

Lecture 25

Laplace-domain solution of the State equations

In the previous lecture, we derived the following state-space model for the uncompensated 2.004 Tower:

$$\dot{\mathbf{q}}(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{b}w(t), \quad [\text{dynamics-equation of motion}] \quad (1)$$

$$y(t) = \mathbf{c}\mathbf{q}(t), \quad [\text{output or observation equation}] \quad (2)$$

where the state vector is

$$\mathbf{q}(t) = \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix} = \begin{pmatrix} x_1(t) \\ v_1(t) \end{pmatrix}. \quad (3)$$

and the system matrix and input vector, respectively, are

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -k_1/m_1 & -b_1/m_1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ 1/m_1 \end{pmatrix}. \quad (4)$$

The wind force (disturbance) is denoted as $w(t)$ while the output vector \mathbf{c} might be $(1 \ 0)$ or $(0 \ 1)$, depending on whether one wishes to define the tower displacement or velocity, respectively, as output; or \mathbf{c} might be any linear combination of displacement and velocity such as $(0.1 \ 0.9)$. In this lecture, we will solve the state equations in the Laplace domain and show how we can obtain the transfer function from the state equations. Before studying these notes, you are advised to make sure that you are thoroughly familiar with matrices. You can find a Math supplement on matrices in the 2.004 Stellar website.

Derivation of the transfer functions for position and velocity

Before we knew about state space, we would have obtained the transfer functions for position or velocity, respectively, directly from the equation of motion as follows. First we would have Laplace-transformed the equation of motion to obtain

$$m_1 s^2 X_1(s) + b_1 s X_1(s) + k_1 X_1(s) = W(s). \quad (5)$$

If the desired output were the displacement $x_1(t)$, then the transfer function would be found directly as

$$\frac{X_1(s)}{W(s)} = \frac{1/m_1}{s^2 + (b_1/m_1)s + (k_1/m_1)}. \quad (6)$$

This is of the form

$$K_0 \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \quad \text{where}$$

$$\omega_n = \sqrt{\frac{k_1}{m_1}} \quad (7)$$

$$\zeta = \frac{b_1}{2\sqrt{k_1 m_1}} \quad (8)$$

$$K_0 = \frac{1}{k_1}. \quad (9)$$

This is our familiar 2nd-order system which is undamped if $\zeta = 0$, underdamped if $0 < \zeta < 1$, critically damped if $\zeta = 1$, and overdamped if $\zeta > 1$.

If the desired output were the velocity $v_1(t)$, taking into account $v_1(t) = \dot{x}_1(t) \Rightarrow V_1(s) = sX_1(s)$, we can easily find

$$\frac{V_1(s)}{W(s)} = \frac{s/m_1}{s^2 + (b_1/m_1)s + (k_1/m_1)}. \quad (10)$$

This is still a 2nd-order system with an additional zero at the origin.

We will now see how both of these transfer functions can be obtained from the state representation in one step using matrix algebra.

Laplace transforming the state equations

We begin by Laplace transforming the state equation (1) which represents the system dynamics. This matrix equation is a shorthand for the fuller version

$$\begin{pmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -k_1/m_1 & -b_1/m_1 \end{pmatrix} \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1/m_1 \end{pmatrix} w(t). \quad (11)$$

Clearly, we can Laplace transform each row of this matrix equation individually. In the matrix notation shorthand, we obtain

$$s\hat{\mathbf{q}}(s) = \mathbf{A}\hat{\mathbf{q}}(s) + \mathbf{b}W(s). \quad (12)$$

Here, we introduced the hat notation

$$\hat{\mathbf{q}}(s) = \begin{pmatrix} Q_1(s) \\ Q_2(s) \end{pmatrix} = \begin{pmatrix} X_1(s) \\ V_1(s) \end{pmatrix} \quad (13)$$

to denote the Laplace transform of the vector \mathbf{q} , since uppercase notation is reserved for matrices. We can solve (12) as follows:

$$s\hat{\mathbf{q}}(s) = s \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \hat{\mathbf{q}}(s) = s\mathbf{I}\hat{\mathbf{q}}(s) \Rightarrow$$

$$(s\mathbf{I} - \mathbf{A})\hat{\mathbf{q}}(s) = \mathbf{b}W(s) \Rightarrow \hat{\mathbf{q}}(s) = (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{b}W(s). \quad (14)$$

Equation 14 is the state-space solution of system dynamics. It provides *simultaneously* the solution for the displacement and velocity of the system.

