

4 RANDOM PROCESSES

From the essential aspects of probability we now move into the time domain, considering random signals. For this, assign to each random event A_i a complete signal, instead of a single scalar: $A_i \rightarrow x_i(t)$. The set of all the functions that are available (or the menu) is called the *ensemble* of the random process. An example case is to roll a die, generating $i = [1, 2, 3, 4, 5, 6]$ and suppose $x_i(t) = t^i$.

In the general case, there could be infinitely many members in the ensemble, and of course these functions could involve some other variables, for example $x_i(t, y, z)$, where y and z are variables not related to the random event A_i . Any particular $x_i(t)$ can be considered a regular, deterministic function, if the event is known. $x(t_o)$, taken at a specific time but without specification of which event has occurred, is a random variable.

4.1 Time Averages

The theory of random processes is built on two kinds of probability calculations: those taken across time and those taken across the ensemble. For time averages to be taken, we have to consider a specific function, indexed by i :

$$\begin{aligned} m(x_i(t)) &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T x_i(t) dt \quad (\text{mean}) \\ V^t(x_i(t)) &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T [x_i(t) - m(x_i(t))]^2 dt \quad (\text{variance on time}) \\ R_i^t(\tau) &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T [x_i(t) - m(x_i(t))][x_i(t + \tau) - m(x_i(t))] dt \quad (\text{autocorrelation}). \end{aligned}$$

The mean and variance have new symbols, but are calculated in a way that is consistent with our prior definitions. The autocorrelation is new and plays a central role in the definition of a spectrum. Notice that is an inner product of the function's deviation from its mean, with a delayed version of the same, such that $R(0) = V^t$.

Consider the roll of a die, and the generation of functions $x_i(t) = a \cos(i\omega_o t)$. We have

$$\begin{aligned} m(x_i(t)) &= \lim_{T \rightarrow \infty} \int_0^T a \cos(i\omega_o t) dt = 0 \\ V^t(x_i(t)) &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T a^2 \cos^2(i\omega_o t) dt = \frac{a^2}{2} \\ R_i^t(\tau) &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T a^2 \cos(i\omega_o t) \cos(i\omega_o(t + \tau)) dt = \frac{a^2}{2} \cos(i\omega_o \tau). \end{aligned}$$

In this case, the autocorrelation depends explicitly on the event index i , and has a peak of $a^2/2$ at $i\omega_o \tau = 2\pi k$, where k is an integer. These values for τ are precisely separated by the period of the i 'th harmonic in the ensemble. When the functions line up, we get a positive R^t ; when they are out of phase, we get a negative R^t .

4.2 Ensemble Averages

The other set of statistics we can compute are across the ensemble, but at a particular time. Set $y_i = x_i(t_o)$ where t_o is a specific time. Then, considering again the six harmonics from above, we have

$$E(y) = \sum p_i y_i = \sum_{i=1}^6 \frac{1}{6} a \cos(i\omega_o t_o)$$

$$E(y^2) = \sum p_i y_i^2 = \sum_{i=1}^6 \frac{1}{6} a^2 \cos^2(i\omega_o t_o).$$

We can see from this simple example that in general, time averages (which are independent of time, but dependent on event) and ensemble statistics (which are independent of event, but dependent on time) are not the same. Certainly one could compute ensemble statistics on time averages, and vice versa, but we will not consider such a procedure specifically here.

The ensemble autocorrelation function is now a function of the time and of the delay:

$$R(t, \tau) = E(x(t)x(t + \tau)) \text{ or}$$

$$R(t, \tau) = E[\{x(t) - E(x(t))\} \{x(t + \tau) - E(x(t + \tau))\}].$$

The second form here explicitly takes the mean values into account, and can be used when the process has nonzero mean. The two versions are not necessarily equal as written.

4.3 Stationarity

A stationary random process is one whose ensemble statistics do not depend on time. Intuitively, this means that if we were to sample a sequence of processes, at the same time within each process, and compute statistics of this data set, we would find no dependence of the statistics on the time of the samples. Aircraft engine noise is a stationary process in level flight, whereas the sound of live human voices is not. For a stationary process, $m(t) = m$, i.e., the ensemble mean has no dependence on time. The same is true for the other statistics: $V(t) = R(t, 0) = V$, and $R(t, \tau) = R(\tau)$. Formally, a stationary process has *all* ensemble statistics independent of time, whereas our case that the mean, variance, and autocorrelation functions are independent of time defines a (weaker) *second-order stationary* process.

