Root Locus sketching rules

Wednesday

- **Rule 1:** # branches = # poles
- **Rule 2:** symmetrical about the real axis
- **Rule 3:** real-axis segments are to the left of an *odd* number of real-axis finite poles/zeros
- **Rule 4:** RL begins at poles, ends at zeros

Today

- **Rule 5:** Asymptotes: angles, real-axis intercept
- **Rule 6:** Real-axis break-in and breakaway points
- **Rule 7:** Imaginary axis crossings *(transition to instability)*

Next week

- Using the root locus: analysis and design examples

Poles and zeros at infinity

$T(s)$ has a *zero at infinity* if $T(s \to \infty) \to 0$.

$T(s)$ has a *pole at infinity* if $T(s \to \infty) \to \infty$.

Example

$$KG(s)H(s) = \frac{K}{s(s+1)(s+2)}.$$

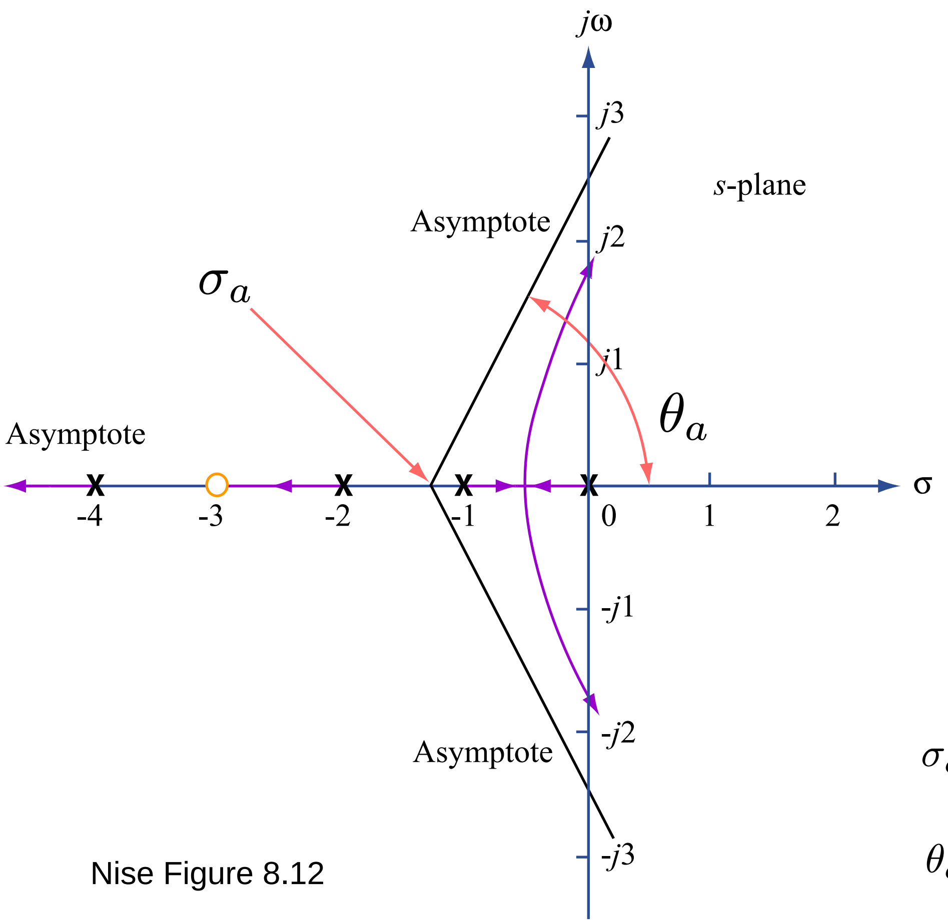
Clearly, this open–loop transfer function has three poles, 0, −1, −2. It has no *finite* zeros.
For large s, we can see that

$$KG(s)H(s) \approx \frac{K}{s^3}.$$

So this open–loop transfer function has **three** *zeros at infinity.*

Root Locus sketching rules

- **Rule 5:** Asymptotes: angles and real-axis intercept



Nise Figure 8.12

Figure by MIT OpenCourseWare.

