Today's goals

- **Last Friday**
  - Time-domain solution for the response of a linear time-invariant system to a sinusoidal input
  - The response is also a sinusoid, but in general its amplitude is attenuated and its phase is delayed compared to the input
  - The phasor $a e^{j\psi} \equiv a\angle\psi$ where a is the attenuation and ψ is the phase delay is sufficient to describe the sinusoidal response of the LTI system
  - The attenuation a and phase delay ψ are both *functions of the frequency* of the applied sinusoid
  - The phasor $a e^{j\psi} \equiv a\angle\psi$ as function of frequency ω is referred to as the Frequency Response of the LTI system
- **Today**
  - From the Transfer Function to the Frequency Response
  - Plotting the frequency response: Bode diagrams
  - Elementary Bode plots of 1st order systems: derivative; integrator; zero; pole

Frequency response in the Laplace domain

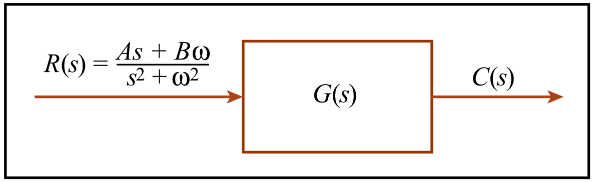

Figure 10.3

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Before we start, note that $r(t)$ is a general sinusoidal function:

$$r(t) = \mathcal{L}^{-1}\left[R(s)\right] = A\cos(\omega t) + B\sin(\omega t) = M_r\cos(\omega t + \phi_r), \quad \text{where}$$

$$M_r = \sqrt{A^2 + B^2}, \qquad \phi = \tan^{-1}\frac{B}{A}.$$

So the input can be represented by a phasor $M_r\angle\phi_r \equiv A - jB$.

The Laplace transform of the output is then

$$C(s) = \frac{As + B\omega}{s^2 + \omega^2}G(s) = \frac{As + B\omega}{(s - j\omega)(s + j\omega)}G(s).$$

To find what the output looks like in the time domain, we must perform a partial fraction expansion on $C(s)$. This process yields

$$C(s) = \frac{K_1}{s + j\omega} + \frac{K_2}{s - j\omega} + \left(\begin{array}{c}\text{additional terms}\\ \text{due to the poles of } G(s)\end{array}\right).$$

The terms due to the poles of the transfer function are, as we have learnt, the **homogeneous response.** Hopefully, the system is stable and so this part of the response will decay to zero after a sufficiently long time. On the other hand, the first two terms due to the poles of the input are the **forced** or **steady–state response**, which we are seeking at the moment. We can see immediately that the forced response is sinusoidal as well, at the same frequency ω as the input.

To complete the solution for the forced response, we need to determine the coefficients K_1, K_2. We do that by using the partial fraction expansion rules,

$$K_1 = \left.\frac{As + B\omega}{s - j\omega}G(s)\right|_{s=-j\omega} = \frac{(A + jB)G(-j\omega)}{2},$$

$$K_2 = \left.\frac{As + B\omega}{s + j\omega}G(s)\right|_{s=j\omega} = \frac{(A - jB)G(j\omega)}{2}.$$

Clearly, the value of the transfer function $G(s)$ at $s = \pm j\omega$ is of interest for determining the steady–state output. Let us denote the complex number $G(j\omega)$as a phasor as well,

$$G(j\omega) = M_G\angle\phi_G.$$

We can then rewrite the coefficients K_1, K_2 as

$$K_1 = \frac{(M_r\angle - \phi_r)\,(M_G\angle - \phi_G)}{2} = \frac{M_r M_G\angle - (\phi_r + \phi_G)}{2},$$

$$K_2 = \frac{(M_r\angle\phi_r)\,(M_G\angle\phi_G)}{2} = \frac{M_r M_G\angle\,(\phi_r + \phi_G)}{2} = K_1^*.$$

Therefore, the Laplace transform of the steady–state (forced) output is

$$C_\infty(s) = \frac{M_r M_G \mathrm{e}^{-j(\phi_r + \phi_G)}}{2(s + j\omega)} + \frac{M_r M_G \mathrm{e}^{j(\phi_r + \phi_G)}}{2(s - j\omega)} \Rightarrow$$

$$\begin{aligned} c(t) &= \frac{M_r M_G}{2}\left[\mathrm{e}^{-j(\omega t + \phi_r + \phi_G)} + \mathrm{e}^{+j(\omega t + \phi_r + \phi_G)}\right] \\ &= M_r M_G \cos\left(\omega t + \phi_r + \phi_G\right). \end{aligned}$$

We can see that the steady–state output is a sinusoid, of magnitude $M_r M_G$ and phase $\phi_r + \phi_G$, whereas the input was a sinusoid of magnitude M_r and phase ϕ_r. Moreover, we can see that the amplitude change M_G and phase change ϕ_G are the magnitude and phase, respectively, of the phasor resulting when the transfer function is computed at $s = j\omega$. If we put M_G and ϕ_G together as a phasor, we obtain $M_G\angle\phi_G \equiv G(j\omega)$.

