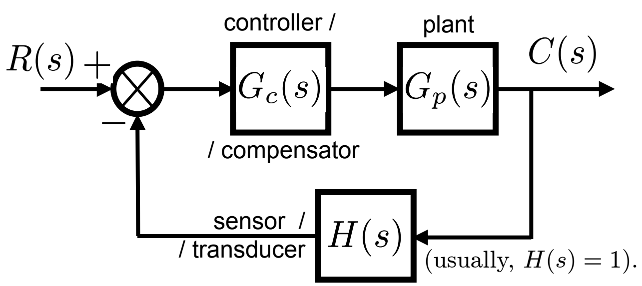
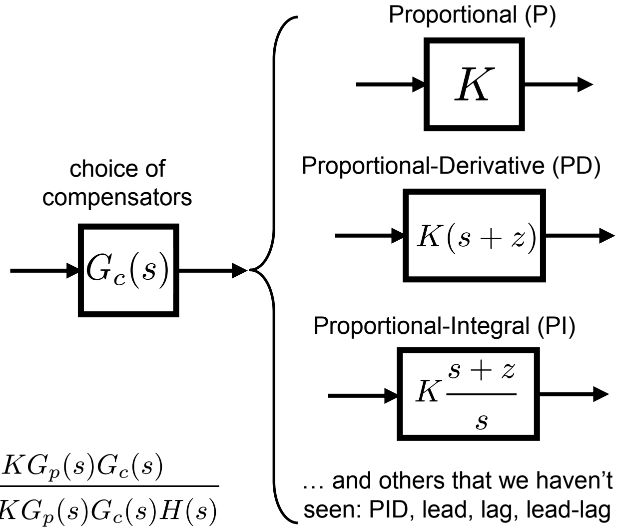
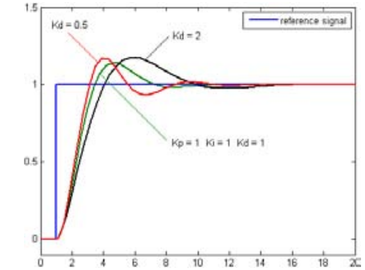
Summary: Compensator design using the Root Locus; State Space



generic system block diagram with controller/compensator and feedback

purpose: to improve the system's dynamics by proper choice of the controller TF and gain





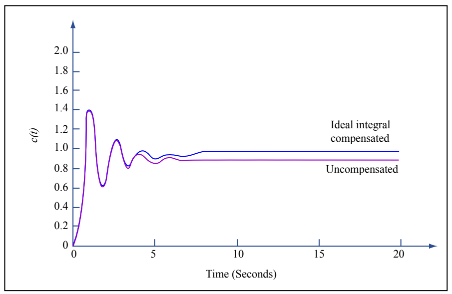


Figure by MIT OpenCourseWare.

Open–Loop TF: $KG_p(s)G_c(s)H(s)$ Closed–Loop TF: $\frac{KG_p(s)G_c(s)}{1+KG_p(s)G_c(s)H(s)}$

Root Locus sketching rules (negative feedback)

- **Rule 1:** # branches = # open-loop poles
- **Rule 2:** symmetrical about the real axis
- **Rule 3:** real-axis segments are to the left of an *odd* number of real-axis finite open-loop poles/zeros
- **Rule 4:** RL begins at open-loop poles (K=0), ends at open-loop zeros (K=∞)
- **Rule 5:** Asymptotes: real-axis intercept σ_a, angles θ_a

$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\text{finite poles} - \#\text{finite zeros}} \qquad \theta_a = \frac{(2m+1)\pi}{\#\text{finite poles} - \#\text{finite zeros}} \qquad m = 0, \pm1, \pm2, \ldots$$

