# 2.035: Midterm Exam - Part 1

# Spring 2007
# SOLUTION

PROBLEM 1:

a) A *vector space* is a set $\mathsf{V}$ of elements called vectors together with operations of addition and multiplication by a scalar, where these operations must have the following properties:

(A) Corresponding to every pair of vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathsf{V}$ there is a vector in $\mathsf{V}$, denoted by $\boldsymbol{x}+\boldsymbol{y}$, and called the sum of $\boldsymbol{x}$ and $\boldsymbol{y}$, with the following properties:

(1) $\boldsymbol{x}+\boldsymbol{y} = \boldsymbol{y}+\boldsymbol{x}$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathsf{V}$;

(2) $\boldsymbol{x}+(\boldsymbol{y}+\boldsymbol{z}) = (\boldsymbol{x}+\boldsymbol{y})+\boldsymbol{z}$ for all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathsf{V}$;

(3) there is a unique vector in $\mathsf{V}$, denoted by $\boldsymbol{o}$ and called the null vector, with the property that $\boldsymbol{x}+\boldsymbol{o} = \boldsymbol{x}$ for all $\boldsymbol{x} \in \mathsf{V}$; and

(4) corresponding to every vector $\boldsymbol{x} \in \mathsf{V}$ there is a unique vector in $\mathsf{V}$, denoted by $-\boldsymbol{x}$ with the property that $\boldsymbol{x}+(-\boldsymbol{x}) = \boldsymbol{o}$.

(B) Corresponding to every real number $\alpha \in \mathbb{R}$ and every vector $\boldsymbol{x} \in \mathsf{V}$ there is a vector in $\mathsf{V}$, denoted by $\alpha\boldsymbol{x}$, and called the product of $\alpha$ and $\boldsymbol{x}$, with the following properties:

(5) $\alpha(\beta\boldsymbol{x}) = (\alpha\beta)\boldsymbol{x}$ for all $\alpha, \beta \in \mathbb{R}$ and all $\boldsymbol{x} \in \mathsf{V}$;

(6) $\alpha(\boldsymbol{x}+\boldsymbol{y}) = \alpha\boldsymbol{x}+\alpha\boldsymbol{y}$ for all $\alpha \in \mathbb{R}$ and all $\boldsymbol{x}, \boldsymbol{y} \in \mathsf{V}$;

(7) $(\alpha+\beta)\boldsymbol{x} = \alpha\boldsymbol{x}+\beta\boldsymbol{x}$ for all $\alpha, \beta \in \mathbb{R}$ and all $\boldsymbol{x} \in \mathsf{V}$; and

(8) $1\boldsymbol{x} = \boldsymbol{x}$ for all $\boldsymbol{x} \in \mathsf{V}$.

b) A set of vectors $\{\boldsymbol{f}_1, \boldsymbol{f}_2, \ldots, \boldsymbol{f}_n\}$ is said to be *linearly independent* if the only scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ for which

$$\alpha_1\boldsymbol{f}_1 \;+\; \alpha_2\boldsymbol{f}_2 \ldots + \alpha_n\boldsymbol{f}_n = \boldsymbol{o}$$

are $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$.

c) If a vector space $\mathsf{V}$ contains a linearly independent set of $n\,(>0)$ vectors but contains no linearly independent set of $n+1$ vectors we say that the *dimension* of $\mathsf{V}$ is $n$.

d) If $\mathsf{V}$ is a $n$-dimensional vector space then any set of $n$ linearly independent vectors is called a *basis* for $\mathsf{V}$.

e) If $\{\boldsymbol{f}_1, \boldsymbol{f}_2, \ldots, \boldsymbol{f}_n\}$ is a basis for an $n$-dimensional vector space $\mathsf{V}$, then any vector $\boldsymbol{x} \in \mathsf{V}$ can be expressed in the form

$$\boldsymbol{x} = \xi_1\boldsymbol{f}_1 + \xi_2\boldsymbol{f}_2 + \ldots + \boldsymbol{\xi}_n\boldsymbol{f}_n$$

where the set of scalars $\xi_1, \xi_2, \ldots, \xi_n$ is unique and are called the *components* of $\boldsymbol{x}$ in the basis $\{\boldsymbol{f}_1, \boldsymbol{f}_2, \ldots, \boldsymbol{f}_n\}$.

f) To every pair of vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathsf{V}$ we associate a real number denoted by $\boldsymbol{x} \cdot \boldsymbol{y}$ and called the *scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ provided that this product has the following properties:

(9) $\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{y} \cdot \boldsymbol{x}$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathsf{V}$;