We conclude that:

$$\left(\begin{array}{c}\text{Frequency}\\ \text{Response}\end{array}\right)(\omega) = G(j\omega).$$

Example: frequency response of a single pole TF

$$G(s) = \frac{1}{s+2}$$

$$\begin{aligned} G(j\omega) &= \frac{1}{j\omega+2} \\ &= \frac{2-j\omega}{\omega^2+4} \\ &= \frac{1}{\sqrt{\omega^2+4}} \angle -\tan^{-1}\frac{\omega}{2}. \end{aligned}$$

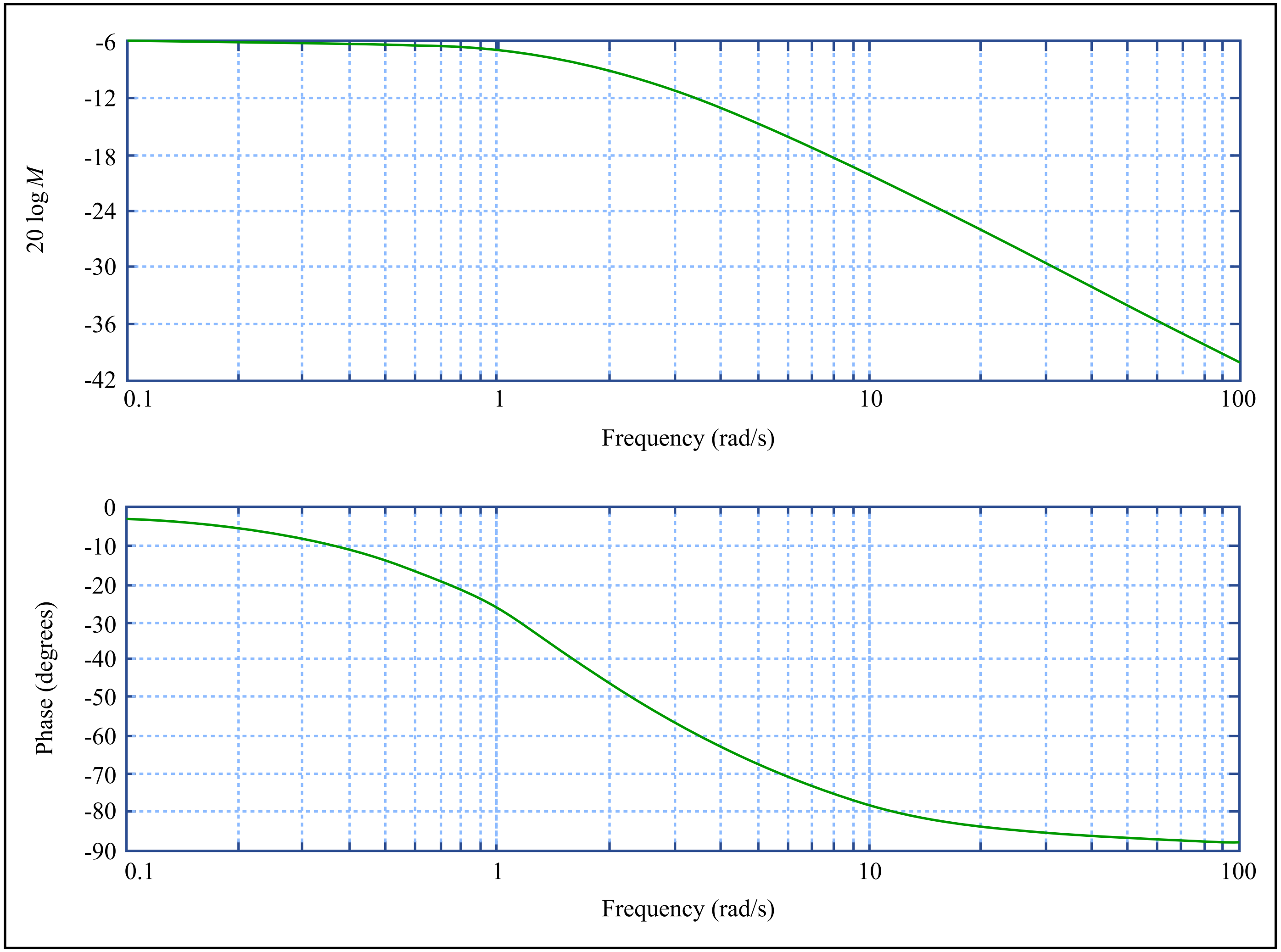

Figure by MIT OpenCourseWare.

