Summary: Root Locus sketching rules

Negative Feedback

- **Rule 1:** # branches = # poles
- **Rule 2:** symmetrical about the real axis
- **Rule 3:** real-axis segments are to the left of an *odd* number of real-axis finite poles/zeros
- **Rule 4:** RL begins at poles, ends at zeros
- **Rule 5:** Asymptotes: real-axis intercept σ_a,angles θ_a

$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\text{finite poles} - \#\text{finite zeros}} \qquad \theta_a = \frac{(2m+1)\pi}{\#\text{finite poles} - \#\text{finite zeros}} \qquad m = 0, \pm 1, \pm 2, \ldots$$

- **Rule 6:** Real-axis break-in and breakaway points

$$\text{Found by setting} \quad K(\sigma) = -\frac{1}{G(\sigma)H(\sigma)} \quad (\sigma \text{ real}) \qquad \text{and solving} \quad \frac{\mathrm{d}K(\sigma)}{\mathrm{d}\sigma} = 0 \quad \text{for real } \sigma.$$

- **Rule 7:** Imaginary axis crossings *(transition to instability)*

$$\text{Found by setting} \quad KG(j\omega)H(j\omega) = -1 \quad \text{and solving} \quad \begin{cases} \mathrm{Re}\left[KG(j\omega)H(j\omega)\right] &= -1, \\ \mathrm{Im}\left[KG(j\omega)H(j\omega)\right] &= 0. \end{cases}$$

- *Today's Goal:* Shaping the transient response by adjusting the feedback gain

Damping ratio and pole location

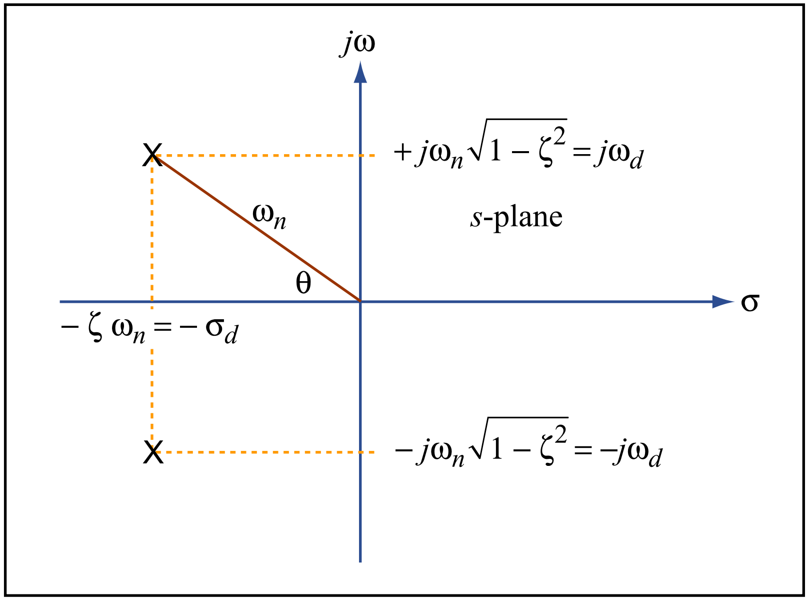

Figure by MIT OpenCourseWare.

Fig. 4.17

Recall 2^{nd}–order underdamped sustem

$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

Complex poles $-\sigma_d \pm j\omega_d$,

$$\text{where} \begin{cases} \sigma_d = \zeta\omega_n, \\ \omega_d = \sqrt{1-\zeta^2}\,\omega_n. \end{cases}$$

From the geometry,

$$\tan\theta = \frac{\sqrt{1-\zeta^2}}{\zeta} \Rightarrow$$

$$\boxed{\cos\theta = \zeta.}$$

The angle θ that a complex pole subtends to the origin of the *s*-plane determines the damping ratio ζ of an underdamped 2^{nd} order system.

The distance from the pole to the origin equals the natural frequency.

Transient response and pole location

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Please see: Fig. 4.19 in Nise, Norman S. *Control Systems Engineering.* 4th ed. Hoboken, NJ: John Wiley, 2004.

