13 MATH FACTS

13.1 Vectors

13.1.1 Definition

We use the overhead arrow to denote a column vector, i.e., a *linear segment with a direction*. For example, in three-space, we write a vector in terms of its components with respect to a reference system as

$$\vec{a} = \left\{ \begin{array}{c} 2 \\ 1 \\ 7 \end{array} \right\}.$$

The elements of a vector have a graphical interpretation, which is particularly easy to see in two or three dimensions.

1. Vector addition:

$$\begin{aligned} \vec{a} + \vec{b} &= \vec{c} \\ \left\{ \begin{array}{c} 2 \\ 1 \\ 7 \end{array} \right\} + \left\{ \begin{array}{c} 3 \\ 3 \\ 2 \end{array} \right\} &= \left\{ \begin{array}{c} 5 \\ 4 \\ 9 \end{array} \right\}. \end{aligned}$$

   Graphically, addition is stringing the vectors together head to tail.

2. Scalar multiplication:

$$-2 \times \left\{ \begin{array}{c} 2 \\ 1 \\ 7 \end{array} \right\} = \left\{ \begin{array}{r} -4 \\ -2 \\ -14 \end{array} \right\}.$$

13.1.2 Vector Magnitude

The total length of a vector of dimension m, its Euclidean norm, is given by

$$||\vec{x}|| = \sqrt{\sum_{i=1}^{m} x_i^2}.$$

This scalar is commonly used to normalize a vector to length one.

13.1.3 Vector Dot or Inner Product

The dot product of two vectors is a scalar equal to the sum of the products of the corresponding components:

$$\vec{x} \cdot \vec{y} = \vec{x}^T \vec{y} = \sum_{i=1}^{m} x_i y_i.$$

The dot product also satisfies

$$\vec{x} \cdot \vec{y} = ||\vec{x}|| ||\vec{y}|| \cos\theta,$$

where θ is the angle between the vectors.

13.1.4 Vector Cross Product

The cross product of two three-dimensional vectors $\vec{x}$ and $\vec{y}$ is another vector $\vec{z}$, $\vec{x} \times \vec{y} = \vec{z}$, whose

1. direction is normal to the plane formed by the other two vectors,
2. direction is given by the right-hand rule, rotating from $\vec{x}$ to $\vec{y}$,
3. magnitude is the area of the parallelogram formed by the two vectors – the cross product of two parallel vectors is zero – and
4. (signed) magnitude is equal to $||\vec{x}|| ||\vec{y}|| \sin\theta$, where θ is the angle between the two vectors, measured from $\vec{x}$ to $\vec{y}$.

In terms of their components,

$$\vec{x} \times \vec{y} \quad = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \left\{ \begin{array}{c} (x_2 y_3 - x_3 y_2)\hat{i} \\ (x_3 y_1 - x_1 y_3)\hat{j} \\ (x_1 y_2 - x_2 y_1)\hat{k} \end{array} \right\}.$$

13.2 Matrices

13.2.1 Definition

A matrix, or array, is equivalent to a set of column vectors of the same dimension, arranged side by side, say

$$A = [\vec{a} \;\; \vec{b}] = \begin{bmatrix} 2 & 3 \\ 1 & 3 \\ 7 & 2 \end{bmatrix}.$$

This matrix has three rows ($m = 3$) and two columns ($n = 2$); a vector is a special case of a matrix with one column. Matrices, like vectors, permit addition and scalar multiplication. We usually use an upper-case symbol to denote a matrix.

13.2.2 Multiplying a Vector by a Matrix

If A_{ij} denotes the element of matrix A in the i'th row and the j'th column, then the multiplication $\vec{c} = A\vec{v}$ is constructed as:

$$c_i = A_{i1}v_1 + A_{i2}v_2 + \cdots + A_{in}v_n = \sum_{j=1}^{n} A_{ij}v_j,$$

where n is the number of columns in A. $\vec{c}$ will have as many rows as A has rows (m). Note that this multiplication is defined only if $\vec{v}$ has as many rows as A has columns; they have consistent *inner dimension* n. The product $\vec{v}A$ would be well-posed only if A had one row, and the proper number of columns. There is another important interpretation of this vector multiplication: Let the subscript : indicate all rows, so that each $A_{:j}$ is the j'th column vector. Then

$$\vec{c} = A\vec{v} = A_{:1}v_1 + A_{:2}v_2 + \cdots + A_{:n}v_n.$$

We are multiplying column vectors of A by the scalar elements of $\vec{v}$.

13.2.3 Multiplying a Matrix by a Matrix

The multiplication $C = AB$ is equivalent to a side-by-side arrangement of column vectors $C_{:j} = AB_{:j}$, so that

$$C = AB = [AB_{:1} \;\; AB_{:2} \;\; \cdots \;\; AB_{:k}],$$

where k is the number of columns in matrix B. The same inner dimension condition applies as noted above: the number of columns in A must equal the number of rows in B. Matrix multiplication is:

1. Associative. $(AB)C = A(BC)$.
2. Distributive. $A(B + C) = AB + AC$, $(B + C)A = BA + CA$.
3. NOT Commutative. $AB \neq BA$, except in special cases.

