1.685J/2.034J/18.377J Nonlinear Dynamics and Waves Spring 2007

Problem Set No. 4

Out: Thursday, April 12, 2007

Due: Thursday, April 26, 2007 *in class*

Problem 1

Consider the system

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial x^2} = \frac{1}{R}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right) - \frac{\partial^4 u}{\partial x^4}$$

and

$$u(x,0,t) = 0, \;\; u(x,1,t) = 1 \;\text{ for } -\infty < x < \infty, \;\; t \geq 0$$

and show that the basic 'flow' $u(x,z,t) = \bar{u} \equiv z$ gives rise to a linearized solution $u = \bar{u} + u'$ with normal modes of the form

$$u'(x,z,t) = \epsilon^{i\alpha x + st}\phi(z),$$

where

$$\phi = \sin\{(n+1)\pi z\}, \;\; s = f(n,\alpha) \equiv \alpha^2(1-\alpha^2) - \{(n+1)^2\pi^2 + \alpha^2\}/R$$

for $n = 0, 1, ...$ and any real wavenumber α. Deduce that the (n,α) mode is stable if and only if $R \leq R_n(\alpha) = \{(n+1)^2\pi^2 + \alpha^2\}/\alpha^2(1-\alpha^2)$. Hence show that the flow is stable if and only if $R \leq R_c = R_0(\alpha_c) = 41.5$ where $\alpha_c^2 = \pi^2\{(1+\pi^{-2})^{1/2} - 1\} = 0.488$.

Show that the Landau equation for the amplitude $A(t)$ of the most unstable mode is

$$\frac{dA}{dt} = f(0, \alpha_c)A - \lambda A^3 \quad \text{for } R \approx R_c,$$

where

$$\lambda = -\frac{1}{4}\pi^2\alpha_c^4 \left[\frac{1}{f(1,0)} - \frac{1}{2f(1,2\alpha_c)}\right]_{R=R_c} > 0.$$

Problem 2

A particle of mass m_1 is attached to a light rigid rod of length l which is free to rotate in the vertical plane as shown below. A bead of mass m_2 is free to slide along the smooth rod under the action of the spring.

(a) Show that the governing equations are

$$\ddot{u} + \omega_1^2 u - u\dot{\theta}^2 + \omega_2^2(1 - \cos\theta) = \omega_1^2 u_e$$

$$(1 + mu^2)\ddot{\theta} + (1 + mu)\omega_2^2 \sin\theta + 2mu\dot{u}\dot{\theta} = 0$$

where $\omega_1^2 = k/m_2$, $\omega_2^2 = g/l, m = m_2/m_1, u = x/l$, and u_e is the equilibrium position.

(b) Retaining up to quadratic terms in the governing equations, determine a uniform expansion when the natural frequency of the bouncing mode is close to twice the natural frequency of the swinging mode. Using computer simulations of the system response, explore the range of validity of this expansion.

(c) What other resonances exist to this order?