So far

- First-order systems (linear)
- Second-order systems (linear)

Today

- Higher-order systems (linear)
  - when can we approximate as second-order?
- Nonlinear systems
  - Review of cases we've seen
  - Linearization
    - Example: pendulum
  - DC motor nonlinearities
    - Example: load connected with gears; saturation, dead zone, backlash

The effect of multiple poles

$$C(s) = \frac{A}{s} + \frac{B\left(s+\zeta\omega_n\right)+C\omega_d}{\left(s+\zeta\omega_n\right)^2+\omega_d^2} + \frac{D}{s+\alpha_r}.$$

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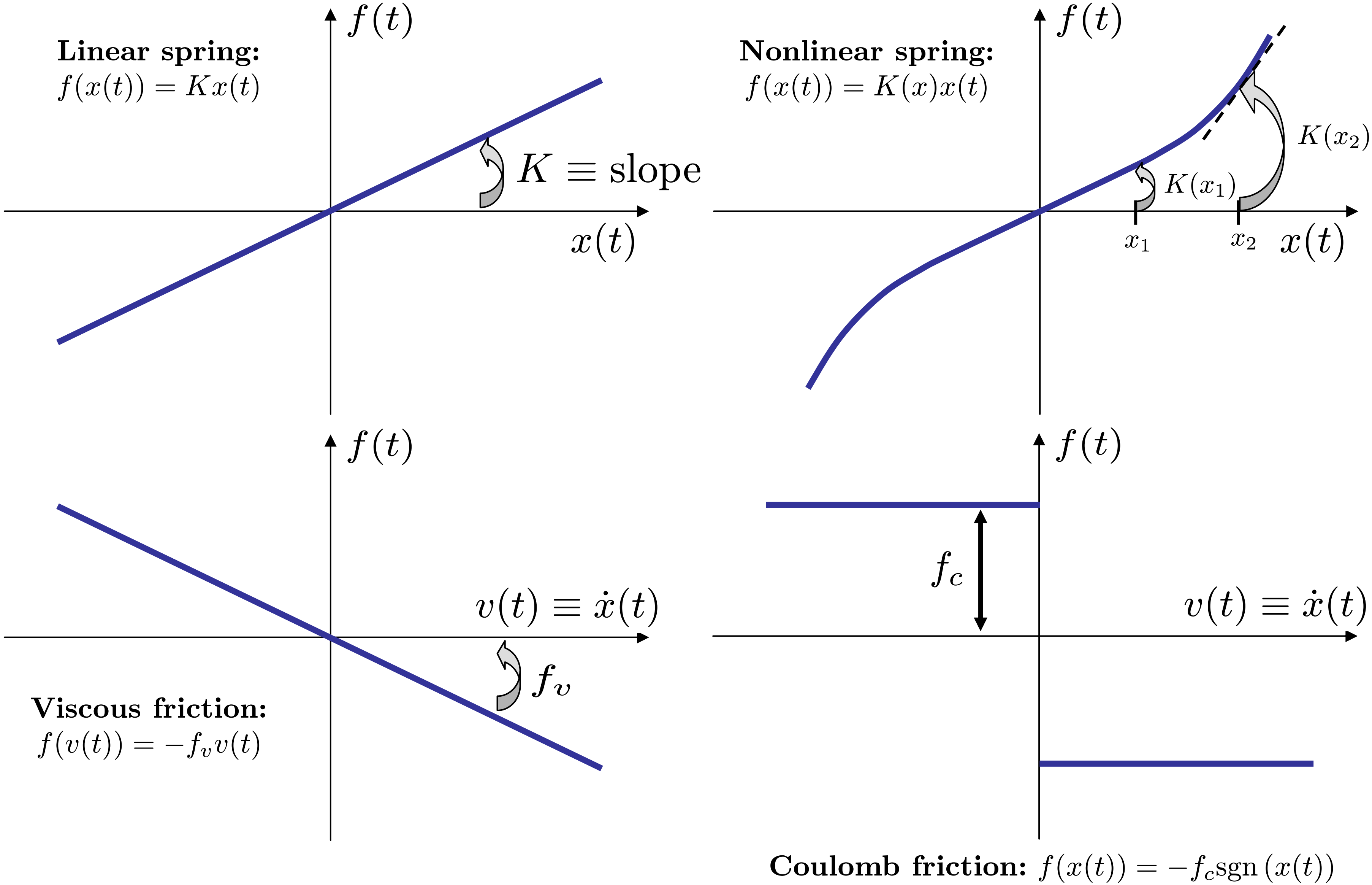
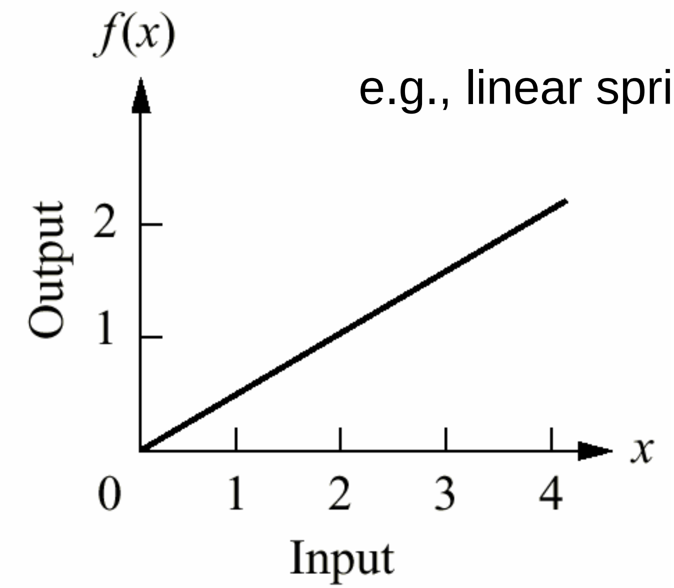
Please see: Fig. 4.23 and 4.24 in Nise, Norman S. *Control Systems Engineering*. 4th ed. Hoboken, NJ: John Wiley, 2004.