13.2.4 Common Matrices

Identity. The identity matrix is usually denoted I, and comprises a square matrix with ones on the diagonal, and zeros elsewhere, e.g.,

$$I_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The identity always satisfies $AI_{n\times n} = I_{m\times m}A = A$.

Diagonal Matrices. A diagonal matrix is square, and has all zeros off the diagonal. For instance, the following is a diagonal matrix:

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

The product of a diagonal matrix with another diagonal matrix is diagonal, and in this case the operation is commutative.

13.2.5 Transpose

The transpose of a vector or matrix, indicated by a T superscript results from simply swapping the row-column indices of each entry; it is equivalent to "flipping" the vector or matrix around the diagonal line. For example,

$$\begin{aligned} \vec{a} &= \left\{ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \right\} \longrightarrow \vec{a}^{\,T} = \{1 \;\; 2 \;\; 3\} \\ A &= \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 8 & 9 \end{bmatrix} \longrightarrow A^T = \begin{bmatrix} 1 & 4 & 8 \\ 2 & 5 & 9 \end{bmatrix}. \end{aligned}$$

A very useful property of the transpose is

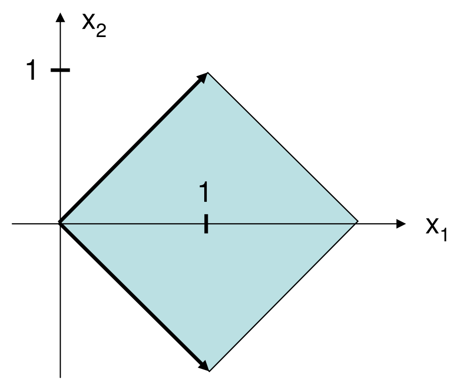
$$(AB)^T = B^T A^T.$$

13.2.6 Determinant

The determinant of a square matrix A is a scalar equal to *the volume* of the parallelepiped enclosed by the constituent vectors. The two-dimensional case is particularly easy to remember, and illustrates the principle of volume:

$$det(A) = A_{11}A_{22} - A_{21}A_{12}$$

$$det\left(\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\right) = 1 + 1 = 2.$$



In higher dimensions, the determinant is more complicated to compute. The general formula allows one to pick a row k, perhaps the one containing the most zeros, and apply

$$det(A) = \sum_{j=1}^{j=n} A_{kj}(-1)^{k+j}\Delta_{kj},$$

where Δ_{kj} is the determinant of the sub-matrix formed by neglecting the k'th row and the j'th column. The formula is symmetric, in the sense that one could also target the k'th column:

$$det(A) = \sum_{j=1}^{j=n} A_{jk}(-1)^{k+j}\Delta_{jk}.$$

If the determinant of a matrix is zero, then the matrix is said to be singular – there is no volume, and this results from the fact that the constituent vectors do not span the matrix dimension. For instance, in two dimensions, a singular matrix has the vectors colinear; in three dimensions, a singular matrix has all its vectors lying in a (two-dimensional) plane. Note also that $det(A) = det(A^T)$. If $det(A) \neq 0$, then the matrix is said to be nonsingular.

13.2.7 Inverse

The inverse of a square matrix A, denoted A^{-1}, satisfies $AA^{-1} = A^{-1}A = I$. Its computation requires the determinant above, and the following definition of the $n \times n$ *adjoint* matrix:

$$adj(A) = \begin{bmatrix} (-1)^{1+1}\Delta_{11} & \cdots & (-1)^{1+n}\Delta_{1n} \\ \cdots & \cdots & \cdots \\ (-1)^{n+1}\Delta_{n1} & \cdots & (-1)^{n+n}\Delta_{nn}. \end{bmatrix}^T.$$

Once this computation is made, the inverse follows from

$$A^{-1} = \frac{adj(A)}{det(A)}.$$

If A is singular, i.e., $det(A) = 0$, then the inverse does not exist. The inverse finds common application in solving systems of linear equations such as

$$A\vec{x} = \vec{b} \longrightarrow \vec{x} = A^{-1}\vec{b}.$$

13.2.8 Eigenvalues and Eigenvectors

A typical eigenvalue problem is stated as

$$A\vec{x} = \lambda\vec{x},$$

where A is an $n \times n$ matrix, $\vec{x}$ is a column vector with n elements, and λ is a scalar. We ask for what nonzero vectors $\vec{x}$ (right eigenvectors), and scalars λ (eigenvalues) will the equation be satisfied. Since the above is equivalent to $(A - \lambda I)\vec{x} = \vec{0}$, it is clear that $det(A - \lambda I) = 0$. This observation leads to the solutions for λ; here is an example for the two-dimensional case:

$$\begin{aligned} A &= \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix} \longrightarrow \\ A - \lambda I &= \begin{bmatrix} 4-\lambda & -5 \\ 2 & -3-\lambda \end{bmatrix} \longrightarrow \\ det(A - \lambda I) &= (4-\lambda)(-3-\lambda) + 10 \\ &= \lambda^2 - \lambda - 2 \\ &= (\lambda + 1)(\lambda - 2). \end{aligned}$$

