1.685J/2.034J/18.377J Nonlinear Dynamics and Waves Spring 2007

Problem Set No. 2

Out: Thursday, March 1, 2007

Due: Thursday, March 15, 2007 *in class*

Problem 1

The cylinder rolls back and forth without slip as shown in the figure below.

(a) Show that the equation of motion can be written in the form

$$\dot{x} + \omega^2[1 - l(1 + x^2)^{-1/2}]x = 0$$

where $\omega^2 = 2k/3M$ and l is the free length of the spring. All lengths were made dimensionless with respect to the radius r.

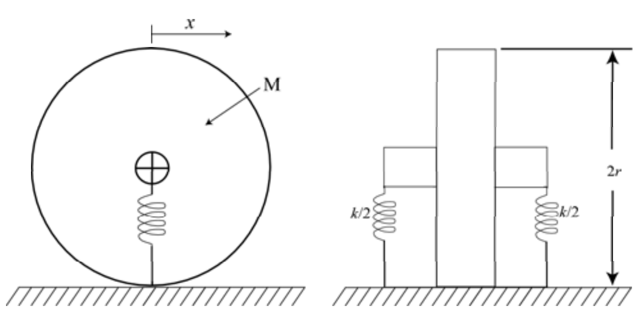
(b) Sketch the potential energy as a function of x for

(i) $1 \leq l$

(ii) $1 > l$

Show the equilibrium positions and indicate whether they are stable or unstable.

(c) For $l = \sqrt{2}$, obtain a two-term frequency-amplitude relationship for small oscillations around the equilibrium position.



Problem 2

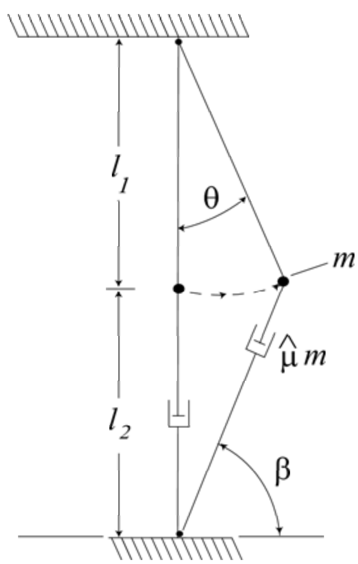
Consider a simple pendulum with a dashpot as shown below.

(a) Show that the equation of motions is

$$ml_1\ddot{\theta} = -mg\sin\theta - \hat{\mu}ml_1\dot{\theta}\cos^2(\beta-\theta). \tag{1}$$

Then show that (1) can also be written as

$$\ddot{\theta} + \omega^2\sin\theta + \frac{\hat{\mu}(l_1+l_2)^2\sin^2\theta}{l_2^2 + 2l_1(l_1+l_2)(1-\cos\theta)}\dot{\theta} = 0. \tag{2}$$



(b) Expanding and retaining through the cubic terms, show that (2) becomes

$$\ddot{\theta} + \omega^2(1 - \frac{1}{6}\theta^2)\theta + 2\mu\theta^2\dot{\theta} = 0 \tag{3}$$

where

$$2\mu = \frac{\hat{\mu}(l_1 + l_2)^2}{l_2^2}. \tag{4}$$

Using (3), obtain the following first approximation for θ when the amplitude of the motion is small but finite:

$$\theta = \frac{a_0}{\sqrt{1 + \frac{1}{2}\mu a_0^2 t}} \cos\left\{\omega\left[t - \frac{\ln(1 + \frac{1}{2}\mu a_0^2 t)}{8\mu}\right] + \beta_0\right\} \tag{5}$$

where a_0 and β_0 are constants of integration. Note that μ is not small and that in this case the frequency is affected by the damping in the first approximation. As a check, show that in the limit as $\mu \to 0$ equation (5) reduces to

$$\theta = a_0 \cos[\omega(1 - \frac{1}{16}\, a_0^2)t + \beta_0]. \tag{6}$$

Problem 3

The response of a nonlinear system to harmonic excitation is governed by the following equation:

$$\ddot{x} + 2\zeta\dot{x}|\dot{x}| + x + \beta\epsilon x^3 = \cos\frac{\Omega}{\omega_0}t,$$

where $\Omega/\omega_0 \approx 1$. Assume light damping ($\zeta \ll 1$) and weak nonlinearity ($0 < \epsilon \ll 1$) with $\beta = O(1)$.

(a) Find the appropriate scaling of the small parameter ζ, in terms of ϵ, so that light damping and weak nonlinearity balance. What is the width and height of the resonance peak in terms of ϵ?

(b) Under the assumptions in (a), derive evolution equations for the response. Which method out of the three we learned in class (Poincaré-Lindstedt, multiple scales, averaging) is best suited for this problem? Why?

(c) Obtain the frequency-response equation. Is there a jump phenomenon associated with this motion? Is this motion bounded?

Problem 4

A two-degree-of-freedom system is governed by the following coupled (dimensionless) equations

$$\begin{cases} \dfrac{d^2x}{dt^2} + \omega_1^2 x = y, \\ \\ \dfrac{d^2y}{dt^2} + \omega_2^2 y = \epsilon^2 \beta x^3, \end{cases}$$

subject to initial conditions

$$x(0) = x_0, \quad y(0) = y_0, \quad \dot{x}(0) = u_0, \quad \dot{y}(0) = v_0.$$

Here ϵ is a measure of nonlinearity and β is an $O(1)$ parameter.

In the limit $\epsilon \to 0$ and away from resonance ($\omega_1 \neq \omega_2$), the linear response of this system consists of two harmonics with frequencies ω_1 and ω_2:

$$y(t) = y_0 \cos \omega_2 t + \frac{v_0}{\omega_2} \sin \omega_2 t,$$

$$x(t) = \frac{y(t)}{\omega_1^2 - \omega_2^2} + \frac{1}{\omega_1} \left(u_0 - \frac{v_0}{\omega_1^2 - \omega_2^2} \right) \sin \omega_1 t + \left(x_0 - \frac{y_0}{\omega_1^2 - \omega_2^2} \right) \cos \omega_1 t.$$

Note that in this expression $x(t)$ becomes unbounded at resonance ($\omega_2 = \omega_2$).

Your job is to construct a uniformly valid expansion that describes the weakly nonlinear ($0 < \epsilon \ll 1$) response of this system near resonance conditions ($\omega_1 \approx \omega_2$).

Clue: Based on the linear response exactly at resonance ($\omega_1 = \omega_2$), use a 'naive' expansion to deduce the timescale on which nonlinear effects come into play as well as the appropriate re-scaling of x and y near resonance.