Today's goals

- **Last week**
  - Frequency response=G(jω)
  - Bode plots
- **Today**
  - Using Bode plots to determine stability
    - Gain margin
    - Phase margin
  - Using frequency response to determine transient characteristics
    - damping ratio / percent overshoot
    - bandwidth / response speed
    - steady-state error
  - Gain adjustment in the frequency domain

Gain and phase margins

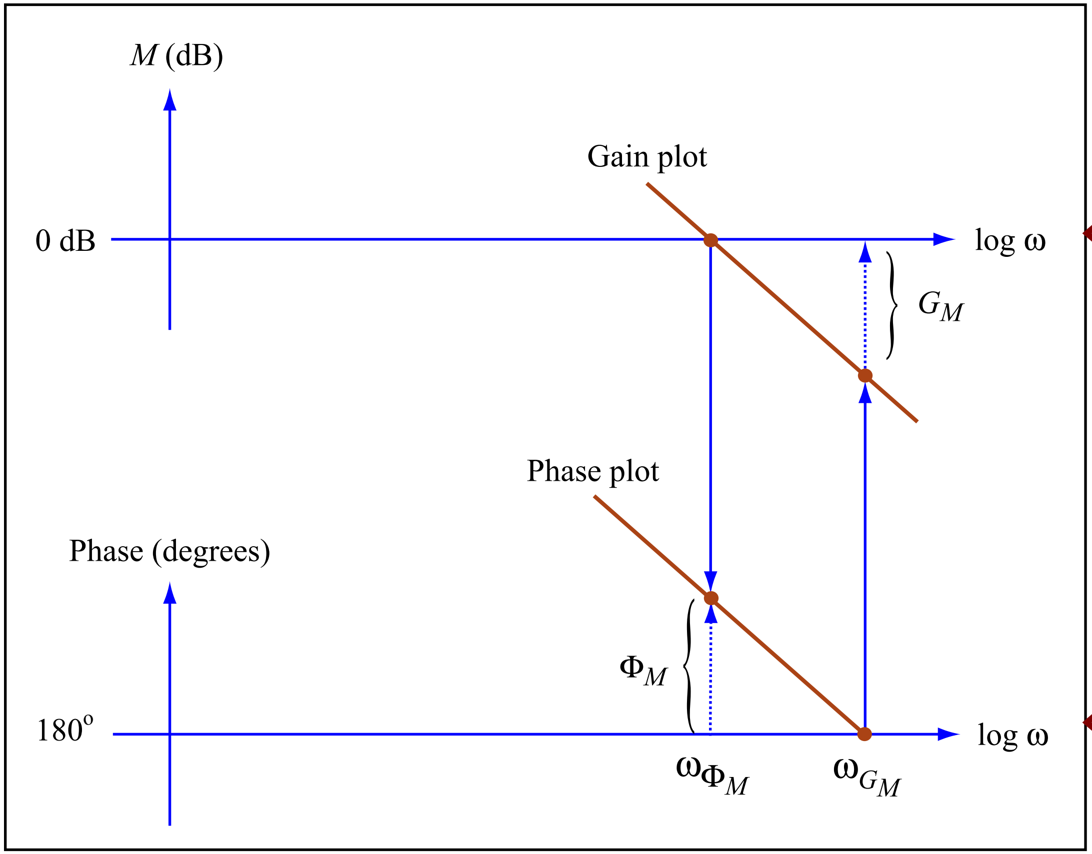


Figure by MIT OpenCourseWare.

<ins>Gain margin:</ins>
the difference (in dB) between 0dB and the system gain, computed at the frequency where the phase is 180°

<ins>Phase margin:</ins>
the difference (in °) between the system phase and 180°, computed at the frequency where the gain is 1 (i.e., 0dB)