Thus, A has two eigenvalues, $\lambda_1 = -1$ and $\lambda_2 = 2$. Each is associated with a *right eigenvector* $\vec{x}$. In this example,

$$\begin{aligned} (A - \lambda_1 I)\vec{x}_1 &= \vec{0} \longrightarrow \\ \begin{bmatrix} 5 & -5 \\ 2 & -2 \end{bmatrix} \vec{x}_1 &= \vec{0} \longrightarrow \\ \vec{x}_1 &= \left\{ \sqrt{2}/2, \ \ \sqrt{2}/2 \right\}^T \end{aligned}$$

$$\begin{aligned} (A - \lambda_2 I)\vec{x}_2 &= \vec{0} \longrightarrow \\ \begin{bmatrix} 2 & -5 \\ 2 & -5 \end{bmatrix} \vec{x}_2 &= \vec{0} \longrightarrow \\ \vec{x}_2 &= \left\{ 5\sqrt{29}/29, \ \ 2\sqrt{29}/29 \right\}^T. \end{aligned}$$

Eigenvectors are defined only within an arbitrary constant, i.e., if $\vec{x}$ is an eigenvector then $c\vec{x}$ is also an eigenvector for any $c \neq 0$. They are often normalized to have unity magnitude, and positive first element (as above). The condition that $rank(A - \lambda_i I) = rank(A) - 1$ indicates that there is only one eigenvector for the eigenvalue λ_i; more precisely, a unique direction for the eigenvector, since the magnitude can be arbitrary. If the left-hand side rank is less than this, then there are multiple eigenvectors that go with λ_i.

The above discussion relates only the right eigenvectors, generated from the equation $A\vec{x} = \lambda\vec{x}$. Left eigenvectors, defined as $\vec{y}^T A = \lambda \vec{y}^T$, are also useful for many problems, and can be defined simply as the right eigenvectors of A^T. A and A^T share the same eigenvalues λ, since they share the same determinant. Example:

$$\begin{aligned}
(A^T - \lambda_1 I)\vec{y}_1 &= \vec{0} \longrightarrow \\
\begin{bmatrix} 5 & 2 \\ -5 & -2 \end{bmatrix} \vec{y}_1 &= \vec{0} \longrightarrow \\
\vec{y}_1 &= \left\{ 2\sqrt{29}/29, \ \ -5\sqrt{29}/29 \right\}^T
\end{aligned}$$

$$\begin{aligned}
(A^T - \lambda_2 I)\vec{y}_2 &= \vec{0} \longrightarrow \\
\begin{bmatrix} 2 & 2 \\ -5 & -5 \end{bmatrix} \vec{y}_2 &= \vec{0} \longrightarrow \\
\vec{y}_2 &= \left\{ \sqrt{2}/2, \ \ -\sqrt{2}/2 \right\}^T .
\end{aligned}$$

13.2.9 Modal Decomposition

For simplicity, we consider matrices that have unique eigenvectors for each eigenvalue. The right and left eigenvectors corresponding to a particular eigenvalue λ can be defined to have unity dot product, that is $\vec{x}_i^T \vec{y}_i = 1$, with the normalization noted above. The dot products of a left eigenvector with the right eigenvectors corresponding to *different eigenvalues* are zero. Thus, if the set of right and left eigenvectors, V and W, respectively, is

$$\begin{aligned}
V &= [\vec{x}_1 \cdots \vec{x}_n] \, , \text{ and} \\
W &= [\vec{y}_1 \cdots \vec{y}_n] \, ,
\end{aligned}$$

then we have

$$\begin{aligned}
W^T V &= I, \text{ or} \\
W^T &= V^{-1}.
\end{aligned}$$

Next, construct a diagonal matrix containing the eigenvalues:

$$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \cdot & \\ 0 & & \lambda_n \end{bmatrix};$$

it follows that

$$\begin{aligned} AV &= V\Lambda \longrightarrow \\ A &= V\Lambda W^T \\ &= \sum_{i=1}^{n} \lambda_i \vec{v}_i \vec{w}_i^T. \end{aligned}$$

Hence A can be written as a sum of modal components.[3]

[3] By carrying out successive multiplications, it can be shown that A^k has its eigenvalues at λ_i^k, and keeps the same eigenvectors as A.