2 Convolution

The step function $s(t)$ is defined as zero when the argument is negative, and one when the argument is zero or positive:

$$s(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

For the LTI systems whose impulse responses are given below, use convolution to determine the system responses to step input, i.e., $u(t) = s(t)$.

1. $h(t) = 1$
   The impulse response is the step function itself - it turns on to one as soon as the impulse is applied, and this makes it a pure *integrator*. We get for the response to step input

$$\begin{aligned} y(t) &= \int_0^t s(\tau)s(t-\tau)d\tau \\ &= \int_0^t s(t-\tau)d\tau \text{ and the integrand is one because always } t \geq \tau \text{ so} \\ &= \int_0^t d\tau = t. \end{aligned}$$

   You recognize this as the integral of the input step.

2. $h(t) = \sin(t)$
   This impulse response is like that of an undamped second-order oscillator, having unity resonance frequency.

$$\begin{aligned} y(t) &= \int_0^t s(t-\tau)\sin(\tau)d\tau \\ &= \int_0^t \sin(\tau)d\tau \\ &= -\cos(\tau)|_0^t = 1 - \cos(t). \end{aligned}$$

3. $h(t) = 2\sin(t)e^{-t/4}$
   This is a typical underdamped response for a second-order system - a sinusoid multiplied by a decaying exponential. We make the substitution and find:

$$\begin{aligned} y(t) &= \int_0^t s(t-\tau)2\sin(\tau)e^{-\tau/4}d\tau \\ &= 2\int_0^t \sin(\tau)e^{-\tau/4}d\tau \\ &= \frac{32}{17}\left(1 - e^{-t/4}[\sin(t)/4 + \cos(t)]\right) \end{aligned}$$

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