A system is <ins>stable</ins> if the gain and phase margins are both positive

Figure 10.37

Example 1

Open–Loop Transfer Function

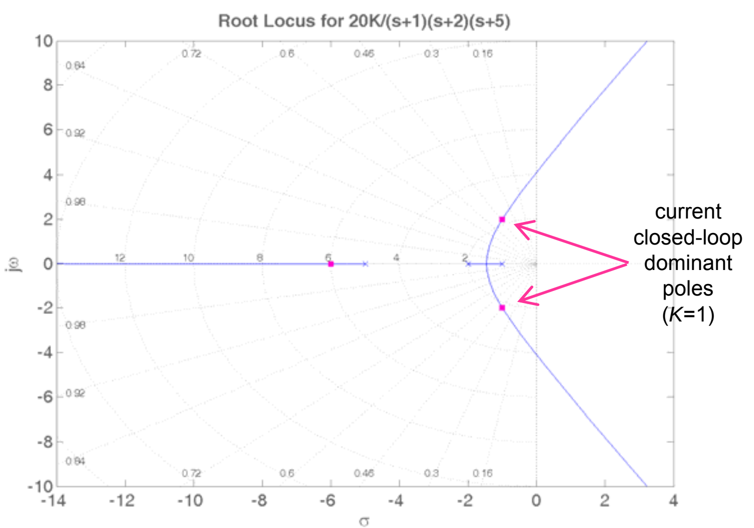
$$KG(s)H(s) = \frac{20K}{(s+1)(s+2)(s+5)}.$$

$$\text{DC gain} = 20 = 20\log_{10}20(\text{dB}) \approx 6\text{dB}$$

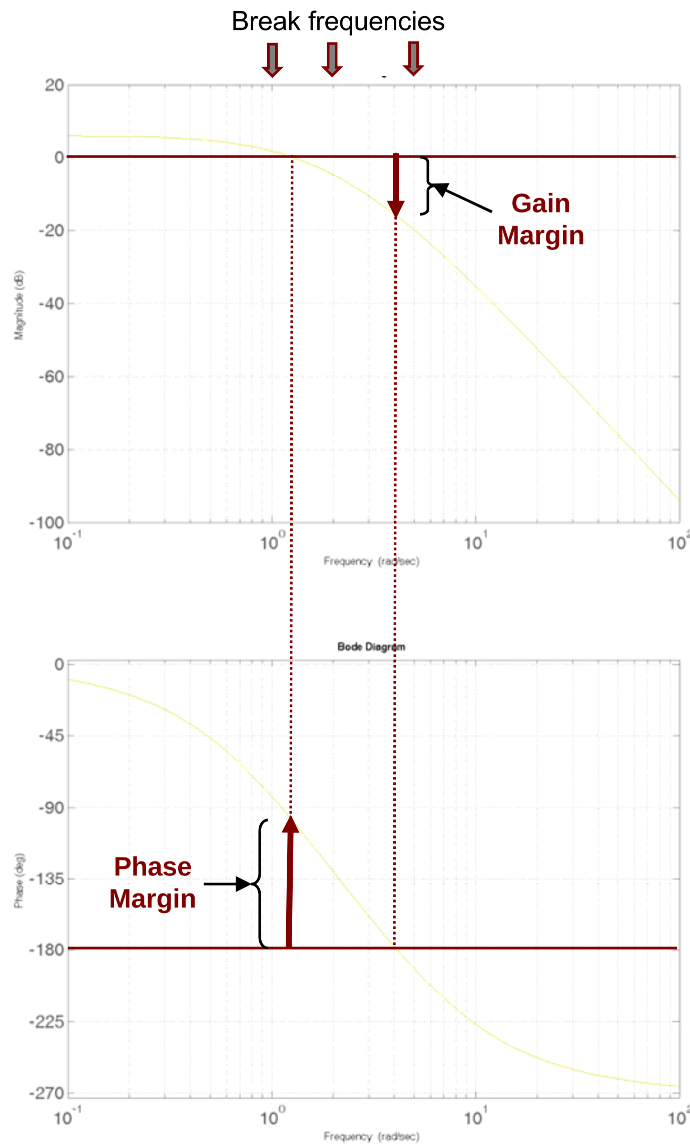
Break (cut–off) frequencies: $1, 2, 5$ rad/sec.

Final gain slope: -60 dB/dec.

Total phase change: -270°.