- Settling time

$$T_s \approx 4/(\zeta\omega_n);$$

- Damped osc. frequency

$$\omega_d = \sqrt{1-\zeta^2}\omega_n$$

- Overshoot %OS

$$\%\mathrm{OS} = \exp\left(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)$$

$$\tan\theta = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

Trends in underdamped response as ζ increases

Fig. 4.14

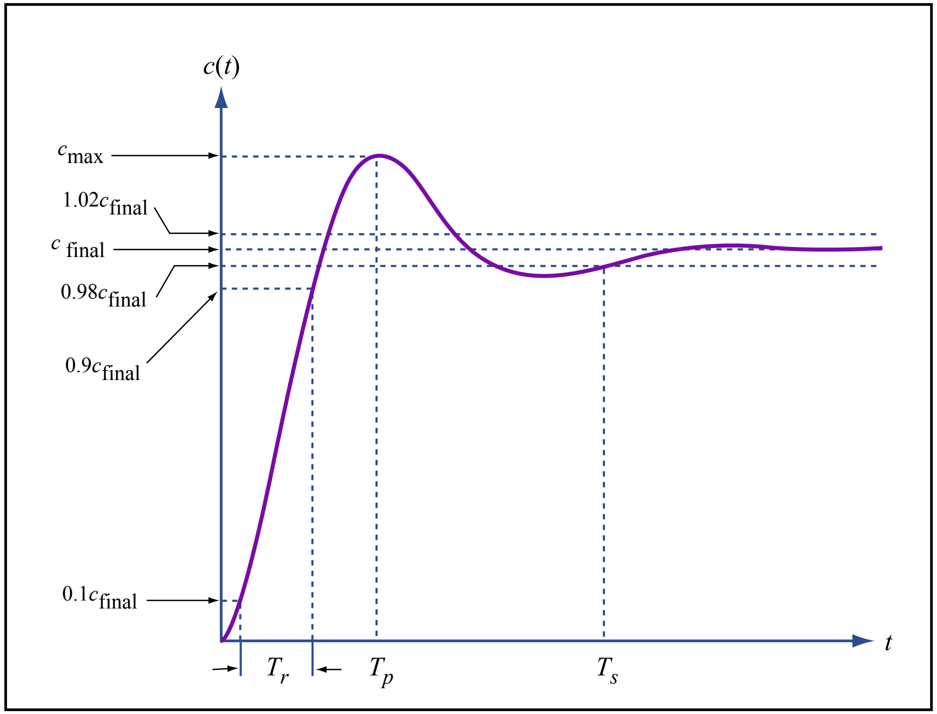

Figure by MIT OpenCourseWare.

As $\zeta \uparrow$,

- Rise time $T_r \uparrow$ (slower);
- Settling time $T_s \approx 4/(\zeta\omega_n) \quad \uparrow$ (slower);
- Peak time $T_p = \pi/(\sqrt{1-\zeta^2}\omega_n) \uparrow$ (slower);
- Overshoot %OS $\downarrow$ (smaller)

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Please see: Fig. 4.15 and 4.16 in Nise, Norman S. *Control Systems Engineering*. 4th ed. Hoboken, NJ: John Wiley, 2004.