$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\text{finite poles} - \#\text{finite zeros}}$$

$$\theta_a = \frac{(2m+1)\pi}{\#\text{finite poles} - \#\text{finite zeros}}$$

$$m = 0, \pm 1, \pm 2, \ldots$$

In this example, poles $= \{0, -1, -2, -4\}$, zeros $= \{-3\}$ so

$$\sigma_a = \frac{[0 + (-1) + (-2) + (-4)] - [(-3)]}{4-1} = -\frac{4}{3}$$

$$\theta_a = \frac{(2m+1)\pi}{4-1} = \left\{\frac{\pi}{3}, \pi, \frac{5\pi}{3}\right\}$$

Root Locus sketching rules

- **Rule 6:** Real axis break-in and breakaway points

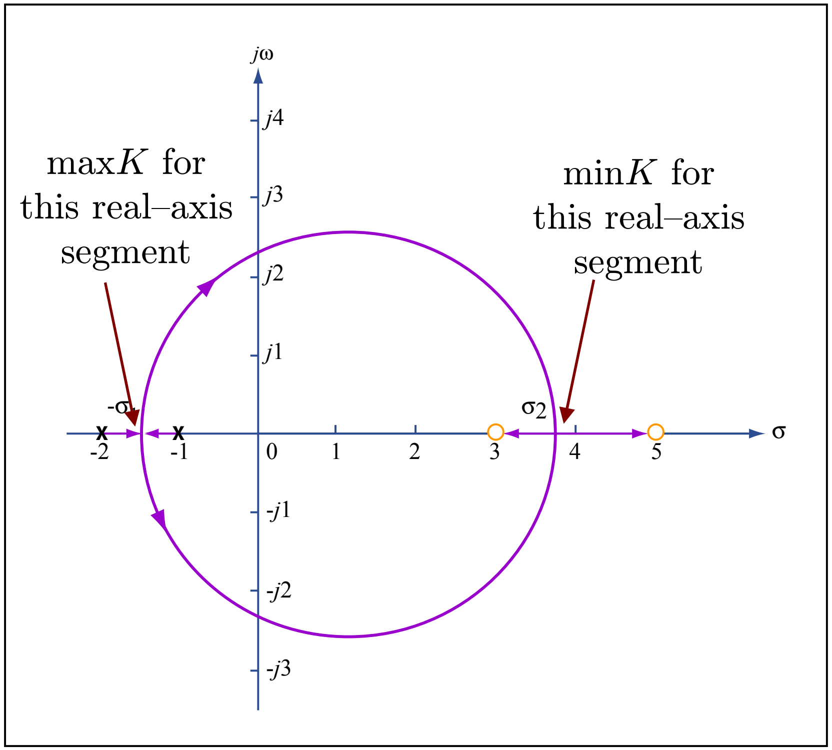


Figure by MIT OpenCourseWare.

Nise Figure 8.13

For each $s=\sigma$ on a real–axis segment of the root locus,

$$KG(\sigma)H(\sigma)=-1 \Rightarrow K=-\frac{1}{G(\sigma)H(\sigma)} \tag{1}$$

Real–axis break–in & breakaway points are the real values of σ for which

$$\frac{\mathrm{d}K(\sigma)}{\mathrm{d}\sigma}=0,$$

where $K(\sigma)$ is given by (1) above. Alternatively, we can solve

$$\sum \frac{1}{\sigma+z_i}=\sum \frac{1}{\sigma+p_i}.$$

for real σ.

Root Locus sketching rules

- **Rule 6:** Real axis break-in and breakaway points

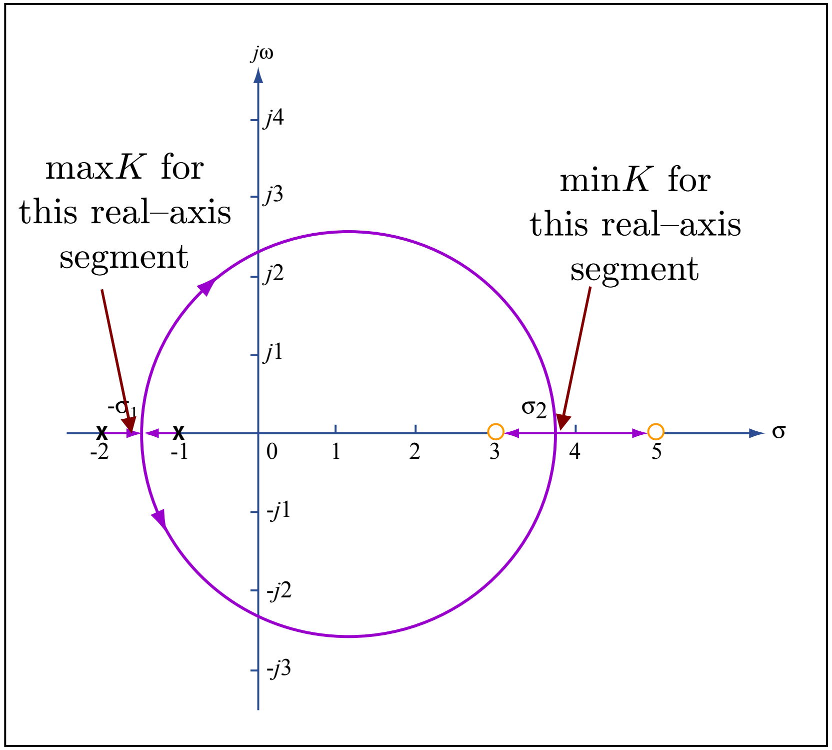


Figure by MIT OpenCourseWare.