Increasing the closed-loop gain by an amount equal to the G.M.
(i.e., setting *K*=+G.M. dB or more) will destabilize the system

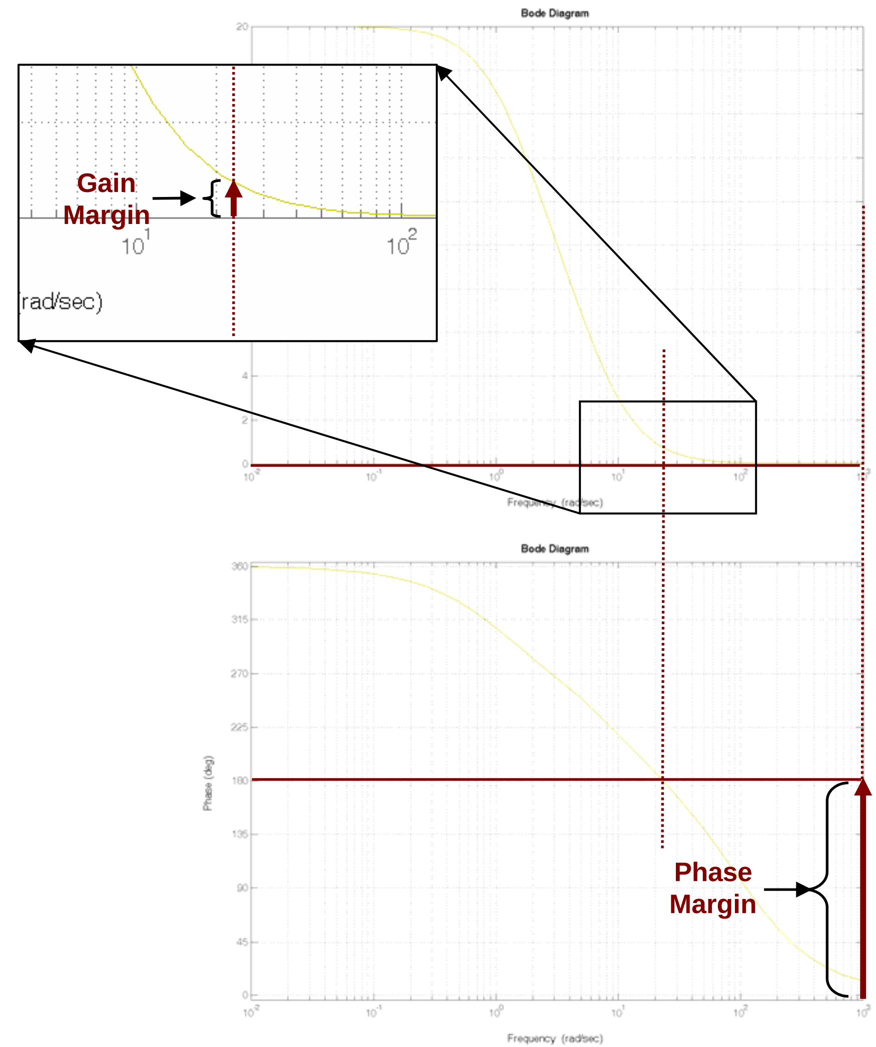


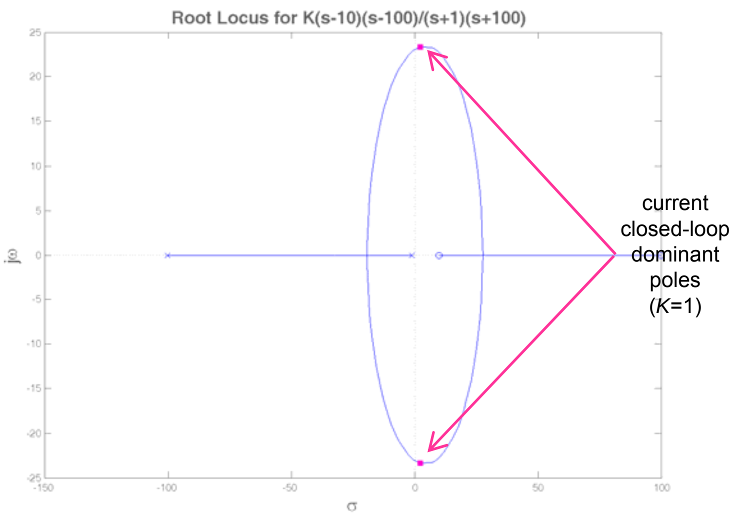
Example 2

Open–Loop Transfer Function

$$KG(s)H(s) = \frac{K(s-10)(s-100)}{(s+1)(s+100)}.$$

negative Gain Margin
→ **closed-loop** system is **unstable**

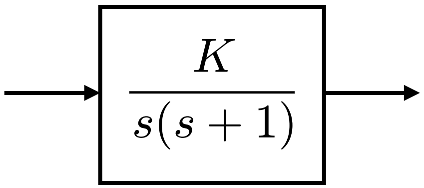




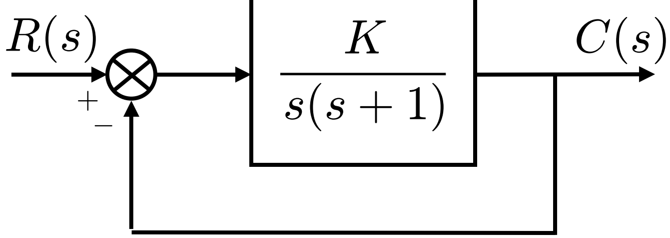
Reducing the closed-loop gain by an amount equal to the G.M.
(i.e., setting *K*=-G.M. dB or less) will stabilize the system

Transient from *closed*-loop frequency response /1

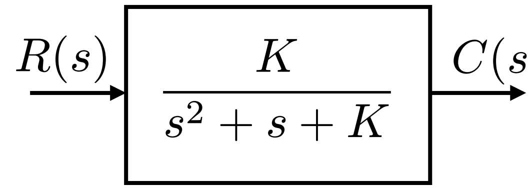
Consider a 1st–order system with ideal integral control:



Open-Loop system

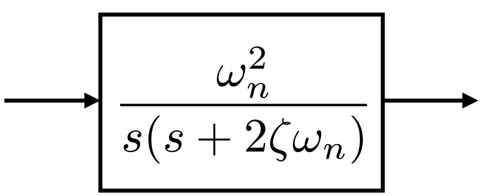


Closed-Loop system

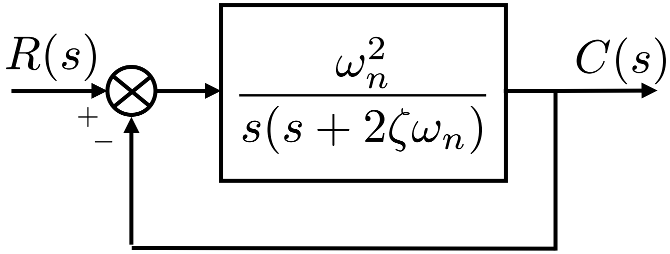


Equivalent block diagram for the Closed-Loop system

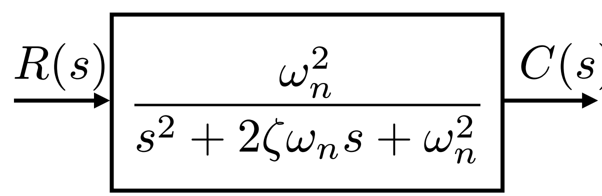
More generally, with the definition $K \equiv \omega_n^2$:



Open-Loop system



Closed-Loop system



Equivalent block diagram for the Closed-Loop system

Transient from *closed*-loop frequency response /2

Closed–Loop Transfer Function: $\frac{C(s)}{R(s)} \equiv T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