Achieving a desired transient with a given RL

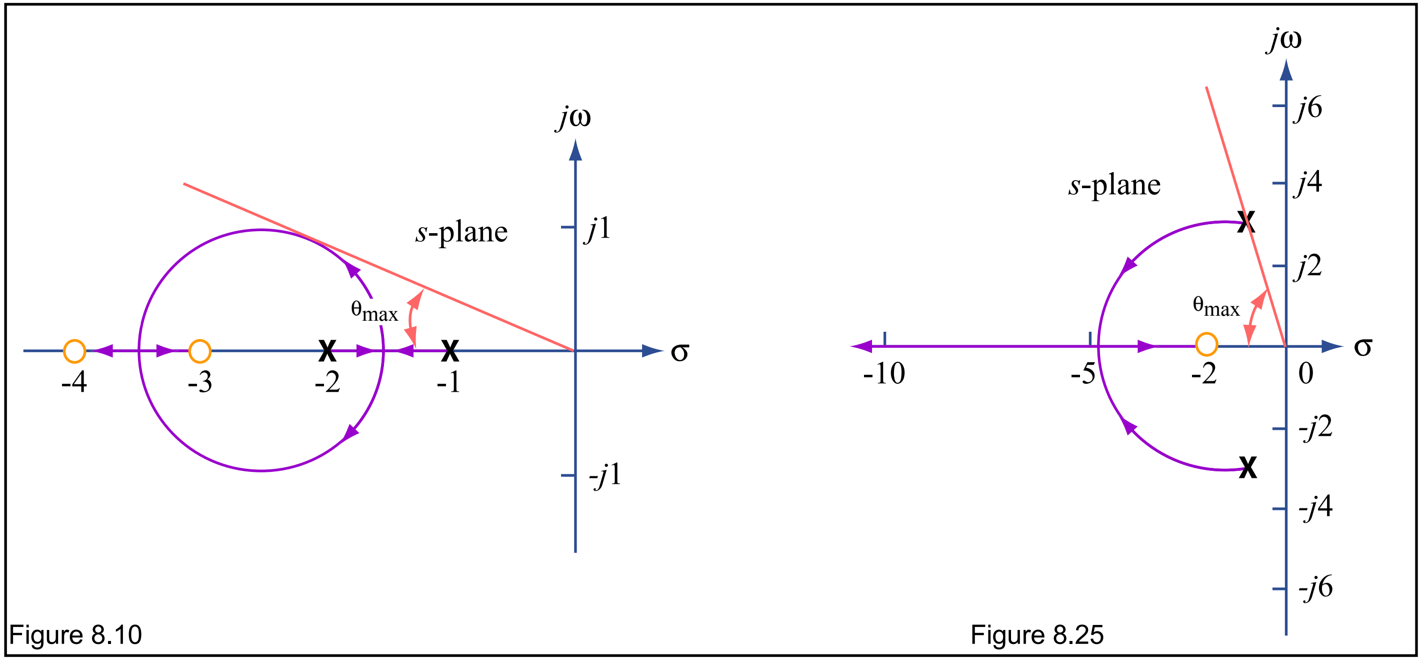


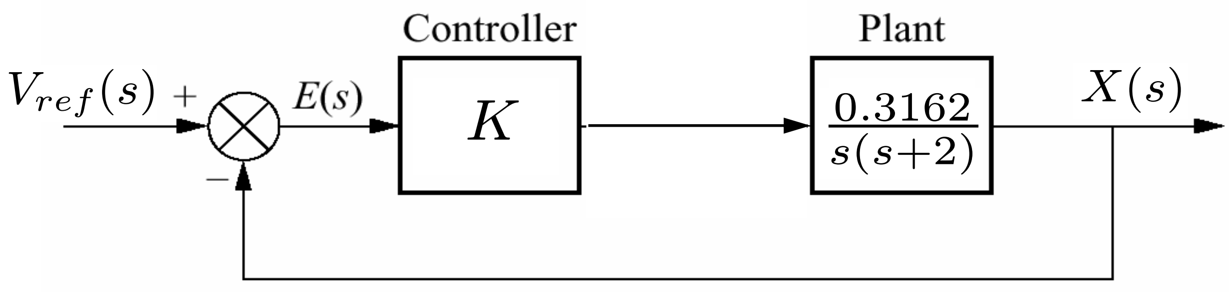
Figure by MIT OpenCourseWare.

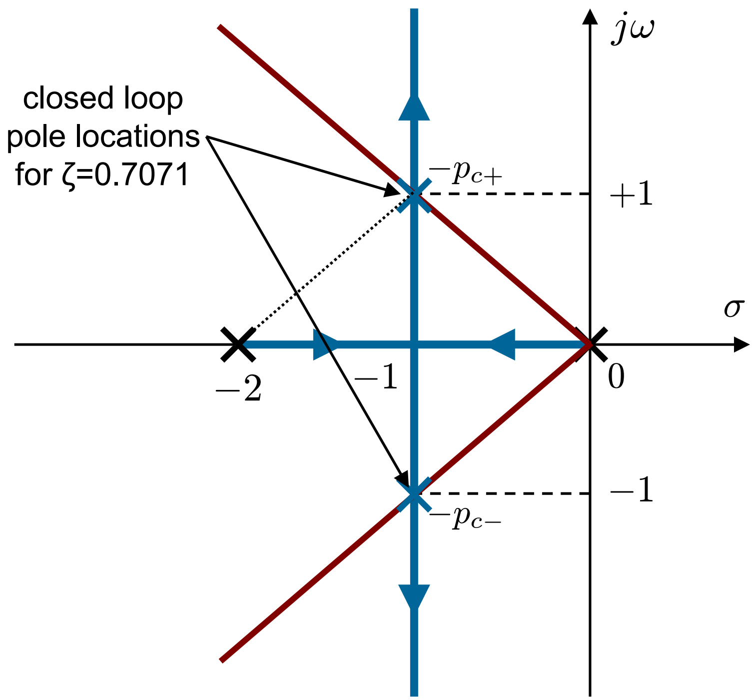
As $\zeta \uparrow \quad \Leftrightarrow \quad \theta \downarrow$,