Linear vs Nonlinear *(revisited)*



Linear vs Nonlinear *(revisited)*



Linear system: $f(x(t)) = kx(t)$

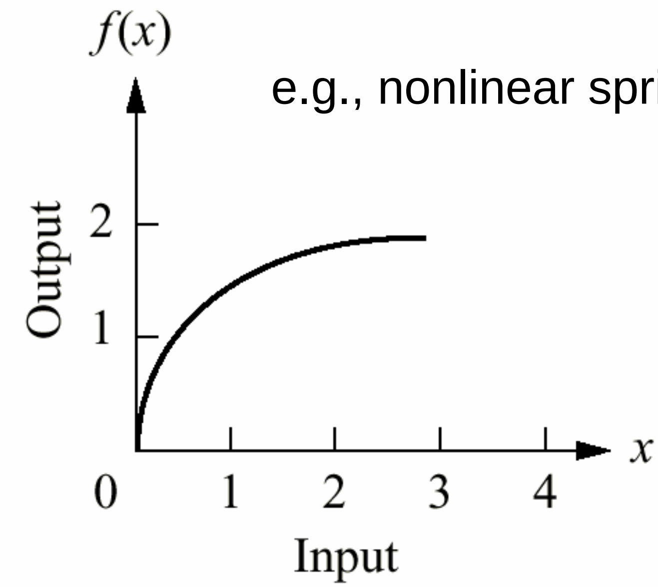
Consider linear combination of inputs

$$a_1 x_1(t) + a_2 x_2(t)$$

Then output is **same**
linear combination of outputs

$$f\left(a_1 x_1(t) + a_2 x_2(t)\right) = k\left(a_1 x_1(t) + a_2 x_2(t)\right) =$$

$$a_1\left(kx_1(t)\right) + a_2\left(kx_2(t)\right) = a_1 f\left(x_1(t)\right) + a_2 f\left(x_2(t)\right).$$



Nonlinear system: $f(x(t)) \neq kx(t)$
e.g., $f(x(t)) = k\sqrt{x(t)}$.

