6 Convolution of Sine and Unit Step

The sine function $q(t)$ has a zero value before zero time, and then is a unit sine wave afterwards:

$$q(t) = \begin{cases} 0 & \text{if } t < 0 \\ \sin(t) & \text{if } t \geq 0 \end{cases}$$

For the LTI systems whose impulse responses $h(t)$ are given below, use convolution to determine the system responses to a sine function input, i.e., $u(t) = q(t)$.

1. $h(t) = 1$
   Solution:
   $$y(t) = \int_0^t h(t-\tau)q(\tau)d\tau = \int_0^t \sin(\tau)d\tau = -\cos(\tau)|_0^t = 1 - \cos t.$$

2. $h(t) = \sin(\alpha t)$, where α is a fixed positive number.
   Solution:
   $$\begin{aligned}
   y(t) &= \int_0^t h(\tau)q(t-\tau)d\tau \\
   &= \int_0^t \sin(\alpha\tau)\sin(t-\tau)d\tau \\
   &= \int_0^t \left[\sin(\alpha\tau)\sin t\cos\tau - \sin(\alpha\tau)\cos t\sin\tau\right]d\tau \qquad \text{(from a trig. identity)} \\
   &= \sin t\int_0^t \sin(\alpha\tau)\cos\tau d\tau - \cos t\int_0^t \sin(\alpha\tau)\sin\tau d\tau \\
   &= \frac{1}{2}\sin t\int_0^t \left[\sin((\alpha+1)\tau) + \sin((\alpha-1)\tau)\right]d\tau - \\
   &\quad \frac{1}{2}\cos t\int_0^t \left[\cos((\alpha-1)\tau) - \cos((\alpha+1)\tau)\right]d\tau \qquad \text{(two more trig. identities)} \\
   &= \frac{1}{2}\sin t\left[-\frac{1}{\alpha+1}\cos((\alpha+1)t) - \frac{1}{\alpha-1}\cos((\alpha-1)t) + \frac{1}{\alpha+1} + \frac{1}{\alpha-1}\right] - \\
   &\quad \frac{1}{2}\cos t\left[\frac{1}{\alpha-1}\sin((\alpha-1)t) - \frac{1}{\alpha+1}\sin((\alpha+1)t)\right] \\
   &= \frac{1}{2}\sin t\left[\frac{1}{\alpha+1} + \frac{1}{\alpha-1}\right] + \\
   &\quad \frac{1}{2}\sin t\left[-\frac{1}{\alpha+1}\cos((\alpha+1)t) - \frac{1}{\alpha-1}\cos((\alpha-1)t)\right] + \\
   &\quad \frac{1}{2}\cos t\left[-\frac{1}{\alpha-1}\sin((\alpha-1)t) + \frac{1}{\alpha+1}\sin((\alpha+1)t)\right] \\
   &= \frac{1}{2}\sin t\left[\frac{1}{\alpha+1} + \frac{1}{\alpha-1}\right] - \\
   &\quad \frac{1}{2}\frac{1}{\alpha-1}\sin(\alpha t) + \frac{1}{2}\frac{1}{\alpha+1}\sin(\alpha t) \qquad \text{(and two more identities)} \\
   &= \frac{1}{\alpha^2-1}\left[\alpha\sin t - \sin(\alpha t)\right].
   \end{aligned}$$

This result can be checked numerically, or with the LaPlace transform. It is important to note that this solution applies only when $\alpha \neq 1$ *- the factors of* $1/(\alpha - 1)$ *after the integrals are computed make no sense. Instead, for this case the* $\sin((\alpha - 1)t)$ *in the integral will be replaced with zero, and* $\cos((\alpha - 1)t)$ *will be replaced with one. Working things out along the same lines gives:*

$$y(t) = \frac{1}{2}(\sin t - t \cos t).$$

Clearly, this is an unbounded response, the usual result of forcing a system exactly at its resonant frequency.

2.017J Design of Electromechanical Robotic Systems
Fall 2009