- Rise time $T_r \uparrow$ (slower);
- Settling time $T_s \uparrow$ (slower);
- Peak time $T_p \uparrow$ (slower);
- Overshoot $\%OS \downarrow$ (smaller)

If the given RL does not allow the desired transient characteristics to be achieved, then we must *modify* the RL by *adding poles/zeros (compensator design)*

Example: 2nd order – type 1 system





We are given $\zeta = 1/\sqrt{2} = 0.7071$. For this value,

$$\%\text{OS} = \exp\left(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right) \times 100 = \mathrm{e}^{-\pi} \times 100 = 4.32\%.$$

Also, $\cos\theta = \zeta \Rightarrow \theta = \pm 45^\circ$.

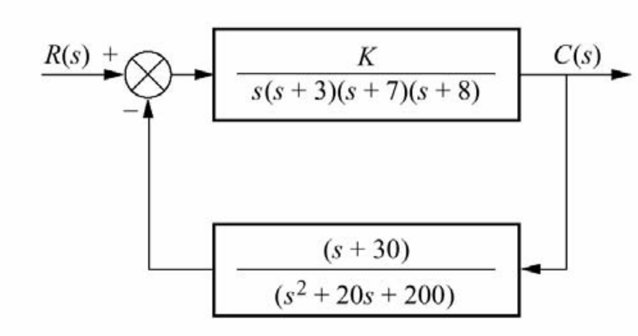
We can locate the closed–loop poles by finding the intersection of the root locus with the lines $\theta = \pm 45^\circ$.

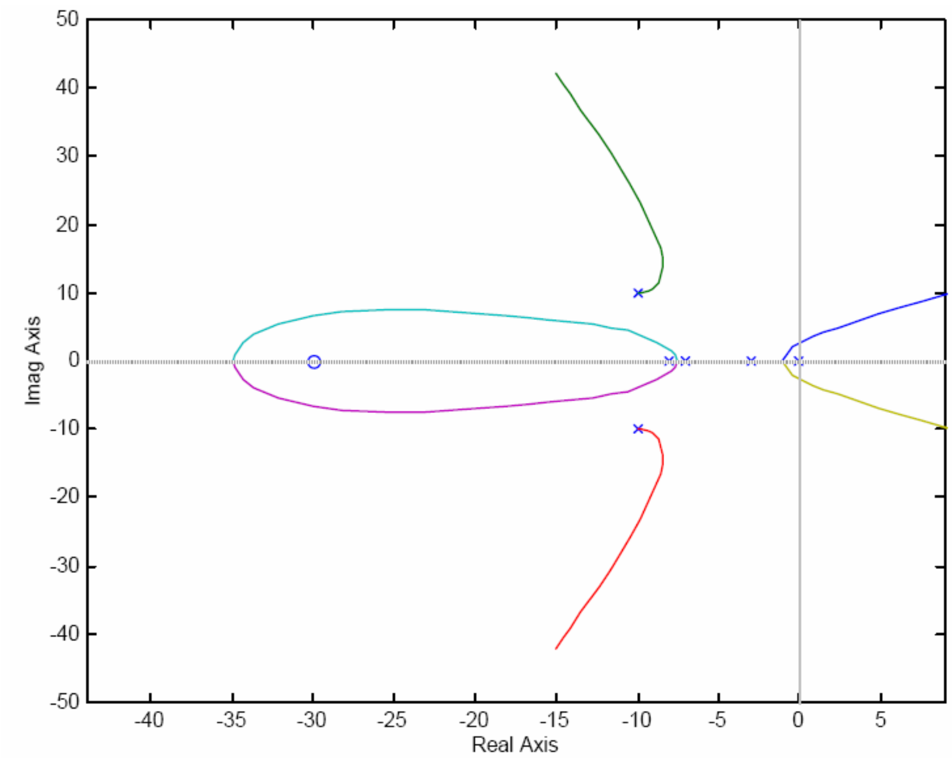
We can also estimate the feedback gain K that will yield the required closed–loop poles $-p_{c+}$, $-p_{c-}$ from the relationship $K = 1/\left|G(-p_{c\pm})H(-p_{c\pm})\right| \Rightarrow$

$$K = \frac{|p_{c\pm}|\,|p_{c\pm}+2|}{0.3162} = \frac{\sqrt{2}\times\sqrt{2}}{0.3162} = 6.325.$$