Consider same linear combination of inputs

$$a_1 x_1(t) + a_2 x_2(t)$$

Then output is **not the same**
linear combination of outputs

$$f\left(a_1 x_1(t) + a_2 x_2(t)\right) = k\sqrt{\left(a_1 x_1(t) + a_2 x_2(t)\right)} \neq$$

$$a_1\sqrt{kx_1(t)} + a_2\sqrt{kx_2(t)} = a_1 f\left(x_1(t)\right) + a_2 f\left(x_2(t)\right).$$

Linearization

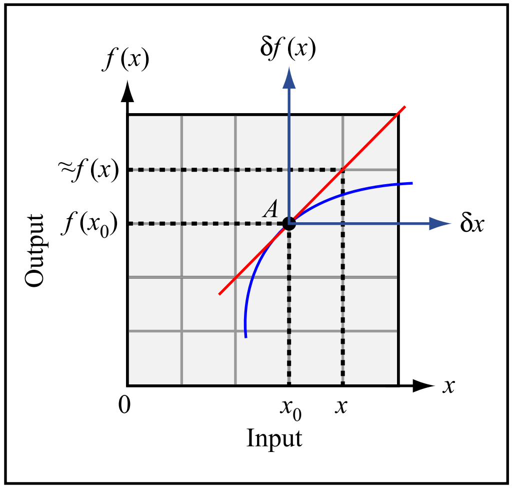


Figure by MIT OpenCourseWare.

$$f(x) \approx f(x_0) + m_a\,(x - x_0)$$

where

$$m_a = \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=x_0}.$$

Example

Linearize $f(x) = 5\cos x$ near $x = \pi/2$.

Answer: We have $f(\pi/2) = 0$, $m_a = -5$, so

$$f(x) \approx -5\left(x - \frac{\pi}{2}\right) \qquad (x \approx \pi/2)$$

Linearizing *systems* : the pendulum

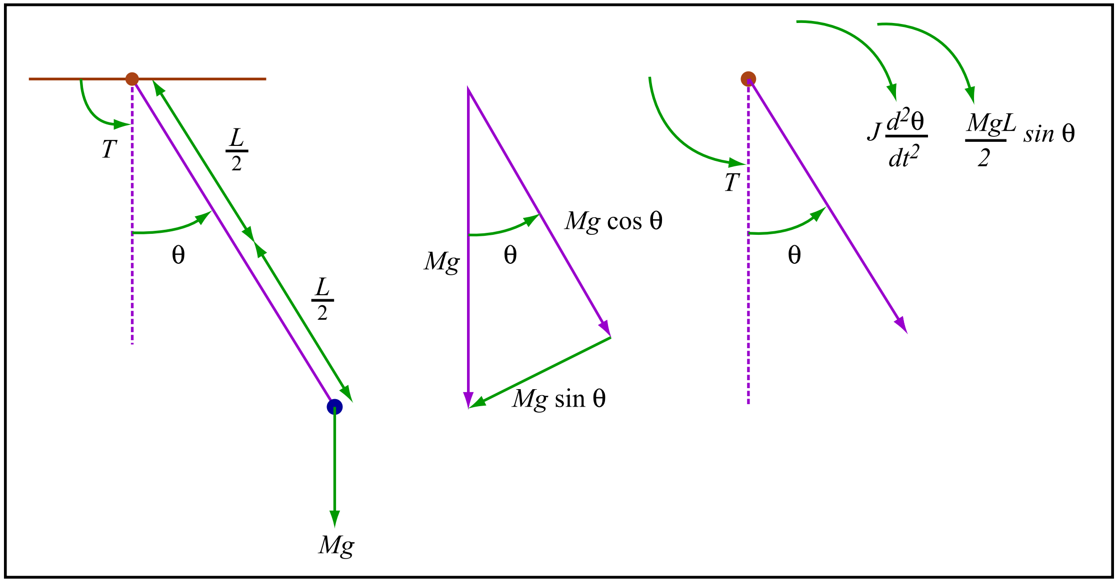


Figure by MIT OpenCourseWare.