Here is an example: $y_i(t) = a \cos(\omega_o t + \theta_i)$, where θ_i is a random variable, distributed uniformly in the range $[0, 2\pi]$. Is this process stationary? We have to show that all three of the ensemble statistics are independent of time:

$$E(y(t)) = \frac{1}{2\pi} \int_0^{2\pi} a \cos(\omega_o t + \theta) d\theta = 0$$

$$R(t, \tau) = E(y(t)y(t + \tau))$$

$$= \frac{1}{2\pi} \int_0^{2\pi} a^2 \cos(\omega_o t + \theta) \cos(\omega_o(t + \tau) + \theta) d\theta$$

$$\begin{aligned}
&= \frac{1}{2}a^2 \cos(\omega_o\tau) \\
V(t) &= R(t, 0).
\end{aligned}$$

Thus the process is second-order stationary.

As noted above, the statistics of a stationary process are not necessarily the same as the time averages. A very simple example of this is a coin toss, in which heads triggers $x_1(t) = 1$ and $x_2(t) = 2$. Clearly the mean on time of $x_1(t)$ is one, but the ensemble mean at any time is $E(x(t_o)) = 1.5$. This difference occurs here even though the process is obviously stationary.

When the ensemble statistics and the time averages are the same, we say that the process is *ergodic*. Continuing our example above, let us calculate now the time averages:

$$\begin{aligned}
m(y_i(t)) &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T a \cos(\omega_o t + \theta_i) dt \\
&= \lim_{T \rightarrow \infty} \frac{1}{T} a \frac{1}{\omega_o} \sin(\omega_o t + \theta_i) \Big|_0^T \\
&= 0; \\
R^t(\tau) &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T a^2 \cos(\omega_o t + \theta_i) \cos(\omega_o(t + \tau) + \theta_i) dt \\
&= \frac{1}{2} a^2 \cos(\omega_o \tau); \\
V^t &= R^t(0) = \frac{a^2}{2}.
\end{aligned}$$

So a sinusoid at random phase is an ergodic process. Indeed, this form is a foundation for modeling natural random processes such as ocean waves, atmospheric conditions, and various types of noise. In particular, it can be verified that the construction

$$y(t) = \sum_{n=1}^N a_n \cos(\omega_n t + \theta_n),$$

where the θ_n are independently and uniformly distributed in $[0, 2\pi]$, is stationary and ergodic. It has mean zero, and autocorrelation

$$R(\tau) = \sum_{n=1}^N \frac{a_n^2}{2} \cos(\omega_n \tau).$$

We now make two side notes. Under stationary and ergodic conditions, the autocorrelation function is symmetric on positive and negative τ because we can always write

$$R(\tau) = E(x(t)x(t + \tau)) = E(x(t' - \tau)x(t')), \text{ where } t' = t + \tau.$$

Furthermore, we have the inequality that $R(0) \geq |R(\tau)|$ for any τ . To see this,

$$0 \leq E[(x(t) + x(t + \tau))^2] = E[x(t)^2] + 2E[x(t)x(t + \tau)] + E[x(t + \tau)^2]$$

$$\begin{aligned}
&= 2R(0) + 2R(\tau); \text{ similarly,} \\
0 \leq E[(x(t) - x(t + \tau))^2] &= E[x(t)^2] - 2E[x(t)x(t + \tau)] + E[x(t + \tau)^2] \\
&= 2R(0) - 2R(\tau).
\end{aligned}$$

The only way both of these can be true is if $R(0) \geq |R(\tau)|$.

4.4 The Spectrum: Definition

Given an ergodic process $y(t)$, with mean zero and autocorrelation $R(\tau)$, the power spectral density of $y(t)$, or the spectrum, is the *Fourier transform of the autocorrelation*:

$$\begin{aligned}
S(\omega) &= \int_{-\infty}^{\infty} R(\tau)e^{-i\omega\tau} d\tau \\
R(\tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)e^{i\omega\tau} d\omega.
\end{aligned}$$

The spectrum is a real and even function of frequency ω , because the autocorrelation is real and even. Expanding the above definition,

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)(\cos \omega\tau - i \sin \omega\tau) d\tau,$$

and clearly only the cosine will create an inner product with $R(\tau)$.

4.5 Wiener-Khinchine Relation

Recall from our discussion of the Fourier transform that convolution in the time domain of the impulse response $h(t)$ and an arbitrary system input $u(t)$, is equivalent to multiplication in the frequency domain of the Fourier transforms. This is a property in particular of linear, time-invariant systems. Now we can make some additional strong statements in the case of random processes.