Frequency Response magnitude: $M(\omega) = |T(s)| = \frac{\omega_n^2}{\left\{\left(\omega_n^2 - \omega^2\right)^2 + 4\zeta^2\omega_n^2\omega^2\right\}^{1/2}}$

Frequency response magnitude peaks at frequency $\omega_p = \omega_n\sqrt{1-2\zeta^2}$.

Frequency response peak magnitude is $M_p = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$.

Bandwidth:
The frequency where the magnitude drops by 3dB below the DC magnitude

$$\omega_{\text{BW}} = \sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

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Please see: Fig. 10.39 in Nise, Norman S. *Control Systems Engineering*. 4th ed. Hoboken, NJ: John Wiley, 2004.

Transient from *closed*-loop frequency response /3

Please see: Fig. 10.40 and 10.41 in Nise, Norman S. *Control Systems Engineering*. 4th ed. Hoboken, NJ: John Wiley, 2004.

Transient from *open*-loop phase diagrams

Relationship between phase margin Φ_M
and damping ratio:

$$\Phi_M = \tan^{-1} \frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1 + 4\zeta^2}}}.$$

Open–Loop gain
vs Open–Loop phase
at frequency $\omega = \omega_{\mathrm{BW}}$
(*i.e.*, when Closed–Loop gain
is 3dB below the Closed–Loop DC gain.)