State–space solution of the uncompensated 2.004 Tower dynamics

Let us see how (14) applies to the uncompensated 2.004 Tower. We have

$$s\mathbf{I} - \mathbf{A} = s \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -k_1/m_1 & -b_1/m_1 \end{pmatrix} = \begin{pmatrix} s & -1 \\ k_1/m_1 & s + b_1/m_1 \end{pmatrix} \Rightarrow$$

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{s(s + b_1/m_1) + k_1/m_1} \begin{pmatrix} s + b_1/m_1 & 1 \\ -k_1/m_1 & s \end{pmatrix}.$$

Substituting into (14) we obtain

$$\hat{\mathbf{q}}(s) = \frac{1}{s(s + b_1/m_1) + k_1/m_1} \begin{pmatrix} s + b_1/m_1 & 1 \\ -k_1/m_1 & s \end{pmatrix} \begin{pmatrix} 0 \\ 1/m_1 \end{pmatrix} W(s) \Rightarrow$$

$$\hat{\mathbf{q}}(s) = \frac{1}{s^2 + (b_1/m_1)s + (k_1/m_1)} \begin{pmatrix} 1/m_1 \\ s/m_1 \end{pmatrix} W(s). \quad (15)$$

Obtaining the transfer function from the state solution

We can easily verify that row–by–row, the state solution (15) above is in agreement with the Laplace–domain solutions for displacement (6) and velocity (10), which we obtained in the traditional way earlier.

More formally, we can use the output state–space equation (2) to obtain a transfer function for an arbitrary output (as long as our chosen output is a linear combination of the state variables.) Laplace transforming (2) we obtain

$$Y(s) = \mathbf{c}\hat{\mathbf{q}}(s). \quad (16)$$

Substituting (14) we find

$$Y(s) = \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{b}W(s). \quad (17)$$

In the case of the 2.004 Tower, should we decide to define the displacement as the output, *i.e.* $Y(s) \equiv X_1(s) = Q_1(s)$, we would have $\mathbf{c} = (1 \ 0)$, so using (15) and (17) we obtain

$$\frac{Y(s)}{W(s)} = \frac{1}{s^2 + (b_1/m_1)s + (k_1/m_1)} (1 \ 0) \begin{pmatrix} 1/m_1 \\ s/m_1 \end{pmatrix} = \frac{1/m_1}{s^2 + (b_1/m_1)s + (k_1/m_1)}. \quad (18)$$

On the other hand, should we decide to define the velocity as the output, *i.e.* $Y(s) \equiv V_1(s) = Q_2(s)$, we would have $\mathbf{c} = (0 \ 1)$, so using (15) and (17) we obtain

$$\frac{Y(s)}{W(s)} = \frac{1}{s^2 + (b_1/m_1)s + (k_1/m_1)} (0 \ 1) \begin{pmatrix} 1/m_1 \\ s/m_1 \end{pmatrix} = \frac{s/m_1}{s^2 + (b_1/m_1)s + (k_1/m_1)}. \quad (19)$$

These results are in agreement with (6) and (10).

The impulse response in state space

The impulse response is particularly simple to find. If the wind disturbance is an impulse $w(t) = \delta(t)$, then its Laplace transform is unity, $W(s) = 1$. Therefore, the impulse response is simply the Laplace transform of the transfer function for either dynamical variable, position or velocity. In state space,

$$\hat{\mathbf{q}}(s) = \begin{pmatrix} Q_1(s) \\ Q_2(s) \end{pmatrix} = \frac{1}{s^2 + (b_1/m_1)s + (k_1/m_1)} \begin{pmatrix} 1/m_1 \\ s/m_1 \end{pmatrix}. \quad (20)$$

Inverse Laplace transforming the above equation can be done row-by-row according to the Laplace transform tables. We will see that below in a special context.

From the state-space solution to the Phase Space diagram

Now that we can obtain simultaneously the solutions for displacement and velocity, it is worth noting an additional way to look at the dynamical response of systems. This new view is called “Phase Space,” and is the bread and butter of dynamicists who study complex phenomena in fields as diverse as weather patterns, fluid dynamics, combustion, predator-prey dynamics, and geometrical optics.

