1.138J/2.062J/18.376J, WAVE PROPAGATION

Fall, 2006 MIT

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Homework set no 3,   Due Oct 24,2006

1 Refraction of obliquely incident water wave in a shallow sea

Read Section 1.5 Chapter 1. Start from eq (5.11).

Consider one dimensional bathymetry: $h = h(x)$ varying slowly from h_0 at $x \sim -\infty$ to h_1 at $x \sim \infty$. A simple harmonic waves arrives from $x \sim -\infty$. at an angle. Let

$$\zeta(x, y, t) = \Re[\eta(x, y)e^{-i\omega t}] \tag{H.1.1}$$

Find the governing equation for $\eta(x, y)$. Assuming oblique incidence so that the incident wave is given by

$$\eta_I(x, y) = Ae^{i\alpha_0 x + i\beta y}, \quad x \sim -\infty \tag{H.1.2}$$

where α is strictly a constant. Show first that in the far field to the left

$$k_0 = \sqrt{\alpha_0^2 + \beta^2} = \frac{\omega}{\sqrt{gh_0}} \tag{H.1.3}$$

then show that the incident wave number vector $\vec{k}_0 = \alpha_0\vec{e}_x + \beta\vec{e}_y$ is inclined with respect to the x axis by the angle of incidence θ_0 where

$$\tan\theta_0 = \beta/\alpha_0 \tag{H.1.4}$$

When the wave enters the zone of slowly varying depth. Try a solution of the form

$$\eta = A(x)\exp\left(i\int^x \alpha(x')dx' + i\beta y\right), \tag{H.1.5}$$

By assuming that $A(x), \alpha(x), h(x)$ vary slowly in x within a wavelength. show that to the leading order

$$k(x) = \sqrt{\alpha(x)^2 + \beta^2} = \frac{\omega}{\sqrt{gh(x)}} \tag{H.1.6}$$

How does the direction of local wave number vector $\vec{k}(x) = \vec{\alpha}(x)\vec{e}_x + \beta\vec{e}_y$ vary with x?

How does the direction of wave, and the wave length and phase velocity change from deeper to shallower water?

2 Reflection and transmission at the interface of two membranes

Consider two semi-infinite membranes joined together at the common boundary boundary $-\infty < x < \infty, z = 0$. The two membranes are kept taut at the same uniform tension T and have different densities per unit area in the x, y plane.: ρ in $y > 0$ and ρ_1 in $y < 0$. The membrane in the lower half plane $y < 0$ is laterally supported by strings of elastic constant K per unit area. The membrane in the upper half plane is free from lateral support.

A monochromatic plane wave of frequency ω arrives tfrom the $z \sim \infty$,

$$V_I(x, y, t) = A_0 e^{ik(x \sin\theta x - y\cos\theta) - i\omega t} \tag{H.2.7}$$

Find the reflected wave on the side $y > 0$ and the transmitted wave (if any) in $y < 0$. Discuss the result for a wide range of density ratios. Examine the effect of the spring elasticity for high and low frequencies.

3 Identities for scattering coefficients in a shallow sea.

Consider normal incidence of a plane monochromatic water wave over a shallow sea where the depth changes only in one direction: x), from one constant h_- at $x \sim -\infty$ to h_+ at $x \sim \infty$. Within certain limited region near the origin, $h = h(x)$ varies smoothly. Let the incident wave be

$$\zeta_I(x, t) = Ae^{i(k_1 x - \omega t)}, \quad x \sim -\infty. \tag{H.3.8}$$

where $\omega = \sqrt{gh_-}$. In general there will be a reflected wave towards $x \sim -\infty$ and a tranmitted waves towards $x \sim \infty$. Let the reflection and tranmission coefficients be denoted by R and T respectively.

1. . For the special case of a depth discontinuity at $x = 0$, i.e.;, $h = h_-, x < 0; h = h_+, x > 0$. Find T and R explicitly.

2. For any smooth $h(x)$ in a finite region of x, one must use numerical method. Deduce however a theoretical relationship between R and T for checking computations.

3. For the same smooth $h(x)$, consider two different scattering problems: In the first the incident wave comes from $x \sim -\infty$. In the second the incident wave comes from $x \sim \infty$ instead. Deduce a relation between T_1 and T_2.