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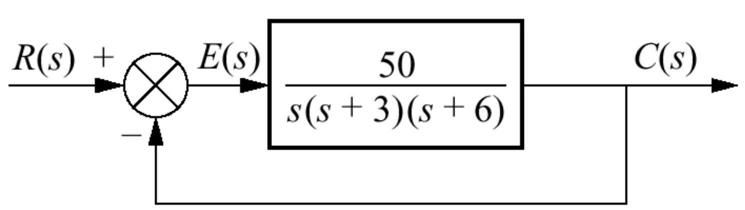
Please see: Fig. 10.48 and 10.49 in Nise, Norman S. *Control Systems Engineering*. 4th ed. Hoboken, NJ: John Wiley, 2004.

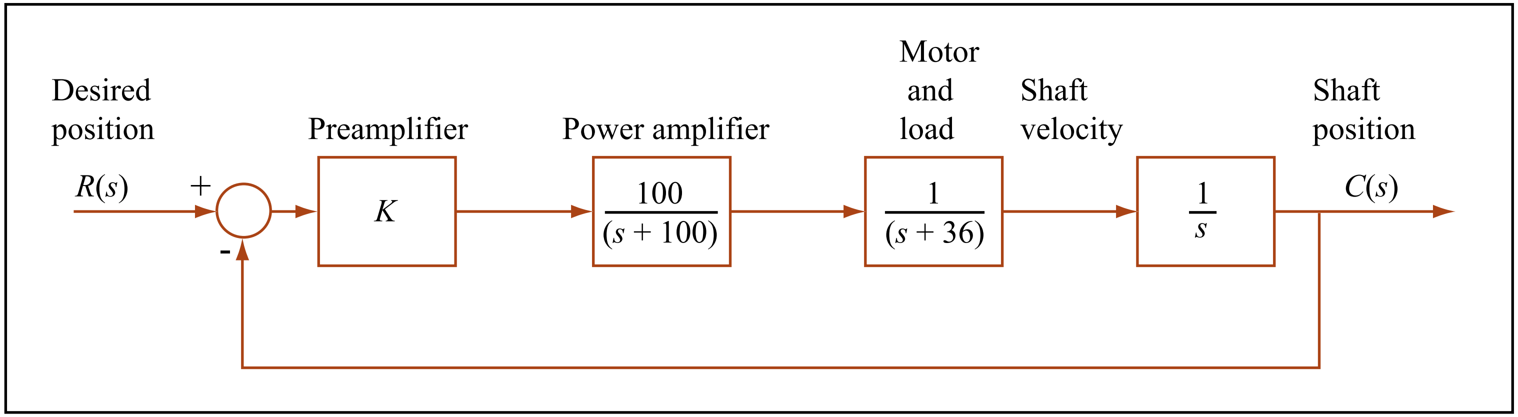
Example



Bandwidth from frequency response:
find where $M = -6 \sim -7.5\text{dB}$ while $\Phi = -135^\circ \sim -225^\circ$
$\Rightarrow \omega_{\text{BW}} \approx 3.5\text{rad/sec.}$

Damping ratio from phase margin:
Find phase margin ($\approx 35^\circ$)
and substitute into plot ($\zeta \approx 0.32$).