(10) $(\boldsymbol{x} + \boldsymbol{y}) \cdot \boldsymbol{z} = \boldsymbol{x} \cdot \boldsymbol{z} + \boldsymbol{y} \cdot \boldsymbol{z}$ for all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathsf{V}$;

(11) $(\alpha \boldsymbol{x}) \cdot \boldsymbol{y} = \alpha(\boldsymbol{x} \cdot \boldsymbol{y})$ for all $\alpha \in \mathbb{R}$ and all vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathsf{V}$; and

(12) $\boldsymbol{x} \cdot \boldsymbol{x} > 0$ for all vectors $\boldsymbol{x} \neq \boldsymbol{o}$ in $\mathsf{V}$.

g) The real number denoted by $|\boldsymbol{x} - \boldsymbol{y}|$ and defined as $|\boldsymbol{x} - \boldsymbol{y}| = \Big( (\boldsymbol{x} - \boldsymbol{y}) \cdot (\boldsymbol{x} - \boldsymbol{y}) \Big)^{1/2}$ is called the *distance* between the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$.

h) If $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ is a basis for an $n$-dimensional vector space and if

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \begin{cases} 0 & \text{if} \quad i \neq j, \\ 1 & \text{if} \quad i = j \end{cases} \qquad i, j = 1, 2, \ldots n,$$

we say that $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ is an *orthonormal basis*.

i) A *linear transformation* $\boldsymbol{A}$ on a vector space $\mathsf{V}$ is a transformation that assigns to each vector $\boldsymbol{x} \in \mathsf{V}$ a unique vector in $\mathsf{V}$ which we denote by $\boldsymbol{A}\boldsymbol{x}$ with the properties:

(13) $\boldsymbol{A}(\boldsymbol{x} + \boldsymbol{y}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{A}\boldsymbol{y}$ for all vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathsf{V}$; and

(14) $\boldsymbol{A}(\alpha \boldsymbol{x}) = \alpha(\boldsymbol{A}\boldsymbol{x})$ for every $\alpha \in \mathbb{R}$ and every vector $\boldsymbol{x} \in \mathsf{V}$.

j) Let $\mathsf{S}$ be a subset of a vector space $\mathsf{V}$. Suppose further that $\mathsf{S}$ itself is in fact a vector space on its own right under the same operations of addition and scalar multiplication as in $\mathsf{V}$. Then $\mathsf{S}$ is said to be a subspace of $\mathsf{V}$. Finally, suppose in addition that $\boldsymbol{A}\boldsymbol{x} \in \mathsf{S}$ for all $\boldsymbol{x} \in \mathsf{S}$. Then we say that $\mathsf{S}$ is an *invariant subspace* of $\boldsymbol{A}$.

k) The set $\mathsf{N}$ of all vectors $\boldsymbol{x}$ for which $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{o}$ is called the *null space* of $\boldsymbol{A}$.

$\ell$) A linear transformation $\boldsymbol{A}$ is said to be *singular* if there is a vector $\boldsymbol{x} \neq \boldsymbol{o}$ for which $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{o}$.

m) The $n^2$ real numbers $A_{ij}$ defined by

$$A_{ij} = \boldsymbol{e}_j \cdot \boldsymbol{A}\boldsymbol{e}_i$$

are called the *components* of the linear transformation $\boldsymbol{A}$ in the basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$.

n) A scalar valued function $\phi$ defined on the set of all linear transformations is said to be a *scalar invariant* if $\phi(\boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^T) = \phi(\boldsymbol{A})$ for every linear transformation $\boldsymbol{A}$ and all orthogonal linear transformations $\boldsymbol{Q}$.

PROBLEM 2:

a) Consider the set $\mathsf{V}$ of all $2 \times 2$ matrices $\boldsymbol{x}$ of the form

$$\boldsymbol{x} = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_1 \end{pmatrix}$$

where $x_1$ and $x_2$ range over all real numbers; let

$$\boldsymbol{o} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

be the null vector; and define addition, $\boldsymbol{x} + \boldsymbol{y}$, and scalar multiplication, $\alpha\boldsymbol{x}$, in the natural way by

$$\begin{pmatrix} x_1 & x_2 \\ x_2 & x_1 \end{pmatrix} + \begin{pmatrix} y_1 & y_2 \\ y_2 & y_1 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 & x_2 + y_2 \\ x_2 + y_2 & x_1 + y_1 \end{pmatrix}, \qquad \alpha \begin{pmatrix} x_1 & x_2 \\ x_2 & x_1 \end{pmatrix} = \begin{pmatrix} \alpha x_1 & \alpha x_2 \\ \alpha x_2 & \alpha x_1 \end{pmatrix}.$$