The equation of motion is found as

$$J\ddot{\theta} + \frac{MgL}{2}\sin\theta = T. \qquad \text{(We \textbf{cannot} Laplace transform!)}$$

For small angles $\theta \approx 0$, we have

$$\sin\theta \approx \left.\frac{\mathrm{d}\sin\theta}{\mathrm{d}\theta}\right|_{\theta=0} \times \theta = \cos\theta|_{\theta=0} \times \theta = 1 \times \theta = \theta.$$

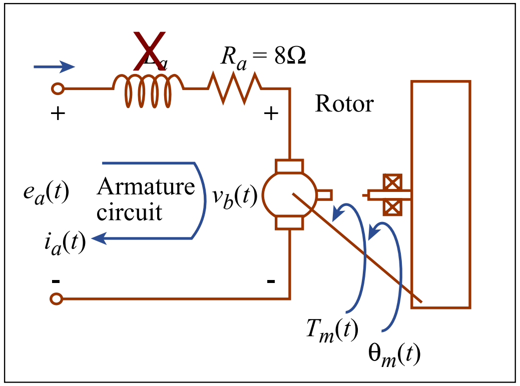
Therefore, the **linearized** equation of motion is

$$J\ddot{\theta} + \frac{MgL}{2}\theta = T \Rightarrow Js^2\Theta(s) + \frac{MgL}{2}\Theta(s) = T(s).$$

Some common nonlinearities

- Saturation
- Dead zone
- Backlash

Case study: motor with gear load



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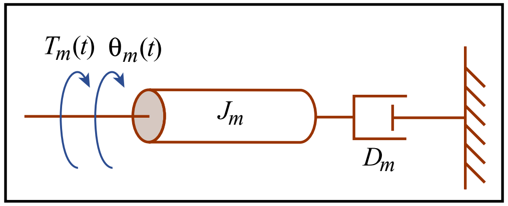
Motor circuit

$$K_m = 0.5\text{N}\cdot\text{m}/\text{A}$$
$$K_b = 0.5\text{V}\cdot\text{sec}/\text{rad}$$
$$K_b = 0.5\text{V}\cdot\text{sec}/\text{rad}$$

Motor loading



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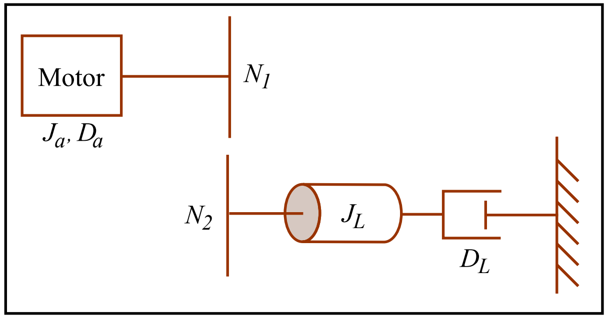
Motor loaded with gears



$$N_1 : N_2 = 25 : 250$$
$$J_a = 0.02\text{kg}\cdot\text{m}^2$$
$$J_L = 1\text{kg}\cdot\text{m}^2$$
$$D_a = 0.01\text{kg}\cdot\text{m}^2/\text{sec}$$
$$D_L = 1\text{kg}\cdot\text{m}^2/\text{sec}$$

$$\frac{\Omega_o(s)}{V_s(s)} = \frac{0.2083}{s+1.71}; \qquad \frac{\Theta_o(s)}{V_s(s)} = \frac{0.2083}{s\,(s+1.71)}$$

Saturation

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Please see Fig. 4.29 in Nise, Norman S. *Control Systems Engineering*. 4th ed. Hoboken, NJ: John Wiley, 2004.

System TF

$$\frac{0.2083}{s+1.71}$$

Input **Step function**

$$u(t) \leftrightarrow \frac{1}{s}$$

Dead zone

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Please see: Fig. 4.30 in Nise, Norman S. *Control Systems Engineering.* 4th ed. Hoboken, NJ: John Wiley, 2004.