Example: Proportional control in the frequency domain



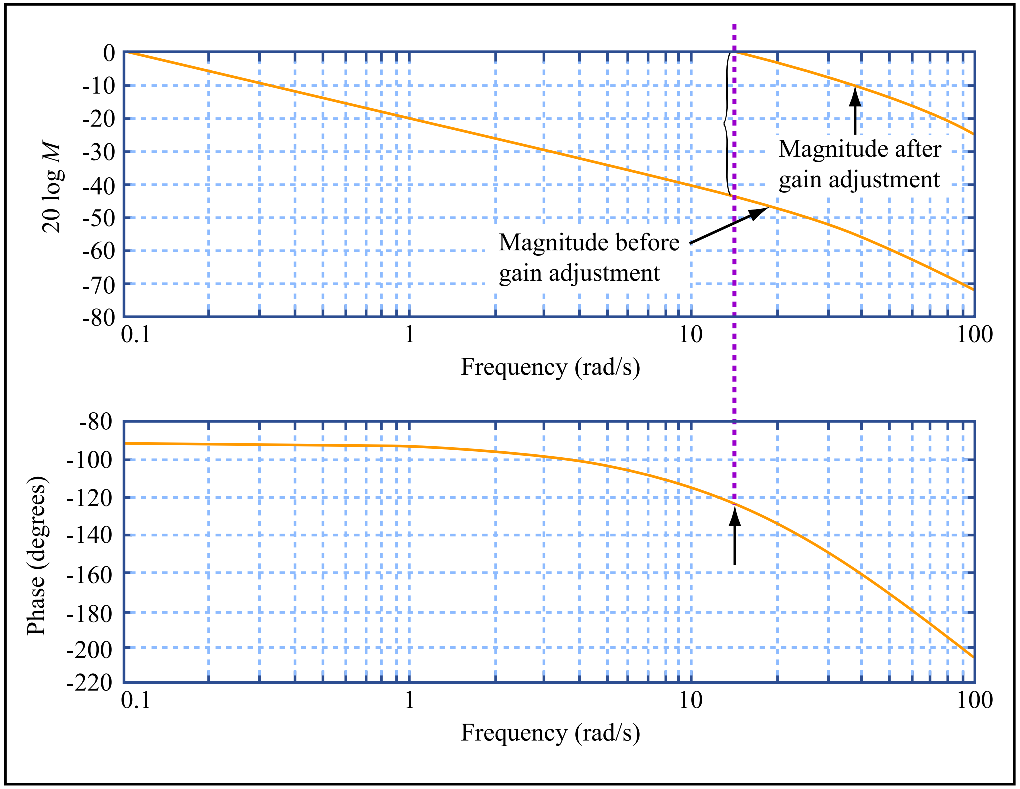
Figures by MIT OpenCourseWare.

Specification: 9.5% overshoot.
For 9.5% overshoot, the required damping ratio is $\zeta = 0.6$.
Using the damping ratio–phase margin relationship, we find

$$\Phi_M = \tan^{-1}\frac{2\zeta}{\sqrt{-2\zeta^2+\sqrt{1+4\zeta^4}}} \Rightarrow \Phi_M = 59.2^\circ.$$

Before compensation, the phase margin was $\approx 85^\circ$ (see the Bode plot on the right.)
We must reduce the phase margin to 59.2°, *i.e.* the Bode magnitude must be 0dB when the Bode phase is $-180^\circ + 59.2^\circ = -120.8^\circ$.
This occurs when $\omega \approx 15^\circ$ and we can see that the required gain adjustment is ≈ 44dB.
What is the total gain for the compensator?
In our uncompensated Bode plot, $M = 1$ when $\omega = 0.1 \Rightarrow$ the uncompensated gain is $K \approx 3.6$.
After compensation, the gain (in dB) should be

$$20\log 3.6 + 44 \approx 11 + 44 = 55 \Rightarrow K \approx 570.$$



Gain adjustment for phase margin specification

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Please see: Fig. 11.1 in Nise, Norman S. *Control Systems Engineering*. 4th ed. Hoboken, NJ: John Wiley, 2004.

Steady-state errors from the frequency response

Type 0 system (no free integrators)

$$G(s) = K\frac{\Pi\,(s+z_k)}{\Pi\,(s+p_k)}$$

Steady–state **position** error

$$\begin{aligned} e_\infty &= \frac{1}{1+K_p}, \text{ where} \\ K_p &\equiv K\frac{\Pi\, z_k}{\Pi\, p_k} \\ &= \text{DC gain.} \end{aligned}$$

Type 1 system (one free integrator)

