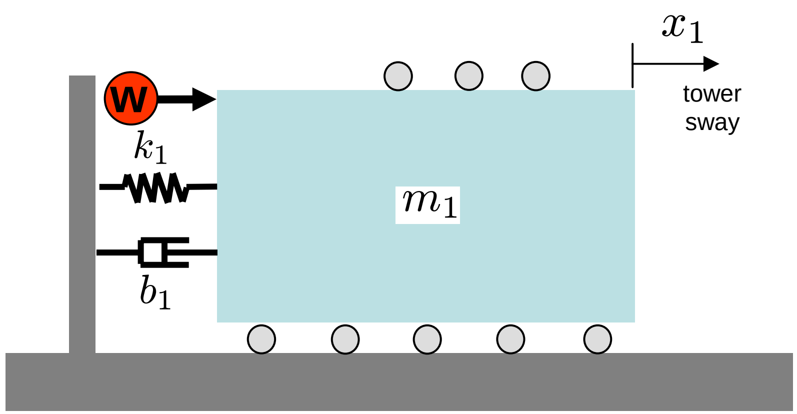
Today's goals

- **State space so far**
  - Definition of state variables
  - Writing the state equations
  - Solution of the state equations in the Laplace domain
  - Phase space and phase diagrams
- **Today**
  - Stability in state space
  - State feedback control

State space overview



From the Equation of Motion to the State–Space representation:

$$m\ddot{x}(t)+b\dot{x}(t)+kx(t) = w(t) \rightarrow \begin{pmatrix} x \\ \dot{x} \end{pmatrix} \equiv \mathbf{q}(t) = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \text{ state,} \quad y(t) \equiv \dot{x}(t) \text{ output}$$

$$\Rightarrow \dot{\mathbf{q}}(t) = \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -k/m & -b/m \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} w(t); \quad y(t) = (0 \quad 1) \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \equiv \mathbf{cq}.$$

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -k/m & -b/m \end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Solution to the state equations:

$$s\hat{\mathbf{q}}(s) = \mathbf{A}\hat{\mathbf{q}}(s) + \mathbf{b}W(s) \Rightarrow$$

$$\hat{\mathbf{q}}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{b}W(s).$$

$$Y(s) = \mathbf{c}\hat{\mathbf{q}}(s) = \mathbf{c}\,(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{b}W(s).$$

State space solution to the uncompensated 2.004 Tower model

$$\hat{\mathbf{q}}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}W(s) = \frac{1}{s^2 + (b_1/m_1)\,s + (k_1/m_1)} \begin{pmatrix} 1/m_1 \\ s/m_1 \end{pmatrix} W(s).$$

From this result we can obtain transfer functions for position, velocity:

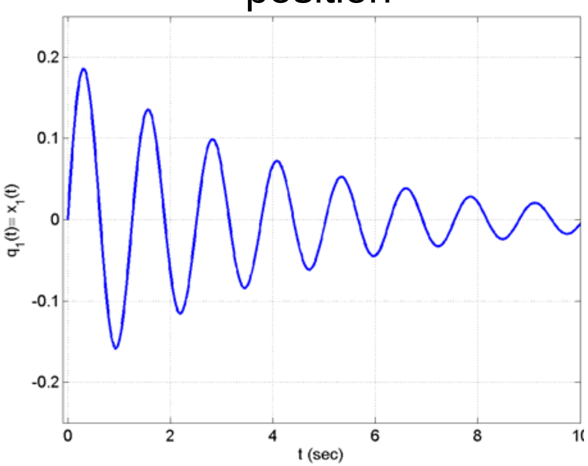
for position choose $\mathbf{c} = \begin{pmatrix} 1 & 0 \end{pmatrix}$, $\quad X(s) \equiv Y(s) = \mathbf{c}\,(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}W(s) \Rightarrow$