The numerator is computed geometrically from the equilateral triangle $\{(-2), (p_{c+}), (0)\}$

Example: higher order system





Positive feedback: sketching the Root Locus

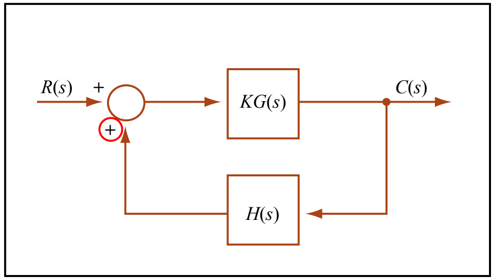


Figure 8.26 Figure by MIT OpenCourseWare.

$$\text{Closed–loop TF}(s) = \frac{KG(s)}{1 - KG(s)H(s)}.$$

- **Rule 1:** # branches = # poles
- **Rule 2:** symmetrical about the real axis
- **Rule 3:** real-axis segments are to the left of an *even* number of real-axis finite poles/zeros
- **Rule 4:** RL begins at poles, ends at zeros
- **Rule 5:** Asymptotes: real-axis intercept σ_a,angles θ_a

$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\text{finite poles} - \#\text{finite zeros}} \qquad \theta_a = \frac{2m\pi}{\#\text{finite poles} - \#\text{finite zeros}} \qquad m = 0, \pm 1, \pm 2, \ldots$$

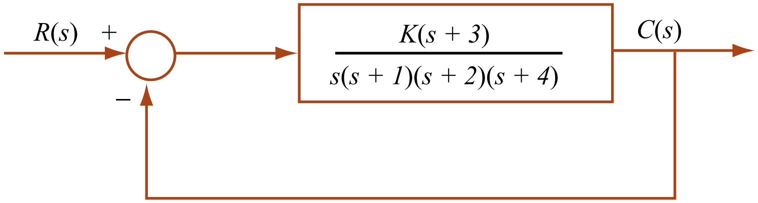
- **Rule 6:** Real-axis break-in and breakaway points

$$\text{Found by setting} \quad K(\sigma) = +\frac{1}{G(\sigma)H(\sigma)} \quad (\sigma \text{ real}) \qquad \text{and solving} \quad \frac{\mathrm{d}K(\sigma)}{\mathrm{d}\sigma} = 0 \quad \text{for real } \sigma.$$

- **Rule 7:** Imaginary axis crossings *(transition to instability)*

$$\text{Found by setting} \quad KG(j\omega)H(j\omega) = +1 \quad \text{and solving} \quad \begin{cases} \mathrm{Re}\left[KG(j\omega)H(j\omega)\right] &= +1, \\ \mathrm{Im}\left[KG(j\omega)H(j\omega)\right] &= 0. \end{cases}$$

Example: positive feedback



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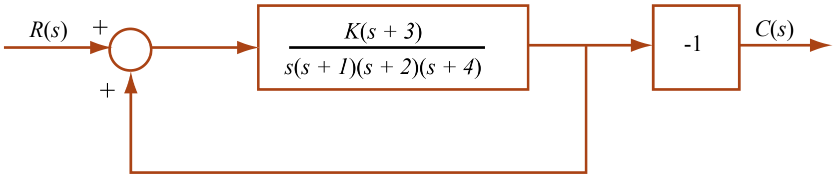
$K < 0 \Leftrightarrow$



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with $K > 0$.

Figure 8.26

Figure 8.11

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Please see Fig. 8.26b in Nise, Norman S. *Control Systems Engineering*. 4th ed. Hoboken, NJ: John Wiley, 2004.

Real–axis asymptote intercept:

$$\sigma_a = \frac{(-1-2-4)-(-3)}{4-1} = -\frac{4}{3}$$

Asymptote angles

$$\begin{aligned} \theta_a &= \frac{2m\pi}{4-1}, \quad m = 0, 1, 2, \ldots \\ &= 0, \quad m = 0, \\ &= 2\pi/3, \quad m = 1, \\ &= 4\pi/3, \quad m = 2. \end{aligned}$$

Breakaway point:
found numerically.