One can verify that all of the requirements (1)–(8) of Problem 1 are satisfied by these operations, and moreover, that $\boldsymbol{x} + \boldsymbol{y}$ and $\alpha\boldsymbol{x}$ are both in $\mathsf{V}$ when $\boldsymbol{x}, \boldsymbol{y} \in \mathsf{V}$ and $\alpha \in \mathbb{R}$. Thus $\mathsf{V}$ is a vector space.

b) Consider the following two vectors $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$:

$$\boldsymbol{f}_1 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \qquad \boldsymbol{f}_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

One can readily verify that if $\alpha_1\boldsymbol{f}_1 + \alpha_2\boldsymbol{f}_2 = \boldsymbol{o}$, then necessarily $\alpha_1 + 2\alpha_2 = 0$ and $2\alpha_1 + \alpha_2 = 0$ which in turn implies that $\alpha_1 = \alpha_2 = 0$. Thus $\{\boldsymbol{f}_1, \boldsymbol{f}_2\}$ is a linearly independent set of vectors.

c) Consider the following three vectors $\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{x}$,

$$\boldsymbol{f}_1 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \qquad \boldsymbol{f}_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \qquad \boldsymbol{x} = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_1 \end{pmatrix}.$$

where $\boldsymbol{x}$ is an arbitrary vector in $\mathsf{V}$. One can readily verify that if $\alpha_1\boldsymbol{f}_1 + \alpha_2\boldsymbol{f}_2 + \alpha_3\boldsymbol{x} = \boldsymbol{o}$ then necessarily $\alpha_1 + 2\alpha_2 + x_1\alpha_3 = 0$ and $2\alpha_1 + \alpha_2 + x_2\alpha_3 = 0$. Observe that the choice

$$\alpha_1 = \frac{1}{3}(2x_2 - x_1), \quad \alpha_2 = \frac{1}{3}(2x_1 - x_2), \quad \alpha_3 = -1$$

satisfies these two scalar equations. Thus if $\alpha_1\boldsymbol{f}_1 + \alpha_2\boldsymbol{f}_2 + \alpha_3\boldsymbol{x} = \boldsymbol{o}$ this does not require that all the $\alpha$'s vanish and so $\{\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{x}\}$ is a linearly dependent set of vectors. Recall that $\{\boldsymbol{f}_1, \boldsymbol{f}_2\}$ is a linearly independent set of vectors. Thus the dimension of $\mathsf{V}$ is 2.

d) Since $\mathsf{V}$ is a 2-dimensional vector space and since the set of vectors $\{\boldsymbol{f}_1, \boldsymbol{f}_2\}$ is linearly independent, it follows that $\{\boldsymbol{f}_1, \boldsymbol{f}_2\}$ is a basis for $\mathsf{V}$.

e) Consider the basis $\{\boldsymbol{f}_1, \boldsymbol{f}_2\}$ and let $\boldsymbol{x}$ be an arbitrary vector in $\mathsf{V}$. Then one can readily verify that

$$\boldsymbol{x} = \xi_1\boldsymbol{f}_1 + \xi_2\boldsymbol{f}_2 \qquad \text{where} \quad \xi_1 = \frac{1}{3}(2x_2 - x_1) \quad \text{and} \quad \xi_2 = \frac{1}{3}(2x_1 - x_2)$$

are the components of $\boldsymbol{x}$ in the basis $\{\boldsymbol{f}_1, \boldsymbol{f}_2\}$.

f) Corresponding to any two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathsf{V}$, where

$$\boldsymbol{x} = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_1 \end{pmatrix}, \qquad \boldsymbol{y} = \begin{pmatrix} y_1 & y_2 \\ y_2 & y_1 \end{pmatrix},$$

tentatively define their scalar product as

$$\boldsymbol{x} \cdot \boldsymbol{y} = x_1 y_1 + x_2 y_2.$$

One can verify that this definition satisfies all of the requirement (9)–(12) of Problem 1 and therefore is in fact a legitimate definition of a scalar product.

g) The distance between the two vectors

$$\boldsymbol{x} = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_1 \end{pmatrix}, \qquad \boldsymbol{y} = \begin{pmatrix} y_1 & y_2 \\ y_2 & y_1 \end{pmatrix}$$

is

$$|\boldsymbol{x} - \boldsymbol{y}| = \Big((\boldsymbol{x} - \boldsymbol{y}) \cdot (\boldsymbol{x} - \boldsymbol{y})\Big)^{1/2} = \Big((x_1 - y_1)^2 + (x_2 - y_2)^2\Big)^{1/2}.$$

h) Consider the two vectors

$$\boldsymbol{e}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \boldsymbol{e}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Observe that $\boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = 0, |\boldsymbol{e}_1| = |\boldsymbol{e}_2| = 1$ and so $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ forms an orthonormal basis for $\mathsf{V}$.