$$\frac{X(s)}{W(s)} = \frac{1/m_1}{s^2 + (b_1/m_1)\,s + (k_1/m_1)}.$$

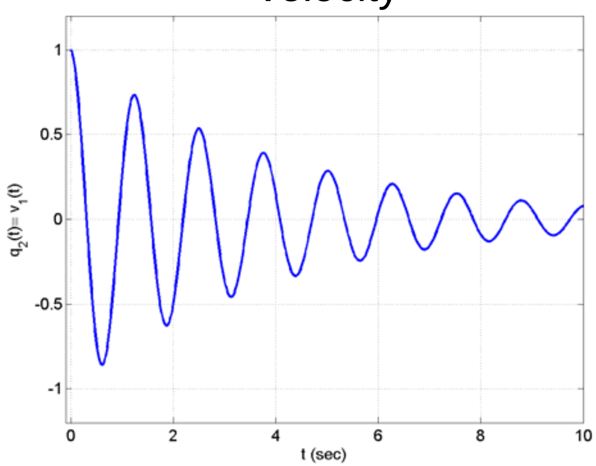
for velocity choose $\mathbf{c} = \begin{pmatrix} 0 & 1 \end{pmatrix}$, $\quad V(s) \equiv Y(s) = \mathbf{c}\,(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}W(s) \Rightarrow$

$$\frac{V(s)}{W(s)} = \frac{s/m_1}{s^2 + (b_1/m_1)\,s + (k_1/m_1)}.$$

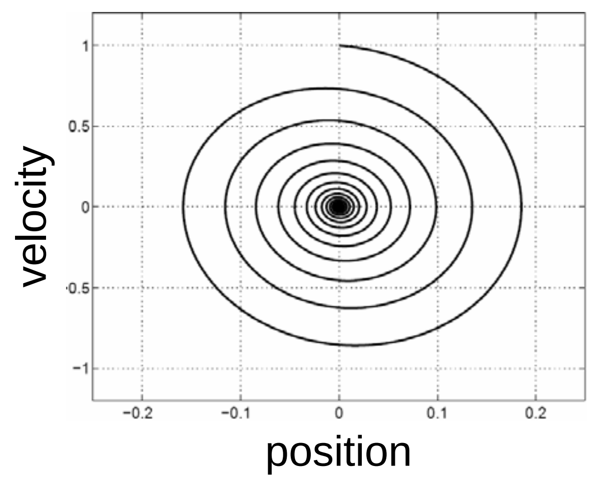
position



velocity



phase diagram



Poles are the eigenvalues of A

Consider the eigevalue problem for the matrix **A**:

$$\mathbf{A}\xi = \mu\xi,$$

where the solutions for μ are the **eigenvalues** and ξ are the **eigenvectors**. To solve the eigenvalue problem, we set $\det(\mu\mathbf{I} - \mathbf{A}) = 0$. That is, the eigenvalues are the roots of the determinant of the matrix $(\mu\mathbf{I} - \mathbf{A})$.

Recall that the state–space solution was

$$\hat{\mathbf{q}}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}W(s) = \frac{\operatorname{adj}(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})}\mathbf{b}W(s) =$$

$$= \frac{1}{s^2 + (b_1/m_1)\,s + (k_1/m_1)}\begin{pmatrix} 1/m_1 \\ s/m_1 \end{pmatrix} W(s),$$

where adj(.) denotes the adjoint. The same denominator $\det(s\mathbf{I} - \mathbf{A})$ appears in the transfer functions for both velocity and position. This denominator is also referred to as **characteristic equation.** Therefore, the poles of the system are the roots of the determinant of the matrix $(s\mathbf{I} - \mathbf{A})$, *i.e.*, the eigenvalues. The uncompensated 2.004 Tower is a 2nd order system with

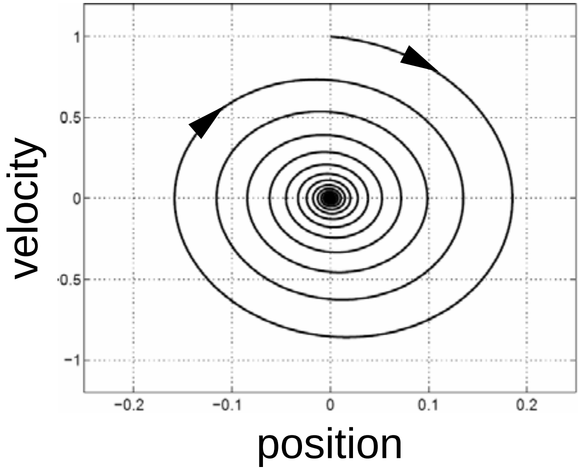
$$\omega_n^2 = \frac{k_1}{m_1}; \qquad \zeta = \frac{b_1}{2\sqrt{k_1 m_1}}$$

Therefore the eigenvalues/poles are

$$s_{\pm} = -\omega_n\left(\zeta \pm \sqrt{\zeta^2 - 1}\right).$$

Stability

The system represented by **A** is **stable** if **A**'s eigenvalues have negative real part (*i.e.*, are on the left–hand half–plane.) The phase diagram is then oriented towards the origin ("sink.")



