

1.685J/2.034J/18.377J Nonlinear Dynamics and Waves

Take-Home Exam

Thursday, April 5, 2007

This is a closed-book exam. You may use your own class notes, problem sets and solutions posted on the 1.685/2.034/18.377 website only. You are not allowed to discuss this exam with anyone else. If you need clarification regarding the problems, please ask me. The exam is due on Tuesday, April 10th, 2007 before 5:00 pm.

Problem 1 (10 points)

A particle of mass m is constrained to move around a smooth circular wire of radius a fixed in a vertical plane. The particle is acted on by gravity and is attached to one end of a linear spring of natural length $l (< 2a)$ and stiffness k , the other end being fixed to the highest point of the circle.

- Derive the governing equations of motion.
- Find the number of equilibrium points and discuss their stability as k varies.
- Sketch the bifurcation diagram describing these equilibrium states as a function of k .
- Sketch the phase plane for representative values of k .

Problem 2 (10 points)

The motion of a particle restrained by a linear spring and under the combined influence of Coulomb and square damping is governed by

$$\ddot{u} + \omega_0^2 u + \epsilon(\mu_0 \operatorname{sgn} \dot{u} + \mu_2 \dot{u} |\dot{u}|) = 0$$

where μ_0 and $\mu_2 > 0$ and $\epsilon \ll 1$.

- Show that

$$u = a \cos(\omega_0 t + \beta) + O(\epsilon)$$

where

$$\dot{a} = -\epsilon \left(\frac{2\mu_0}{\pi\omega_0} + \frac{4}{3\pi}\mu_2\omega_0 a^2 \right)$$

and

$$\dot{\beta} = 0.$$

(b) Show that

$$a = \frac{1}{\omega_0} \sqrt{\frac{3\mu_0}{2\mu_2}} \tan \left(c - \frac{\epsilon}{\pi} \sqrt{\frac{8\mu_2\mu_0}{3}} t \right)$$

where c is a constant of integration.

(c) Explain how the motion decays. When does it stop?

Problem 3 (10 points)

A certain nonlinear vibratory system is governed by the equation

$$\frac{d^2x}{dt^2} - (\epsilon - \beta|x|) \frac{dx}{dt} + x = 0,$$

where ϵ and β are dimensionless parameters.

(a) The origin, $(x, \frac{dx}{dt}) = (0, 0)$, of the phase plane is an equilibrium point. Based on a linear stability analysis, classify this equilibrium point and state when it is stable depending on the value of ϵ .

(b) Determine the bifurcating finite-amplitude solution branches in the vicinity of the critical value $\epsilon = \epsilon_{crit}$ (i.e. the bifurcation point) at which the equilibrium $(x, \frac{dx}{dt}) = (0, 0)$ becomes unstable. Do the bifurcating solutions correspond to finite-amplitude equilibria (static bifurcation) or finite-amplitude periodic motions (Hopf bifurcation)? Determine the stability of these finite-amplitude states.