2.035: Midterm Exam - Part 2 (Take home)

Spring 2004

> " Examinations are formidable even to the best prepared, for the greatest fool may ask more than the wisest person may answer."
>
> Charles Caleb Colton (1780-1832)

INSTRUCTIONS:

- **Do not spend more than 3 hours.**
- **Please give reasons justifying each (nontrivial) step in your calculations.**
- **You may use the <u>notes</u> you took in class, Chapters 1, 2 and 3 of the <u>text</u>, and any <u>handouts</u> originating from me.**
- **No other sources are to be used (not even the appendices of the text).**
- **Your completed solutions are due no later than 9:30 AM on Wednesday April 7.**
- **Please include, on the first page of your solutions, a signed statement confirming that you adhered to the time limit and the permitted resources.**

Problem 1: (Knolwes 3.24) A tensor $\boldsymbol{T}$ is symmetric, orthogonal and positive definite. Determine $\boldsymbol{T}$.

Problem 2: (Essentially Knowles 1.18) Let R be the 3-dimensional Euclidean vector space of polynomials of degree not exceeding two, where the scalar product between two vectors $\boldsymbol{f} = f(t)$ and $\boldsymbol{g} = g(t)$ is defined by

$$\boldsymbol{f} \cdot \boldsymbol{g} = \int_{-1}^{1} f(t)g(t)\, dt.$$

i) Show that $\boldsymbol{f}_1 = 1, \boldsymbol{f}_2 = t, \boldsymbol{f}_3 = t^2$ is a basis for R.

ii) Find an orthonormal basis for R.

Problem 3: (Based on Knowles 3.17) Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two symmetric tensors whose matrices of components in an orthonormal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ are

$$[A] = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \qquad \text{and} \qquad [B] = \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix} \qquad \text{respectively.}$$

i) Do $\boldsymbol{A}$ and $\boldsymbol{B}$ have a common principal basis?

ii) Determine a principal basis for $\boldsymbol{A}$.

Problem 4: (Essentially Knowles 3.18) If a tensor $\boldsymbol{P}$ has the property $\boldsymbol{P}\boldsymbol{P}^T = \boldsymbol{I}$ show that

i) $\boldsymbol{P}$ is nonsingular,

ii) $\boldsymbol{P}^T\boldsymbol{P} = \boldsymbol{I}$, and

iii) $\boldsymbol{P}$ is orthogonal, i.e. that $\boldsymbol{P}$ preserves length ($\Leftrightarrow |\boldsymbol{P}\boldsymbol{x}| = |\boldsymbol{x}|$ for all vectors $\boldsymbol{x}$).

Problem 5: (Essentially Knowles 3.10, 3.26). Let $\boldsymbol{A}$ be a skew-symmetric tensor on a finite dimensional Euclidean vector space.

i) If $\boldsymbol{A}$ has a real eigenvalue α, show that $\alpha = 0$.

ii) Show that $\boldsymbol{I} + \boldsymbol{A}$ and $\boldsymbol{I} - \boldsymbol{A}$ are both nonsingular tensors.

iii) Show that $\boldsymbol{I} + \boldsymbol{A}$ and $\boldsymbol{I} - \boldsymbol{A}$ commute, i.e. that $(\boldsymbol{I} + \boldsymbol{A})(\boldsymbol{I} - \boldsymbol{A}) = (\boldsymbol{I} + \boldsymbol{A})(\boldsymbol{I} - \boldsymbol{A})$.

iv) Show that $(\boldsymbol{I} - \boldsymbol{A})(\boldsymbol{I} + \boldsymbol{A})^{-1}$ is an orthogonal tensor.

Problem 6: (Essentially Knowles 2.18) Let $\boldsymbol{A}$ be a symmetric tensor on a n-dimensional Euclidean vector space. Suppose that $\boldsymbol{A}$ has distinct eigenvalues $\alpha_1, \alpha_2, \ldots, \alpha_n$ and a corresponding set of orthonormal eigenvectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$.

i) For any positive integer m, show that $\boldsymbol{A}^m$ (defined as $\underbrace{\boldsymbol{A}\boldsymbol{A}\ldots\boldsymbol{A}}_{\text{m times}}$) has eigenvalues $\alpha_1^m, \alpha_2^m, \ldots, \alpha_n^m$ and corresponding eigenvectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$.

ii) Let $p(x) = \sum_{k=0}^{n} c_k x^k$ be an arbitrary polynomial of degree n where the c's are real numbers. Let $\boldsymbol{P}$ be the tensor defined by $\boldsymbol{P} = \boldsymbol{P}(\boldsymbol{A}) = \sum_{k=0}^{n} c_k \boldsymbol{A}^k$ where $\boldsymbol{A}^0 = \boldsymbol{I}$. Show that the eigenvalues of $\boldsymbol{P}$ are $p(\alpha_1), p(\alpha_2), \ldots, p(\alpha_n)$ and that the corresponding eignvectors are $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$.

iii) Consider the special case where $p(x)$ is the characteristic polynomial of $\boldsymbol{A}$, i.e. $p(x) = \det[\boldsymbol{A} - x\boldsymbol{I}]$. Show that that the corresponding tensor $\boldsymbol{P}(\boldsymbol{A})$ is the null tensor $\boldsymbol{O}$.