The system represented by **A** is **unstable** if **A**'s eigenvalues have positive real part (phase diagram explodes outwards – "source") and **marginally stable** if **A**'s eigenvalues have zero real part (phase diagram rotates around the origin without either approaching or moving away.)

Eigenvectors and modes

Let's do a specific example: $m_1 = 1$, $b_1 = 1$, $k_1 = 1$,
that is $\omega_n = 1$, $\zeta = 1/2$, $s_\pm = -(1/2) \pm j\sqrt{3/2}$.
This system is stable and, in fact, underdamped,
consistent with $\zeta < 1$ and poles off the real axis.

Now let's compute the eigenvectors, starting with $\xi^{(+)}$
corresponding to the eigenvalue s_+:

$$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \xi^{(+)} = s_+ \xi^{(+)} \Rightarrow \begin{cases} \xi_2^{(+)} = \left(-1 + j\sqrt{3}\right) \xi_1^{(+)}/2 \\ \xi_1^{(+)} + \xi_2^{(+)} = \left(-1 + j\sqrt{3}\right) \xi_2^{(+)}/2. \end{cases}$$

It can be verified that the two equations are equivalent.
Therefore, the eigenvector corresponding to s_+ is

$$\xi^{(+)} = \begin{pmatrix} \dfrac{1}{\sqrt{2}} \\ -\dfrac{1}{\sqrt{2}} + j\dfrac{\sqrt{3}}{\sqrt{2}} \end{pmatrix} \alpha^{(+)},$$

where $\alpha^{(+)}$ is any arbitrary real number. By convention,
the eigenvector is written so that if $\alpha^{(+)} = 1 \Rightarrow \left|\xi^{(+)}\right| = 1$.
Similarly we can find the eigenvector $\xi^{(-)}$
corresponding to the eigenvalue s_-:

$$\xi^{(-)} = \begin{pmatrix} \dfrac{1}{\sqrt{2}} \\ -\dfrac{1}{\sqrt{2}} - j\dfrac{\sqrt{3}}{\sqrt{2}} \end{pmatrix} \alpha^{(-)}.$$

The two eigenvectors $\xi^{(+)}$, $\xi^{(-)}$
are referred to as the **modes** of the system.
The imaginary parts of the corresponding poles
are the **eigenfrequencies** of the modes.
In the uncompensated 2.004 Tower,
the two modes are **degenerate**
because the two poles are conjugate
(*i.e.*, they have the same imaginary
parts with $\pm$ signs.)
This is true for any 2nd order system.
The compensated 2.004 Tower
is a 4th order system, and so it has
two non–degenerate modes.

The significance of a mode
is that if the system is excited
with a sinusoid of frequency equal
to the mode's eigenfrequency,
then the response of the system will
be the mode itself (*i.e.*, the eigenvector.)
At other frequencies, the response
is a **mixture of modes**.

State space representation as block diagram

$$\dot{\mathbf{q}}(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{B}w(t),$$
$$y(t) = \mathbf{C}\mathbf{q}(t).$$