i) Consider a transformation $\boldsymbol{A}$ that takes the vector

$$\boldsymbol{x} = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_1 \end{pmatrix} \qquad \text{into the vector} \quad \boldsymbol{A}\boldsymbol{x} = \begin{pmatrix} x_2 & x_1 \\ x_1 & x_2 \end{pmatrix}$$

One can verify that the requirements (13), (14) of Problem 1 are satisfied, and moreover that $\boldsymbol{A}\boldsymbol{x} \in \mathsf{V}$ for all $\boldsymbol{x} \in \mathsf{V}$. Therefore $\boldsymbol{A}$ is a linear transformation.

j) Consider the set $\mathsf{S}$ of all vectors $\boldsymbol{x}$ of the form

$$\boldsymbol{x} = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$$

where $x$ ranges over all real numbers. Clearly $\mathsf{S}$ is a subset of $\mathsf{V}$. Moreover, one can verify that $\mathsf{S}$ itself is a vector space on its own right under the same operations of addition and scalar multiplication as in $\mathsf{V}$. Thus $\mathsf{S}$ is a subspace of $\mathsf{V}$. Furthermore, observe that $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}$ for all vectors $\boldsymbol{x} \in \mathsf{S}$, so that in particular $\boldsymbol{A}\boldsymbol{x} \in \mathsf{S}$ for all $\boldsymbol{x} \in \mathsf{S}$. Thus $\mathsf{S}$ is an *invariant subspace* of $\boldsymbol{A}$. (In fact it is a one-dimensional invariant subspace associated with the eigenvalue +1).

k) From item (i) we see that if $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{o}$ then necessarily $\boldsymbol{x} = \boldsymbol{o}$. Thus the null space of $\boldsymbol{A}$ is comprised of a single vector, the null vector: $\mathsf{N} = \{\boldsymbol{o}\}$.

$\ell$) As noted in the preceding item, $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{o}$ implies that necessarily $\boldsymbol{x} = \boldsymbol{o}$. Therefore $\boldsymbol{A}$ is nonsingular.

m) Observe from the definitions of $\boldsymbol{A}, \boldsymbol{e}_1$ and $\boldsymbol{e}_2$ that $\boldsymbol{A}\boldsymbol{e}_1 = \boldsymbol{e}_2$ and $\boldsymbol{A}\boldsymbol{e}_2 = \boldsymbol{e}_1$. Thus the components of $\boldsymbol{A}$ in the basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ are

$$A_{11} = \boldsymbol{e}_1 \cdot \boldsymbol{A}\boldsymbol{e}_1 = \boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = 0, \qquad A_{12} = \boldsymbol{e}_1 \cdot \boldsymbol{A}\boldsymbol{e}_2 = \boldsymbol{e}_1 \cdot \boldsymbol{e}_1 = 1,$$

$$A_{22} = \boldsymbol{e}_2 \cdot \boldsymbol{A}\boldsymbol{e}_2 = \boldsymbol{e}_2 \cdot \boldsymbol{e}_1 = 0, \qquad A_{21} = \boldsymbol{e}_2 \cdot \boldsymbol{A}\boldsymbol{e}_1 = \boldsymbol{e}_2 \cdot \boldsymbol{e}_2 = 1.$$

n) Consider the scalar-valued function $\phi(\boldsymbol{A}) = \det \boldsymbol{A}$ defined for all linear transformations $\boldsymbol{A}$. Then for any linear transformation $\boldsymbol{A}$ and any orthogonal transformation $\boldsymbol{Q}$ we have $\phi(\boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^T) = \det(\boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^T) = \det(\boldsymbol{Q}) \det(\boldsymbol{A}) \det(\boldsymbol{Q}^T) = \det(\boldsymbol{Q}) \det(\boldsymbol{A}) \det(\boldsymbol{Q}) = (\pm 1)^2 \det \boldsymbol{A} = \det \boldsymbol{A}$. Thus the function $\phi(\boldsymbol{A}) = \det \boldsymbol{A}$ has the property that $\phi(\boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^T) = \phi(\boldsymbol{A})$ for every linear transformation $\boldsymbol{A}$ and all orthogonal linear transformations $\boldsymbol{Q}$. Thus $\det \boldsymbol{A}$ is a scalar invariant of $\boldsymbol{A}$.