Figure 10.4

“Asymptotic approximation” : Bode plot

$$G(s) = \frac{1}{s+2}$$

$$\begin{aligned} G(j\omega) &= \frac{1}{j\omega+2} \\ &= \frac{2-j\omega}{\omega^2+4} \\ &= \frac{1}{\sqrt{\omega^2+4}} \angle -\tan^{-1}\frac{\omega}{2}. \end{aligned}$$

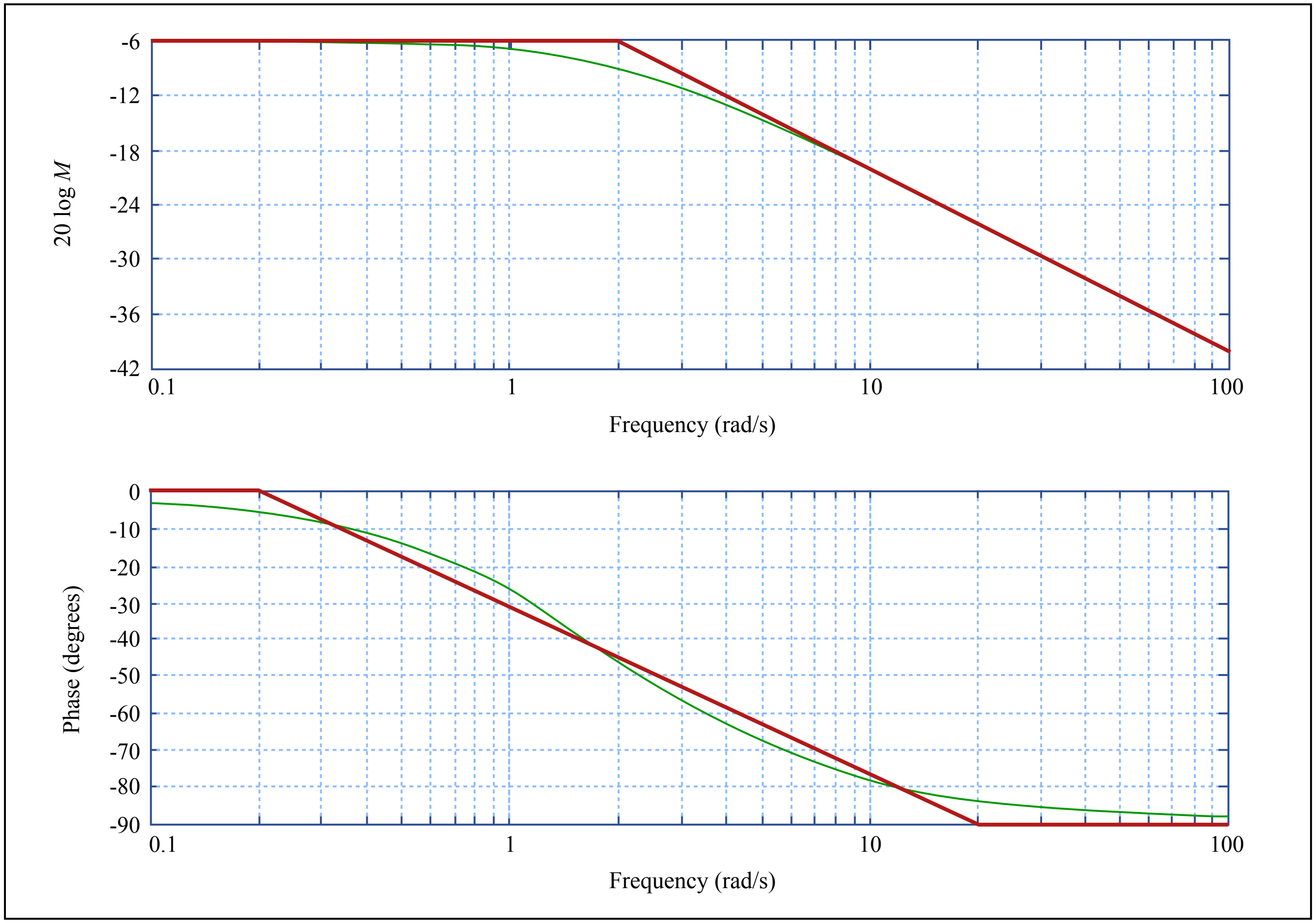

Figure by MIT OpenCourseWare.

Figure 10.4

Elementary Bode plots: 1st order

Normalized and scaled
Bode plots for
a. $G(s) = s$;
b. $G(s) = 1/s$;
c. $G(s) = (s + a)$;
d. $G(s) = 1/(s + a)$