos(\theta - \alpha)$. The points of stationary phase must be found from

$$f_1'(\theta) = -k\sin(\theta - \alpha) = 0, \tag{10.1}$$

hence $\theta = \alpha, \alpha \pm \pi$. Only the first at $\theta_1 = \alpha$ lies in the range of integration $(0, \pi - \alpha)$ and is the stationary point. Since

$$f_1''(\theta_1) = -k\cos(\theta_1 - \alpha) = -k < 0 \tag{10.2}$$

the integral is approximately

$$I_1(\theta) \approx \cos(\frac{n\theta_1}{\nu})e^{-ikr\cos(\theta_1-\alpha)+\frac{i\pi}{4}}\left[\frac{2\pi}{kr}\right]^{\frac{1}{2}} = \cos(\frac{n\alpha}{\nu})e^{-ikr+\frac{i\pi}{4}}\left[\frac{2\pi}{kr}\right]^{\frac{1}{2}} \tag{10.3}$$

For the second integral I_2, we take the phase to be $f_2(\theta) = k\cos(\theta + \alpha)$. The stationary phase point must be the root of

$$f_2'(\theta) = -k\sin(\theta + \alpha) = 0 \tag{10.4}$$

or $\theta = -\alpha, -\alpha \pm \imath$. The stationary point is at $\theta_2 = \pi - \alpha$ which is the the upper limit of integration. Since

$$f_2''(\theta_2) = -k\cos(\theta_2 + \alpha) = k > 0 \tag{10.5}$$

I_2 is approximately

$$I_2(\theta) \approx \frac{1}{2}\cos(\frac{n\theta_2}{\nu})e^{-ikr\cos(\theta_2+\alpha)-\frac{i\pi}{4}}\left[\frac{2\pi}{kr}\right]^{\frac{1}{2}} = \frac{1}{2}\cos(\frac{n(\pi-\alpha)}{\nu})e^{ikr-\frac{i\pi}{4}}\left[\frac{2\pi}{kr}\right]^{\frac{1}{2}} \tag{10.6}$$

Lastly for the third integral I_3, the phase is $f_3(\theta) = k\cos(\theta - \alpha)$. The point of stationary phase is found from

$$f_3'(\theta) = -k\sin(\theta - \alpha) = 0 \tag{10.7}$$

or $\theta = \pi, \pm\pi + \alpha$. Only the point $\theta_3 = \pi + \alpha$ is acceptable and coincides with the upper limit of integration. Since

$$f_3''(\theta_3) = -k\cos(\theta_3 - \alpha) = k > 0, \tag{10.8}$$

I_3 is approximately

$$I_3(\theta) \approx \frac{1}{2}\cos(\frac{n(\pi+\alpha)}{\nu})e^{ikr-\frac{i\pi}{4}}\left[\frac{2\pi}{kr}\right]^{\frac{1}{2}} \tag{10.9}$$

10.3 Two limiting cases

(1) A thin barrier. Let the wedge angle be 0 by setting $\nu = 2$. Equation (10.37) then becomes

$$\phi(r,\theta) = J_0(kr) + 2\sum_{n=1}^{\infty} e^{-i\frac{n\pi}{4}} J_{n/2}(kr)\cos\frac{n\alpha}{2}\cos\frac{n\theta}{2} \tag{10.10}$$

(see Stoker (1957)).

(2) An infinite wall extending from $x = -\infty$ to ∞. Water occupying only the half plane of $y \geq 0$ and the wedge angle is 180 degrees. The diffracted wave is absent from the solution, and the total wave is only the sum of the incident and reflected waves :

$$\phi(r,\theta) = e^{-ikr\cos(\theta-\alpha)} + e^{-ikr\cos(\theta+\alpha)} \tag{10.11}$$

By employing the partial-wave expansion theorm, (Abramowitz and Stegun 1964), the preceding equaion becomes

$$\phi(r,\theta) = 2\left[J_0(kr) + 2\sum_{n=1}^{\infty}(-i)^n J_n(kr)\cos n\alpha\cos n\theta\right] \tag{10.12}$$

which agree with (10.37) for $\nu = 1$.

For the animated version of the wave patterns for different wedge angles snd angle of incidence, please visit the website.

A Partial wave expansion

A useful result in wave theory is the expansion of the plane wave in a Fourier series of the polar angle θ. In polar coordinates the spatial factor of a plane wave of unit amplitude is

$$e^{ikx} = e^{ikr\cos\theta}.$$

Consider the following product of exponential functions

$$e^{zt/2}e^{-z/2t} = \left[\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{zt}{2}\right)^n\right]\left[\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-z}{2t}\right)^n\right]$$
$$\sum_{-\infty}^{\infty} t^n\left[\frac{(z/2)^n}{n!} - \frac{(z/2)^{n+2}}{1!(n+1)!} + \frac{(z/2)^{n+4}}{2!(n+2)!} + \cdots + (-1)^r\frac{(z/2)^{n+2r}}{r!(n+r)!} + \cdots\right].$$

The coefficient of t^n is nothing but $J_n(z)$, hence

$$\exp\left[\frac{z}{2}\left(t - \frac{1}{t}\right)\right] = \sum_{-\infty}^{\infty} t^n J_n(z).$$

Now we set

$$t = ie^{i\theta} \quad z = kr.$$

The plane wave then becomes

$$e^{ikx} = \sum_{N=-\infty}^{\infty} e^{in(\theta+\pi/2}J_n(z).$$

Using the fact that $J_{-n} = (-1)^n J_n$, we finally get

$$e^{ikx} = e^{ikr\cos\theta} = \sum_{n=0}^{\infty}\epsilon_n i^n J_n(kr)\cos n\theta, \tag{A.1}$$

where ϵ_n is the Jacobi symbol. The above result may be viewed as the Fourier expansion of the plane wave with Bessel functions being the expansion coefficients. In wave propagation theories, each term in the series represents a distinct angular variation and is called a *partial wave*.

Using the orthogonality of $\cos n\theta$, we may evaluate the Fourier coefficient

$$J_n(kr) = \frac{2}{\epsilon_n i^n \pi}\int_0^{\pi} e^{ikr\cos\theta}\cos n\theta d\theta, \tag{A.2}$$

which is one of a host of integral representations of Bessel functions.

B Approximate evaluation of an integral

Consider the integral

$$\int_0^{2\pi} d\theta[1+\cos(\theta-\theta_o)]e^{ikr(1-\cos(\theta-\theta_o))}$$

For large kr the stationary phase points are found from

$$\frac{\partial}{\partial\theta}[1-\cos(\theta-\theta_o)] = \sin(\theta-\theta_o) = 0$$

or $\theta = \theta_o, \theta_o + \pi$ within the range $[0, 2\pi]$. Near the first stationary point the integrand is dominated by

$$2\mathcal{A}(\theta_o)e^{ikt(\theta-\theta_o)^2/2}.$$

When the limits are approximated by $(-\infty, \infty)$, the inegral can be evaluated to give

$$\mathcal{A}(\theta_o)\int_{-\infty}^{\infty} e^{ikr\theta^2/2}d\theta = \sqrt{\frac{2\pi}{kr}}e^{i\pi/4}\mathcal{A}(\theta_o)$$

Near the second stationary point the integral vanishes since $1+\cos(\theta-\theta_o) = 1-1 = 0$. Hence the result (8.5) follows.