- **Rule 6:** Real-axis break-in and breakaway points

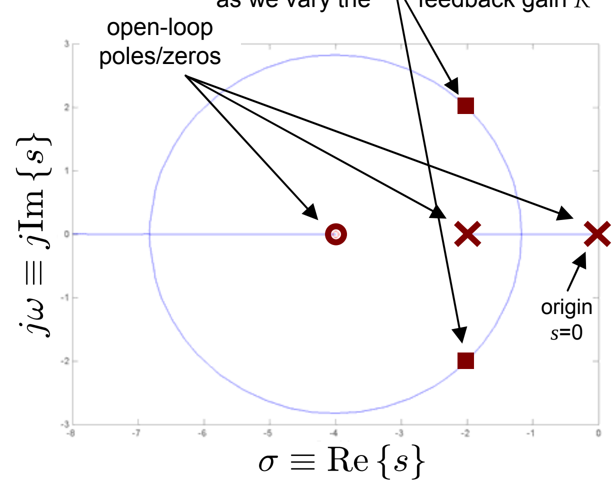
$$\text{Found by setting} \quad K(\sigma) = -\frac{1}{G(\sigma)H(\sigma)} \quad (\sigma \text{ real}) \qquad \text{and solving} \quad \frac{\mathrm{d}K(\sigma)}{\mathrm{d}\sigma} = 0 \quad \text{for real } \sigma.$$

- **Rule 7:** Imaginary axis crossings *(transition to instability)*

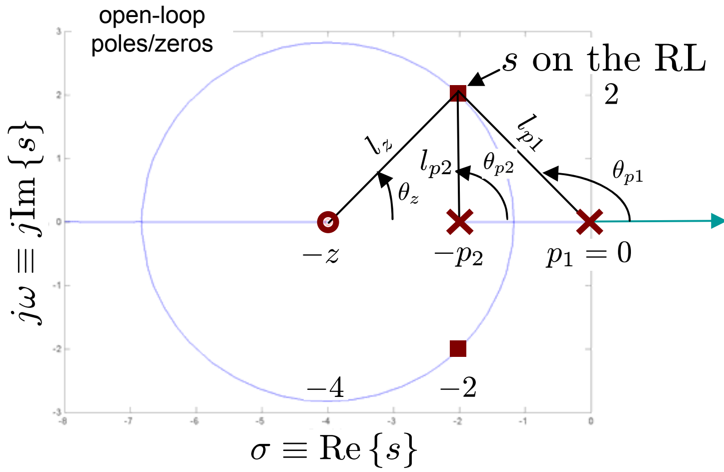
$$\text{Found by setting} \quad KG(j\omega)H(j\omega) = -1 \quad \text{and solving} \quad \begin{cases} \text{Re}\left[KG(j\omega)H(j\omega)\right] = -1, \\ \text{Im}\left[KG(j\omega)H(j\omega)\right] = 0. \end{cases}$$

What is Root Locus?

RL: locations on the s-plane where the closed-loop poles move as we vary the feedback gain K



Summary: Compensator design using the Root Locus; State Space



s on the RL $\Rightarrow 1+KG_c(s)G_p(s)H(s)=0 \Rightarrow KG_c(s)G_p(s)H(s)=-1$

$$\Rightarrow \begin{cases} |KG_c(s)G_p(s)H(s)|=1 \\ \angle\{KG_c(s)G_p(s)H(s)\}=(2m+1)\pi \end{cases}$$

$$\Rightarrow \begin{cases} K=\dfrac{1}{|G_c(s)G_p(s)H(s)|} \\ \sum\angle(s+z)-\sum\angle(s+p)=(2m+1)\pi \end{cases}$$

sums taken over **Open Loop zeros/poles**

Here, Open Loop poles are $p_1=0$, $-p_2=-2$, Open Loop zero is $-z=-4$.
Geometrical interpretation of the amplitude and phase contributions to s:

$$l_{p1}=|s+p_1|=|s|; \qquad l_{p2}=|s+p_2|=|s+2|; \qquad l_z=|s+z|=|s+4|;$$
$$\theta_{p1}=\angle(s+p_1)=\angle s; \quad \theta_{p2}=\angle(s+p_2)=\angle(s+2); \quad \theta_z=\angle(s+z)=\angle(s+4).$$