System TF

$$\frac{0.2083}{s\left(s+1.71\right)}$$

Input **Sinusoid**

$$5\sin\left(2\pi t\right)u(t)$$

Backlash

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Please see: Fig. 4.31 in Nise, Norman S. *Control Systems Engineering*. 4th ed. Hoboken, NJ: John Wiley, 2004.

System TF

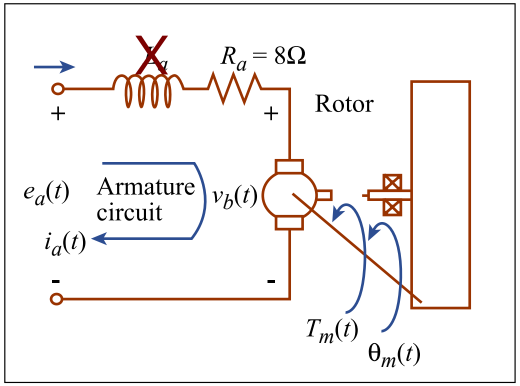
$$\frac{0.2083}{s\left(s+1.71\right)}$$

Input **Sinusoid**

$$5\sin\left(2\pi t\right)u(t)$$

Case study solution \1: electro-mechanical model

Motor circuit



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Recall DC motor equations
(in the Laplace domain)

$$\begin{aligned} T_m(s) &= K_m I_a(s); \\ V_b(s) &= K_b \Omega_m(s). \end{aligned}$$

Since $\omega_m(t) = \dot{\theta}_m(t) \Leftrightarrow \Omega_m(s) = s\Theta_m(s)$,
we can rewrite the second motor equation as

$$V_b(s) = sK_b\Theta_m(s).$$

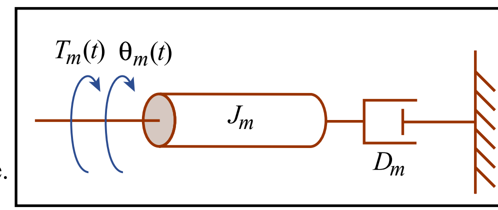
To find an equation relating the source $E_a(s)$
to the motor output angle $\Theta_m(s)$,
we must relate the source $E_a(s)$ to the torque $T_m(s)$,
and the torque $T_m(s)$ to the angle $\Theta_m(s)$.

KVL around the DC motor circuit loop
(neglecting the inductance L_a) yields

$$R_a I_a(s) + V_b(s) = E_a(s).$$

Substituting I_a, V_b from the motor equations,

$$R_a \frac{T_m(s)}{K_m} + K_b s\Theta_m(s) = E_a(s).$$



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Assume an equivalent load of inertia J_m and
damping D_m, subject to the motor's torque $T_m(s)$.
Torque balance on this system yields

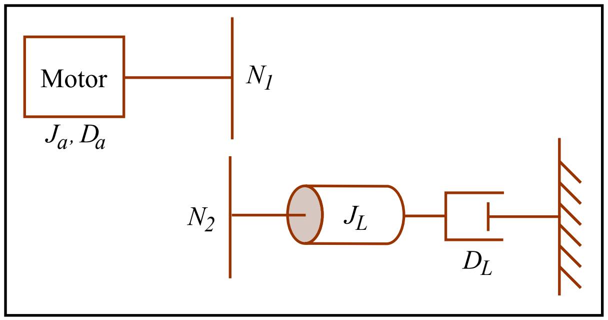
$$\begin{aligned} T_m(t) &= J_m\ddot{\theta}_m(t) + D_m\dot{\theta}_m(t) \Rightarrow \\ T_m(s) &= J_m s^2\Theta_m(s) + D_m s\Theta_m(s). \end{aligned}$$

Substituting into the electrical equation from above,

$$\left[\frac{R_a}{K_m}\left(J_m s^2 + D_m s\right) + K_b s\right]\Theta_m(s) = E_a(s)$$