To develop the Phase Space, we will use the uncompensated 2.004 Tower model, *i.e.* equations (6) and (10) for position and velocity, respectively; or directly the matrix-vector shorthand result (15). First, we assume that damping is negligible ($b_1 \approx 0 \Rightarrow \zeta \approx 0$), *i.e.* our system is *undamped*. We then have

$$\hat{\mathbf{q}}(s) = \begin{pmatrix} Q_1(s) \\ Q_2(s) \end{pmatrix} = \frac{1}{s^2 + (k_1/m_1)} \begin{pmatrix} 1/m_1 \\ s/m_1 \end{pmatrix}. \quad (21)$$

Row-by-row, this equation is rewritten as

$$Q_1(s) = \frac{1/m_1}{s^2 + (k_1/m_1)}, \quad (22)$$

$$Q_2(s) = \frac{s/m_1}{s^2 + (k_1/m_1)}. \quad (23)$$

Using the Laplace transforms for the sine and cosine functions (Nise Table 2,1), we find

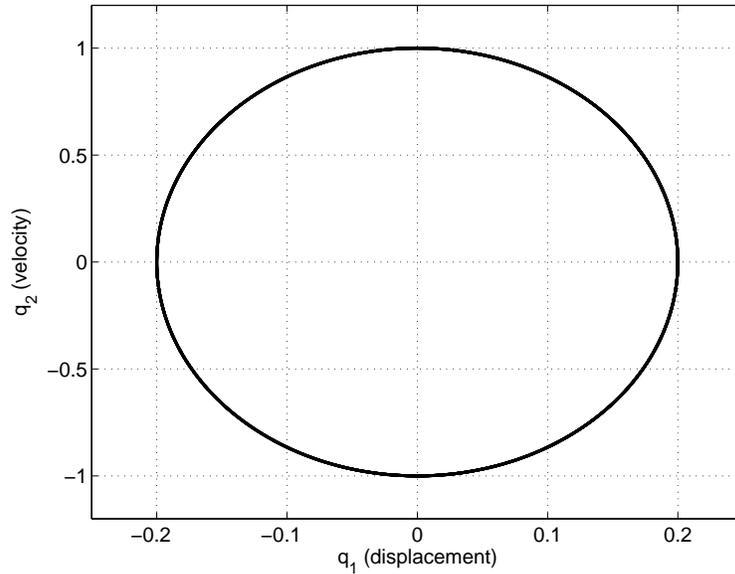
$$\mathbf{q}(t) = \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix} = \begin{pmatrix} x_1(t) \\ v_1(t) \end{pmatrix} = \begin{pmatrix} \sin\left(\sqrt{k_1/m_1}t\right)/\sqrt{k_1m_1} \\ \cos\left(\sqrt{k_1/m_1}t\right)/m_1 \end{pmatrix}. \quad (24)$$

By simple algebraic manipulation, we obtain

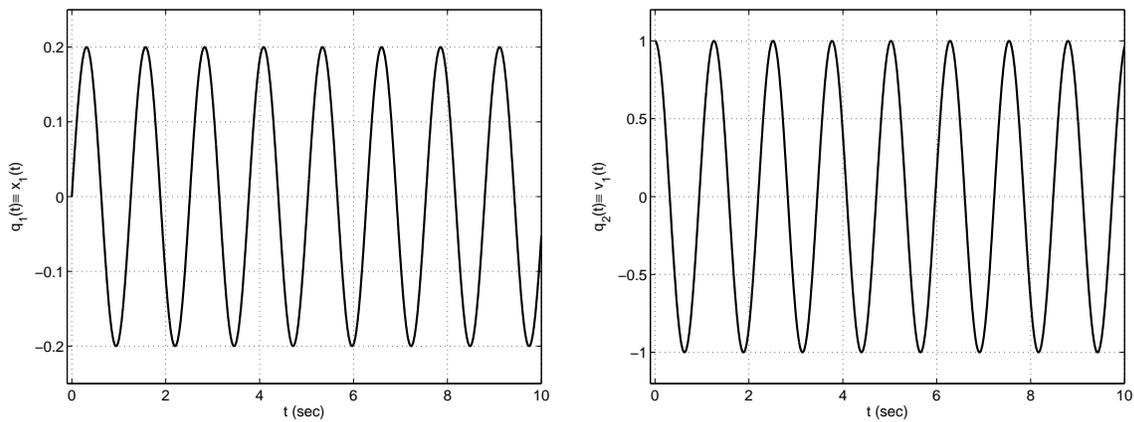
$$\frac{q_1^2(t)}{1/(k_1m_1)} + \frac{q_2^2(t)}{1/m_1^2} = 1. \quad (25)$$

This means that if we plot $q_2(t)$ against $q_1(t)$, we will see an ellipse with half-axes $1/m_1$ and $1/\sqrt{k_1m_1}$. This plot is referred to as “Phase Space diagram,” or simply “Phase Space.” Experimentally it can be observed as follows: we use two transducers, one