Nise Figure 8.13

In this example,

$$KG(s)H(s) = \frac{K(s-3)(s-5)}{(s+1)(s+2)}$$

so on the real–axis segments we have

$$K(\sigma) = -\frac{(\sigma+1)(\sigma+2)}{(\sigma-3)(\sigma-5)} = -\frac{\sigma^2+3\sigma+2}{\sigma^2-8\sigma+15}$$

Taking the derivative,

$$\frac{\mathrm{d}K}{\mathrm{d}\sigma} = -\frac{11\sigma^2-26\sigma-61}{\left(\sigma^2-8\sigma+15\right)^2}$$

and setting $\mathrm{d}K/\mathrm{d}\sigma = 0$ we find

$$\sigma_1 = -1.45 \qquad \sigma_2 = 3.82$$

Alternatively, poles $= \{-1, -2\}$, zeros $= \{+3, +5\}$ so we must solve

$$\frac{1}{\sigma-3} + \frac{1}{\sigma-5} = \frac{1}{\sigma+1} + \frac{1}{\sigma+2} \Rightarrow$$

$$11\sigma^2 - 26\sigma - 61 = 0.$$

This is the same equation as before.

Root Locus sketching rules

- **Rule 7:** Imaginary axis crossings

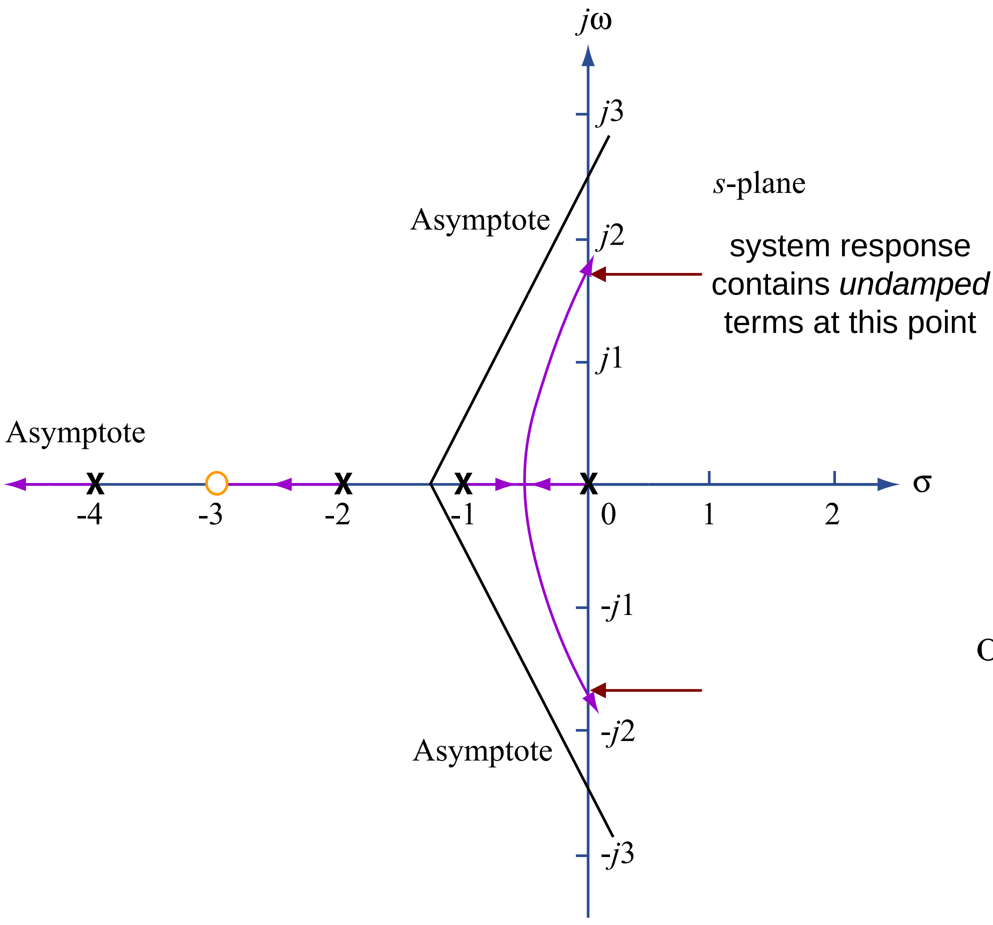


Figure by MIT OpenCourseWare.

If $s = j\omega$ is a closed–loop pole on the imaginary axis, then

$$KG(j\omega)H(j\omega) = -1 \qquad (2)$$

The real and imaginary parts of (2) provide us with a 2×2 system of equations, which we can solve for the two unknowns K and ω (*i.e.*, the critical gain beyond which the system goes unstable, and the oscillation frequency at the critical gain.)