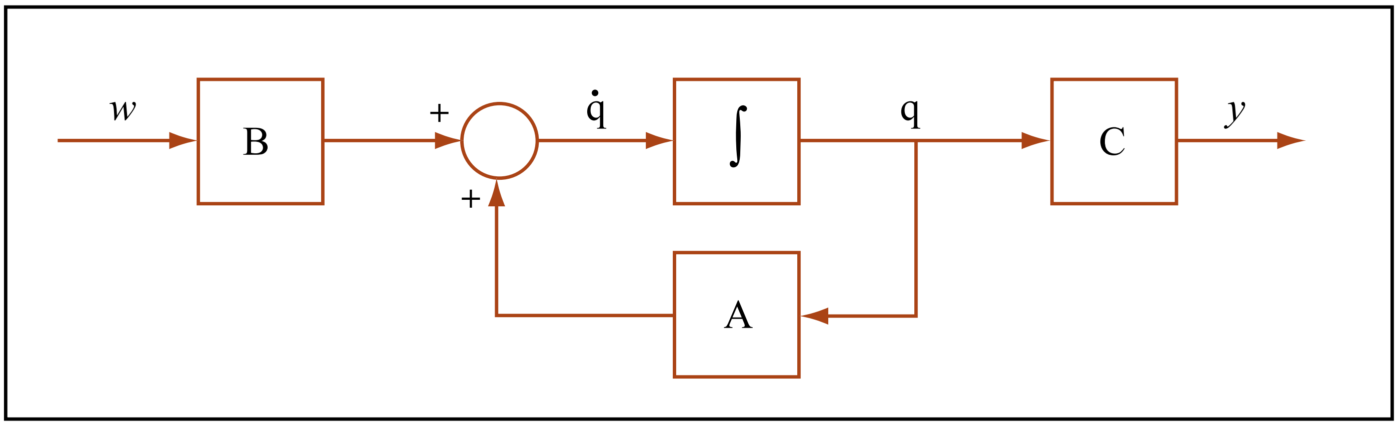
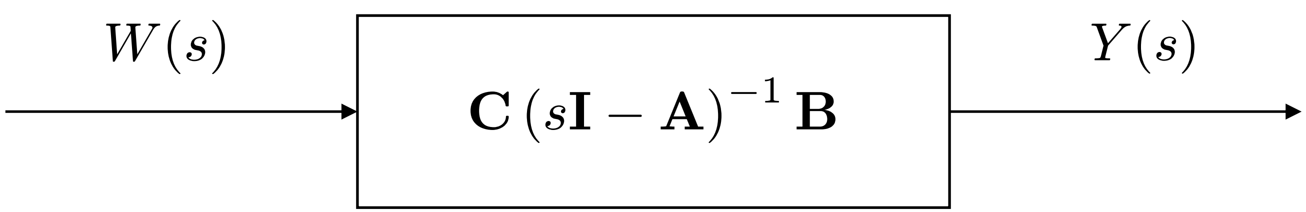


Figure by MIT OpenCourseWare.

Equivalent block diagram representation as transfer function:



State feedback

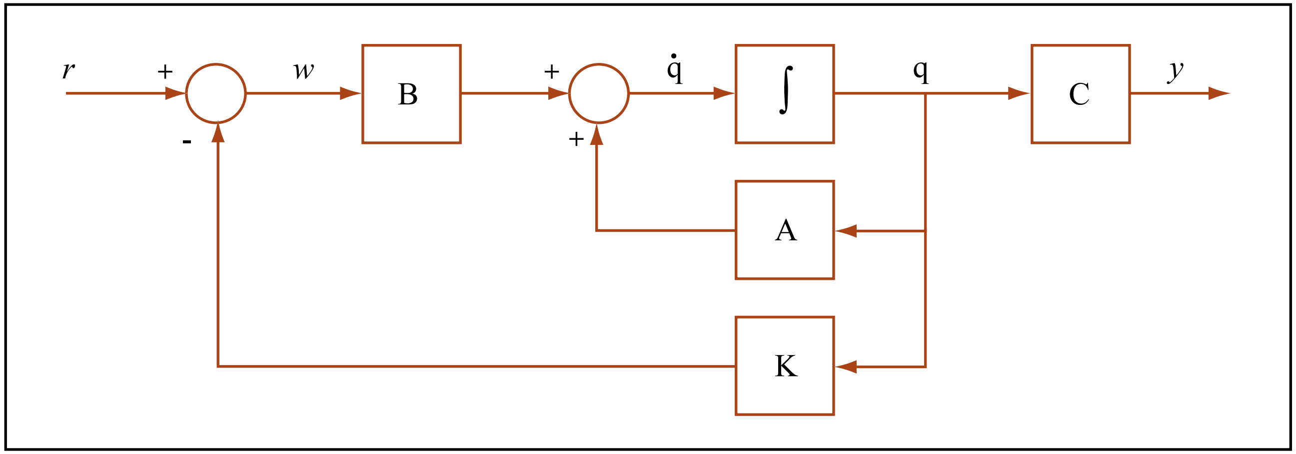


Figure by MIT OpenCourseWare.

$$w(t) = r(t) - \mathbf{K}\mathbf{q}(t) \Rightarrow$$

$$\begin{aligned} \dot{\mathbf{q}}(t) &= \mathbf{A}\mathbf{q}(t) + \mathbf{B}\left(r(t) - \mathbf{K}\mathbf{q}\right), \\ y(t) &= \mathbf{C}\mathbf{q}(t). \end{aligned} \Rightarrow \begin{aligned} \dot{\mathbf{q}}(t) &= \left(\mathbf{A} - \mathbf{B}\mathbf{K}\right)\mathbf{q}(t) + \mathbf{B}r(t), \\ y(t) &= \mathbf{C}\mathbf{q}(t). \end{aligned}$$

$$\text{Closed–Loop TF:} \quad \frac{Y(s)}{R(s)} = \mathbf{C}\left(s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}\right)^{-1}\mathbf{B}$$

