2.035: Midterm Exam - Part 2 (Take home)
Spring 2007

My education was dismal. I went to a school for mentally disturbed teachers.

Woody Allen

INSTRUCTIONS:

- **Do not spend more than 4 hours.**
- **Please give reasons justifying each (nontrivial) step in your calculations.**
- **You may use**
  - **(i) the notes you took in class,**
  - **(ii) any other handwritten notes you may have made in your own handwriting,**
  - **(iii) any handouts I gave out including the bound set of notes, and**
  - **(iv) Chapters 1, 2 and 3 only of the textbook by Knowles.**
- **You should not use any other portions of Knowles' book (not even the appendices).**
- **No other sources are to be used.**
- **Your completed solutions are due no later than 11:00 AM on Thursday April 5.**
- **Please include, on the first page of your solutions, a signed statement confirming that you adhered to all of the instruction here especially the time limit and the permitted resources.**

Problem 1: Consider the set V of all 2×2 skew symmetric matrices

$$\boldsymbol{x} = \begin{pmatrix} 0 & x_{12} \\ x_{21} & 0 \end{pmatrix}, \qquad x_{12} = -x_{21},$$

with addition and multiplication by a scalar defined in the "natural way".

a) Show that V is a vector space.

b) Give an example of a set of 2 linearly *dependent* vectors in V, and an example of 2 linearly *independent* vectors in V.

c) What is the dimension of V?

d) If $\boldsymbol{x}$ and $\boldsymbol{y}$ are two vectors in V, show that

$$\boldsymbol{x} \cdot \boldsymbol{y} = x_{12}y_{12} + x_{21}y_{21}$$

is a proper definition of a scalar product on V. **From hereon assume that** the vector space V has been made Euclidean with this scalar product.

e) Find an orthonormal basis for V.

f) Let $\boldsymbol{A}$ be the transformation defined by

$$\boldsymbol{A}\boldsymbol{x} = \begin{pmatrix} 0 & 2x_{12} \\ 2x_{21} & 0 \end{pmatrix} \qquad \text{for all vectors} \quad \boldsymbol{x} \in \mathsf{V}.$$

Show that $\boldsymbol{A}$ is a tensor.

g) Show that the tensor $\boldsymbol{A}$ is symmetric.

h) Is $\boldsymbol{A}$ singular or nonsingular?

i) Calculate the eigenvalues of $\boldsymbol{A}$.

Problem 2: Consider the 3-dimensional Euclidean vector space V which is comprised of all polynomials of degree ≤ 2; a typical vector in V has the form

$$\boldsymbol{x} = x(t) = c_o + c_1 t + c_2 t^2.$$

Addition and multiplication by a scalar are defined in the "natural way". The scalar product between two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined as

$$\boldsymbol{x} \cdot \boldsymbol{y} = \int_{-1}^{1} x(t)y(t)dt.$$

Determine an orthonormal basis for V. (Hint: First find any basis for V and then use the Gram-Schmidt process described in Problem 1.17 of Knowles.)

Problem 3: Let $\boldsymbol{A}$ be a symmetric positive definite tensor on a n-dimensional vector space V. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the eigenvalues of $\boldsymbol{A}$ where the eigenvalues are ordered according to $0 < \alpha_1 \leq \alpha_2 \ldots \leq \alpha_n$. Show that the smallest eigenvalue

$$\alpha_1 = \min\Big(\boldsymbol{x} \cdot \boldsymbol{A}\boldsymbol{x}\Big)$$

where the minimization is taken over all unit vectors $\boldsymbol{x} \in \mathsf{V}$. (Hint: Work using the components of $\boldsymbol{A}$ and $\boldsymbol{x}$ in a principal basis of $\boldsymbol{A}$.)

Show similarly that the largest eigenvalue

$$\alpha_n = \max\Big(\boldsymbol{x} \cdot \boldsymbol{A}\boldsymbol{x}\Big)$$

where the maximization is taken over all unit vectors $\boldsymbol{x} \in \mathsf{V}$.

Problem 4: If $\mathbf{A}$ is an arbitrary nonsingular tensor, show that

$$\Big(\boldsymbol{A}^{-1}\Big)^T = \Big(\boldsymbol{A}^T\Big)^{-1}.$$

Problem 5: Give an example of a tensor $\boldsymbol{A}$ (that is NOT the identity tensor $\boldsymbol{I}$) which has the property

$$\boldsymbol{A} = \boldsymbol{A}^2 = \boldsymbol{A}^3 = \boldsymbol{A}^4 = \ldots$$

Hint: Think geometrically.

Problem 6: Let $\boldsymbol{c}$ and $\boldsymbol{d}$ be two distinct non-zero vectors belonging to a 3-dimensional Euclidean vector space. Show (geometrically by drawing arrows or some other way) that one can always find a unit vector $\boldsymbol{x}$ such that $(\boldsymbol{c} \cdot \boldsymbol{x})(\boldsymbol{d} \cdot \boldsymbol{x}) > 0$; and that one can always find some other unit vector $\boldsymbol{x}$ such that $(\boldsymbol{c} \cdot \boldsymbol{x})(\boldsymbol{d} \cdot \boldsymbol{x}) < 0$.

Let $\boldsymbol{C}$ be the symmetric tensor defined by

$$\boldsymbol{C} = \boldsymbol{I} + \boldsymbol{c} \otimes \boldsymbol{d} + \boldsymbol{d} \otimes \boldsymbol{c}.$$

Show (using the result of Problem 3 or otherwise) that the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of $\boldsymbol{C}$ have the property that

$$\lambda_1 < 0, \qquad \lambda_2 = 0, \qquad \lambda_3 > 0.$$

(*Remark*: This result is the key to showing the Ball and James Theorem mentioned in class when we were discussing material microstructures.)