$$G(s) = K\frac{\Pi\,(s+z_k)}{s\Pi\,(s+p_k)}$$

Steady–state **velocity** error

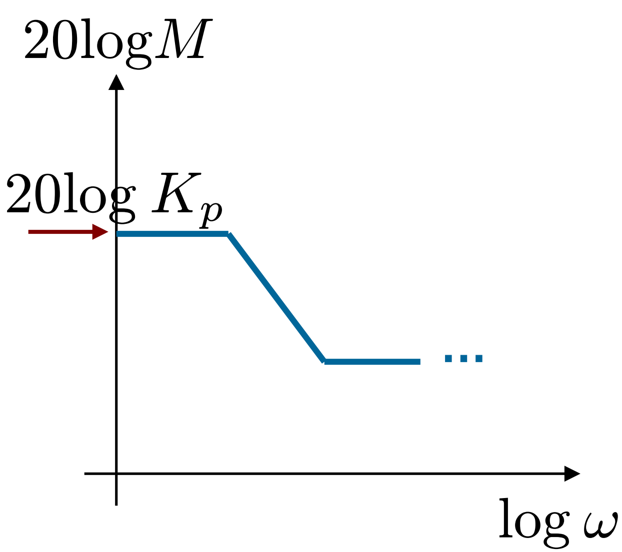
$$\begin{aligned} e_\infty &= \frac{1}{K_v}, \text{ where} \\ K_v &\equiv K\frac{\Pi\, z_k}{\Pi\, p_k} \\ &= \omega\text{–axis intercept.} \end{aligned}$$

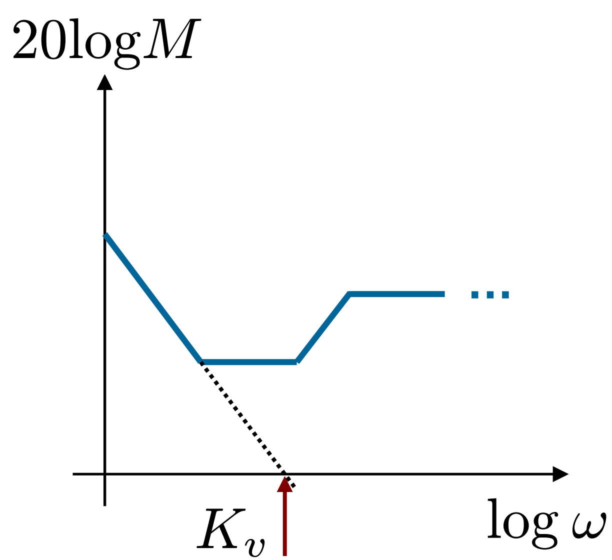
Type 2 system (two free integrators)

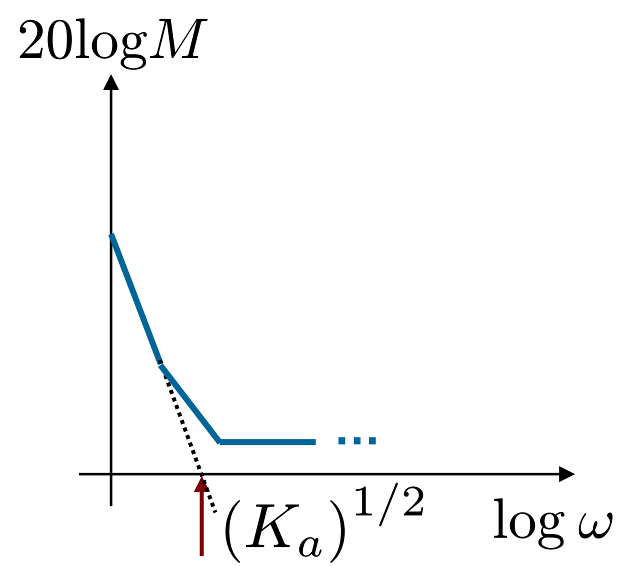
$$G(s) = K\frac{\Pi\,(s+z_k)}{s^2\Pi\,(s+p_k)}$$

Steady–state **acceleration** error

$$\begin{aligned} e_\infty &= \frac{1}{K_a}, \text{ where} \\ K_a &\equiv K\frac{\Pi\, z_k}{\Pi\, p_k} \\ &= (\omega\text{–axis intercept})^2\,. \end{aligned}$$







Example

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Please see: Fig. 10.52 in Nise, Norman S. *Control Systems Engineering*. 4th ed. Hoboken, NJ: John Wiley, 2004.

Type 0; steady–state position error

$$20\log K_p = 25 \Rightarrow e_\infty = 0.0532$$

Type 1; steady–state velocity error

$$M = 0\text{dB when } \omega = 0.55 \Rightarrow e_\infty = 1.818$$

Type 2; steady–state acceleration error

$$M = 0\text{dB when } \omega = 3 \Rightarrow e_\infty = 0.111$$