Example

(Nise 12.1)

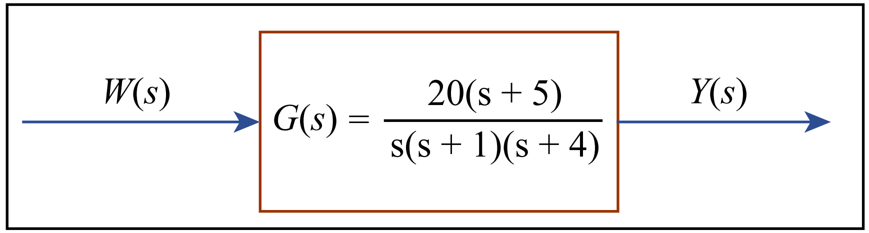


Figure by MIT OpenCourseWare.

Design problem: We are given transient response requirements of 9.5% overshoot and 0.74sec settling time. Moreover, we would like to approximately cancel the zero in the opn–loop transfer function.

To meet these goals, we select two closed–loop poles at $-5.4 \pm j7.2$. These meet the transient reponse requirements. Moreover, we select an additional closed–loop pole at -5.1 to approximately cancel the open–loop zero. Therefore, the desired closed–loop transfer function should be *proportional* to

$$\frac{(s+5)}{(s+5.1)(s+5.4-j7.2)(s+5.4+j7.2)} = \frac{(s+5)}{s^3+15.9s^2+136s+413}.$$

Next we convert the given transfer function to a state–space representation. This is done by first considering a system *without* the open–loop zeros, *i.e.* of the form

$$\frac{X(s)}{W(s)} = \frac{1}{s(s+1)(s+4)} = \frac{1}{s^3+5s^2+4s} \Leftrightarrow x^{(3)} + 5\ddot{x} + 4\dot{x} = w.$$

The state variable are selected as

$$\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} x \\ \dot{x} \\ \ddot{x} \end{pmatrix} \Rightarrow \mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -4 & -5 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

This choise of state variables is also known as **phase–variable form**, because it agrees with the phase diagram representation that we saw earlier (except in this case we have a 3rd order system, and hence three state/phase variables: position, velocity, acceleration.)

We also need to determine the observation matrix $\mathbf{C}$. Since the open–loop transfer function has a zero, the response includes a derivative term; that is,

$$Y(s) = 20(s+5)X(s) \Rightarrow y(t) = 20\left[\dot{x}(t) + 5x(t)\right] = 20(q_2 + 5q_1) \Rightarrow \mathbf{c} = (100 \quad 20 \quad 0).$$

Let the **gain matrix** be

$$\mathbf{K} = (k_1 \quad k_2 \quad k_3) \Rightarrow \mathbf{BK} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} (k_1 \quad k_2 \quad k_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k_1 & k_2 & k_3 \end{pmatrix}$$

$$\Rightarrow \mathbf{A} - \mathbf{BK} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_1 & -(4+k_2) & -(5+k_3) \end{pmatrix}$$

The denominator of the transfer function is the determinant of the matrix

$$s\mathbf{I} - \mathbf{A} + \mathbf{BK} = \begin{pmatrix} s & -1 & 0 \\ 0 & s & -1 \\ s+k_1 & s+(4+k_2) & s+(5+k_3) \end{pmatrix}$$

$$\det\left(s\mathbf{I} - \mathbf{A} + \mathbf{BK}\right) = s^3 + (5+k_3)s^2 + (4+k_2)s + k_1.$$

Equating coefficients with the desired denominator (characteristic equation) $s^3 + 15.9s^2 + 136s + 413$, we obtain the gains

$$k_1 = 413; \quad k_2 = 132; \quad k_3 = 10.9.$$

The state space representation of the *closed–loop* system is

$$\dot{\mathbf{q}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -413 & -136 & -15.9 \end{pmatrix} \mathbf{q} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} r.$$

$$y = (100 \quad 20 \quad 0)\,\mathbf{q}.$$

The *closed–loop* transfer function is

$$\frac{Y(s)}{R(s)} = \frac{20(s+5)}{s^3+15.9s^2+136s+413}.$$