Case study solution \2: load model

Motor loaded with gears



$$N_1 : N_2 = 25 : 250$$
$$J_a = 0.02\text{kg}\cdot\text{m}^2$$
$$J_L = 1\text{kg}\cdot\text{m}^2$$
$$D_a = 0.01\text{kg}\cdot\text{m}^2/\text{sec}$$
$$D_L = 1\text{kg}\cdot\text{m}^2/\text{sec}$$

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The last equation from the previous page can be rearranged and rewritten as a transfer function

$$\frac{\Theta_m(s)}{E_a(s)} = \frac{\dfrac{K_t}{R_a J_m}}{s\left[s + \dfrac{1}{J_m}\left(D_m + \dfrac{K_m K_b}{R_a}\right)\right]}$$

We must now relate the equivalent loads J_m, D_m to the actual load that consists of the motor's own armature inertia J_a and compliance D_a, as well as the load's inertia J_L and compliance D_L. The load is connected to the motor via a gear–pair of ratio $N_1 : N_2$.

Recalling the gear loading relationships from lecture 2,

$$J_m = J_a + \left(\frac{N_1}{N_2}\right)^2 J_L; \quad D_m = D_a + \left(\frac{N_1}{N_2}\right)^2 D_L.$$

Also note that the load shaft angle is related to the motor shaft angle via

$$\Theta_o(s) = \left(\frac{N_1}{N_2}\right)\Theta_m(s).$$

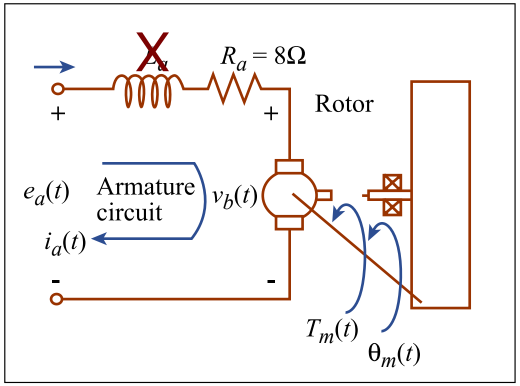
After substituting the numerical values, we find that the transfer function is of the form

$$\frac{\Theta_m(s)}{E_a(s)} = \frac{K}{s\,(s+p)} = \frac{0.2083}{s\,(s+1.71)},$$

where the system **gain** $K = 0.2083\text{rad}/\left(\text{V}\cdot\text{sec}^2\right)$ and the system **pole** $p = -1.71\text{Hz}$.
The extra s in the transfer functions' denominator indicates that **the system includes an integrator.**
We can also obtain the TF for $\Omega_o(s) = s\Theta_o(s)$

$$\frac{\Omega_m(s)}{E_a(s)} = \frac{K}{s+p} = \frac{0.2083}{s+1.71}.$$

Case study solution \3: torque-speed curve



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Please see: Fig. 2.38 in Nise, Norman S. *Control Systems Engineering*. 4th ed. Hoboken, NJ: John Wiley, 2004.

Recall the relationship we obtained from KVL and the motor equations,

$$\frac{R_a}{K_m} T_m(s) + K_b s \Theta_m(s) = E_a(s) \Rightarrow$$

$$\frac{R_a}{K_m} T_m(s) + K_b \Omega_m(s) = E_a(s).$$

Inverse Laplace–transforming,

$$\frac{R_a}{K_m} T_m(t) + K_b \omega(t) = e_a(t) \Rightarrow$$

$$T_m = -\frac{K_b K_m}{R_a} \omega_m + \frac{K_m}{R_a} e_a.$$

This relationship on the $\omega_m - T_m$ plane represents a straight line called **torque–speed curve**, with slope $-K_b K_m / R_a$ and offset K_m / R_a.

$$\omega_m = 0 \Leftrightarrow T_{stall} = \frac{K_m}{R_a} e_a \quad \text{Stall torque;}$$

$$T_m = 0 \Leftrightarrow \omega_{no-load} = \frac{e_a}{K_b} \quad \text{No–load speed.}$$