Note: Nise suggests using the Ruth–Hurwitz criterion for the same purpose. Since we did not cover Ruth–Hurwitz, we present here an alternative but just as effective method.

Root Locus sketching rules

- **Rule 7:** Imaginary axis crossings

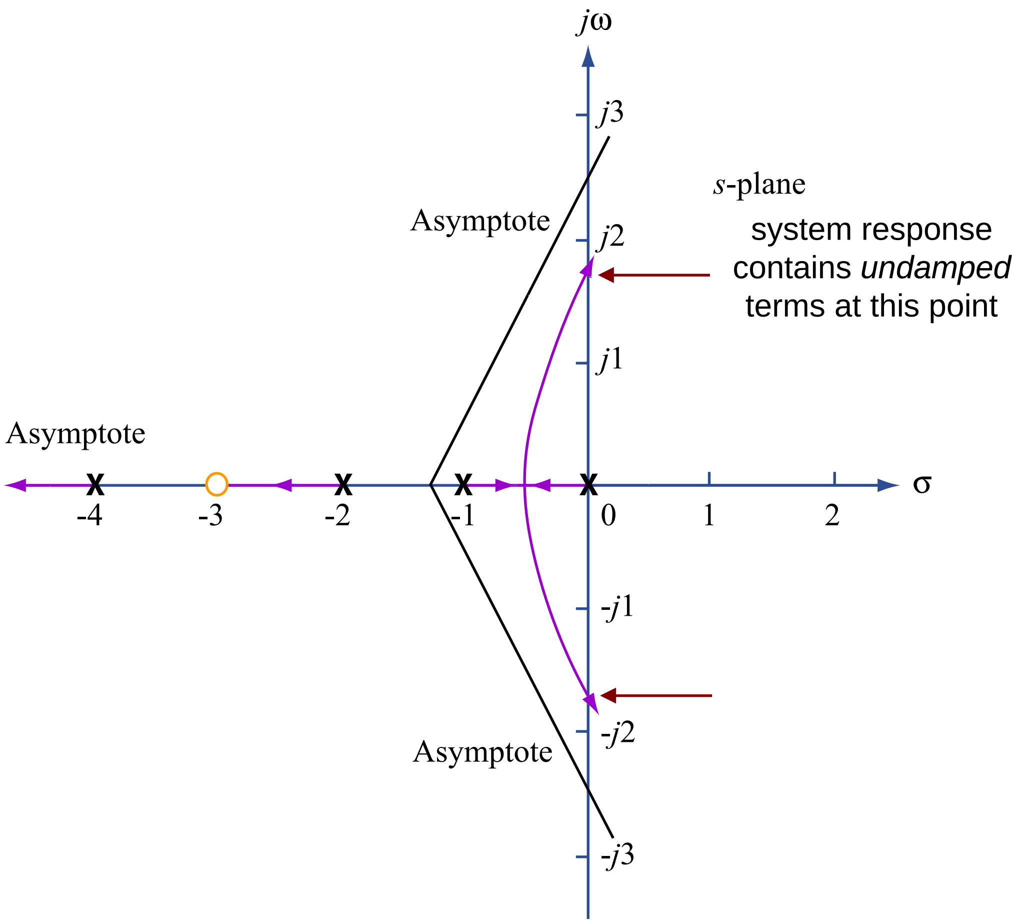


Figure by MIT OpenCourseWare.

In this example,

$$\begin{aligned} KG(s)H(s) &= \frac{K(s+3)}{s(s+1)(s+2)(s+4)} \\ &= \frac{Ks+3K}{s^4+7s^3+14s^2+8s} \Rightarrow \\ KG(j\omega)H(j\omega) &= \frac{jK\omega+3K}{\omega^4-j7\omega^3-14\omega^2+j8\omega}. \end{aligned}$$

Setting $KG(j\omega)H(j\omega)=-1$,

$$-\omega^4+j7\omega^3+14\omega^2-j(8+K)\omega-3K=0.$$

Separating real and imaginary parts,

$$\begin{cases} -\omega^4+14\omega^2-3K = 0, \\ 7\omega^3-(8+K)\omega = 0. \end{cases}$$

In the second equation, we can discard the trivial solution $\omega=0$. It then yields

$$\omega^2=\frac{8+K}{7}.$$

Substituting into the first equation,

$$\begin{aligned} -\left(\frac{8+K}{7}\right)^2+14\left(\frac{8+K}{7}\right)-3K &= 0 \Rightarrow \\ K^2+65K-720 &= 0. \end{aligned}$$

Of the two solutions $K=-74.65$, $K=9.65$ we can discard the negative one (negative feedback $\Rightarrow K>0$). Thus, $K=9.65$ and $\omega=\sqrt{(8+9.65)/7}=1.59$.