Since the point s shown as crimson block belongs to the Root Locus,

$$\Rightarrow \begin{cases} K=\dfrac{|s|\,|s+2|}{|s+4|}=\dfrac{l_{p1}l_{p2}}{l_z} \\ \angle(s+4)-\angle s-\angle(s+2)=\theta_z-\theta_{p1}-\theta_{p2}=-\pi \end{cases}$$

The crimson block is at $s=-2+j2$ on the Root Locus. Using geometry,

$$l_{p1}=|s|=2\sqrt{2}; \qquad l_{p2}=|s+2|=2; \qquad l_z=|s+4|=2\sqrt{2}$$
$$\theta_{p1}=\angle s=3\pi/4; \quad \theta_{p2}=\angle(s+2)=\pi/2; \quad \theta_z=\angle(s+4)=\pi/4.$$

We can see that indeed the angular contributions add up as
$\theta_z-\theta_{p1}-\theta_{p2}=-\pi$,
while the amplitude contributions give $K=\left(2\sqrt{2}\times 2\right)/\left(2\sqrt{2}\right)\Rightarrow K=2.$

s-space geometry and transient characteristics

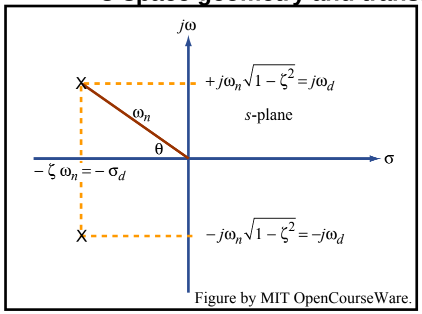


Figure by MIT OpenCourseWare.

$$\mathrm{TF}=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}.$$

- Settling time $T_s \approx 4/(\zeta\omega_n)$;
- Damped osc. frequency $\omega_d=\sqrt{1-\zeta^2}\,\omega_n$
- Overshoot %OS

$\cos\theta=\zeta$ $\quad \tan\theta=\dfrac{\sqrt{1-\zeta^2}}{\zeta}$ $\quad \%\mathrm{OS}=\exp\left(-\dfrac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)$

State Space & Phase Space

From the Equation of Motion to the State–Space representation:

$m\ddot{x}(t)+b\dot{x}(t)+kx(t)=w(t)\rightarrow\begin{pmatrix}x\\ \dot{x}\end{pmatrix}\equiv\mathbf{q}(t)=\begin{pmatrix}q_1\\ q_2\end{pmatrix}$ state, $\quad y(t)\equiv\dot{x}(t)$ output

$$\Rightarrow\dot{\mathbf{q}}(t)=\begin{pmatrix}\dot{q}_1\\ \dot{q}_2\end{pmatrix}=\begin{pmatrix}0 & 1\\ -k/m & -b/m\end{pmatrix}\begin{pmatrix}q_1\\ q_2\end{pmatrix}+\begin{pmatrix}0\\ 1\end{pmatrix}w(t); \quad y(t)=\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}q_1\\ q_2\end{pmatrix}\equiv\mathbf{cq}.$$

$$\mathbf{A}=\begin{pmatrix}0 & 1\\ -k/m & -b/m\end{pmatrix} \qquad \mathbf{b}=\begin{pmatrix}0\\ 1\end{pmatrix}$$

Solution to the state equations:

$$s\hat{\mathbf{q}}(s)=\mathbf{A}\hat{\mathbf{q}}(s)+\mathbf{b}W(s)\Rightarrow$$
$$\hat{\mathbf{q}}(s)=(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{b}W(s).$$
$$Y(s)=\mathbf{c}\hat{\mathbf{q}}(s)=\mathbf{c}\,(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{b}W(s).$$