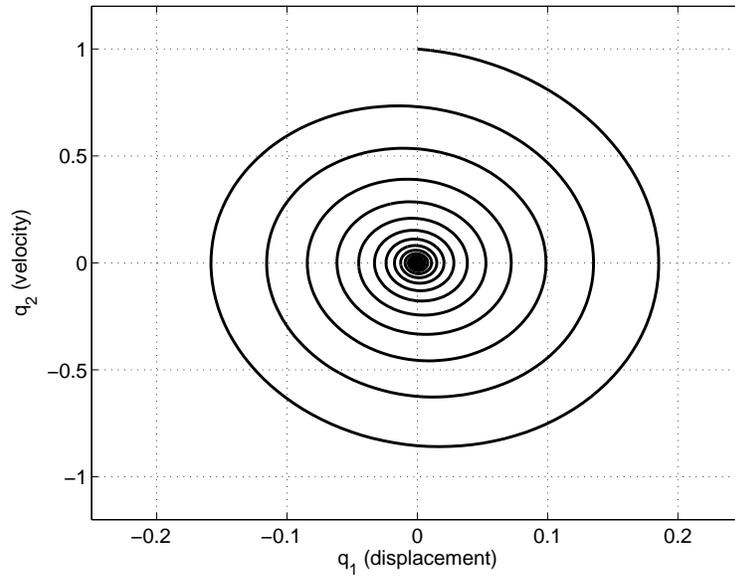
converting displacement to voltage V_d and another converting velocity to voltage V_v ; we then connect the V_d signal to the “X” (horizontal) input of the oscilloscope, and V_v to the “Y” (vertical) input. Considering $m_1 = 1$, $k_1 = 25$, on the scope screen, we should then see something like this: (note the difference in horizontal and vertical scales!)



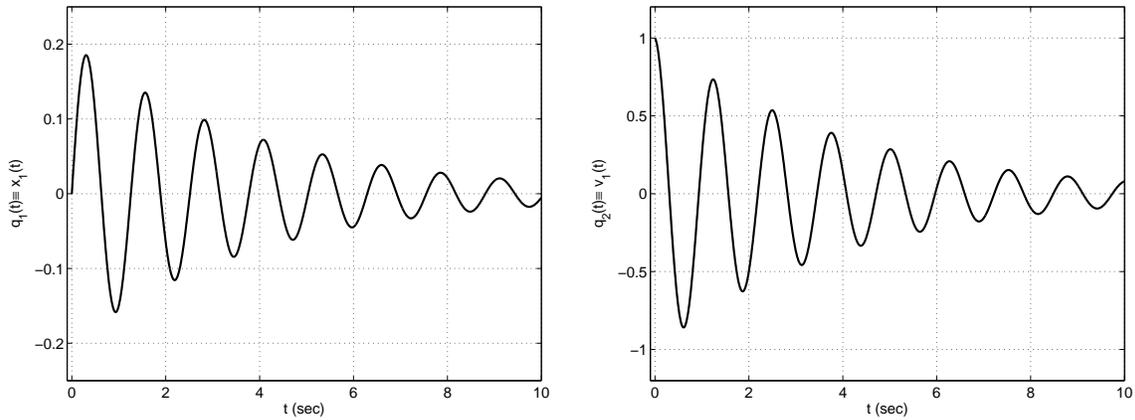
This makes more sense if we compare it to the impulse responses $q_1(t) \equiv x_1(t)$, $q_2(t) \equiv v_1(t)$. We’ve plotted them below side-by-side:



We can generalize the same formulation to non-negligible damping. The algebra in that case turns out to be slightly more tedious. However, some thought will convince us that the underdamped Phase Space diagram will no more be an ellipse; rather, it will spiral inwards, *i.e.* towards the origin $\mathbf{q} = (0 \ 0)^t$, because the damping in the system causes both position and velocity impulse responses to decay as time progresses. This is verified in the next page, where we used $b_1 = 0.5$ (again, note the difference in horizontal and vertical scales!)



The impulse responses $q_1(t) \equiv x_1(t)$, $q_2(t) \equiv v_1(t)$. are again plotted below side-by-side:



We leave the overdamped case for you to ponder. Some thought should convince you that the Phase Space diagram would then look like a curved line, simpler than a spiral, but still converging to the origin $\mathbf{q} = (0 \ 0)^t$ as $t \rightarrow \infty$. You can also verify your conclusion in MATLAB by using the `tf` command to define an overdamped transfer function of your choice and then the `impz` command to generate its impulse response.