If $u(t)$ is stationary and ergodic, and the system is LTI, then the output $y(t)$ is also stationary and ergodic. The statistics are related using the spectrum:

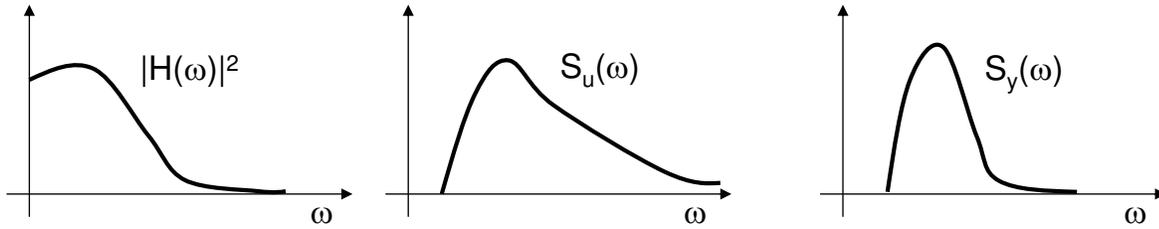
$$S_y(\omega) = |H(\omega)|^2 S_u(\omega).$$

This can be seen as a variant on the transfer function from the Fourier transform. Here, the quantity $|H(\omega)|^2$ transforms the spectrum of the input to the spectrum of the output. It can be used to map the statistical properties of the input (such as an ocean wave field) to statistical properties of the output. In ocean engineering, this is termed the response amplitude operator, or RAO.

To prove this, we will use the convolution property of LTI systems.

$$\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau, \text{ so that} \\
R_y(t, \tau) &= E[y(t)y(t+\tau)], \\
&= E\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)u(t-\tau_1)h(\tau_2)u(t+\tau-\tau_2)d\tau_1d\tau_2\right\} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\tau_1d\tau_2h(\tau_1)h(\tau_2)E[u(t-\tau_1)u(t+\tau-\tau_2)] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\tau_1d\tau_2h(\tau_1)h(\tau_2)R_u(\tau-\tau_2+\tau_1) \\
&\quad (\text{because the input is stationary and ergodic, } R_u \text{ does not depend on time}) \\
S_y(\omega) &= \int_{-\infty}^{\infty} R_y(\tau)e^{-i\omega\tau}d\tau \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\tau d\tau_1d\tau_2e^{-i\omega\tau}h(\tau_1)h(\tau_2)R_u(\tau-\tau_2+\tau_1); \text{ now let } \xi = \tau-\tau_2+\tau_1 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\xi d\tau_1d\tau_2e^{-i\omega(\xi+\tau_2-\tau_1)}h(\tau_1)h(\tau_2)R_u(\xi) \\
&= \int_{-\infty}^{\infty} d\xi e^{-i\omega\xi}R_u(\xi) \int_{-\infty}^{\infty} e^{i\omega\tau_1}h(\tau_1) \int_{-\infty}^{\infty} d\tau_2e^{-i\omega\tau_2}h(\tau_2) \\
&= S_u(\omega)H^*(\omega)H(\omega).
\end{aligned}$$

Here we used the *-superscript to denote the complex conjugate, and finally we note that $H^*H = |H|^2$.



4.6 Spectrum Interpretation

Let us consider now the stationary and ergodic random process with description:

$$y(t) = \sum_{n=1}^N a_n \cos(\omega_n t + \psi_n),$$

where ψ_n is a random variable with uniform distribution in the range $[0, 2\pi]$. As mentioned previously, this process has autocorrelation

$$R(\tau) = \frac{1}{2} \sum_{n=1}^N a_n^2 \cos \omega_n \tau;$$

and then

$$S(\omega) = \frac{1}{2} \sum_{n=1}^N a_n^2 \pi [\delta(\omega - \omega_n) + \delta(\omega + \omega_n)].$$

As with the Fourier transform, each harmonic in the time domain maps to a pair of delta functions in the frequency domain. However, unlike the Fourier transform, there is no phase angle associated with the spectrum - the two delta functions are both positive and both real.

Further, a real process has infinitely many frequency components, so that the spectrum really become a continuous curve. For example the Bretschneider wave spectrum in ocean engineering is given by

$$S^+(\omega) = \frac{5}{16} \frac{\omega_m^4}{\omega^5} H_{1/3}^2 e^{-5\omega_m^4/4\omega^4}$$

where ω is frequency in radians per second, ω_m is the modal (most likely) frequency of any given wave, and $H_{1/3}$ is the significant wave height. The + superscript on $S(\omega)$ indicates a "one-sided spectrum," wherein all the energy at positive and negative frequencies has been collected into the positive frequencies. We also take into account a factor of $1/2\pi$ (for reasons given below), to make the formal definition

$$S^+(\omega) = \begin{cases} \frac{1}{\pi} S(\omega), & \text{for } \omega \geq 0, \text{ and} \\ 0, & \text{for } \omega < 0. \end{cases}$$

What is the justification for the factor of $1/2\pi$? Consider that

$$\begin{aligned} R(\tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{i\omega\tau} d\omega \longrightarrow \\ R(0) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) d\omega \\ &= \frac{2}{2\pi} \int_0^{\infty} S(\omega) d\omega, \end{aligned}$$

and therefore that

$$\sigma^2 = R(0) = \int_0^{\infty} S^+(\omega) d\omega.$$

In words, the area under the one-sided spectrum is exactly equal to the variance, or the square of the standard deviation of the process.

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