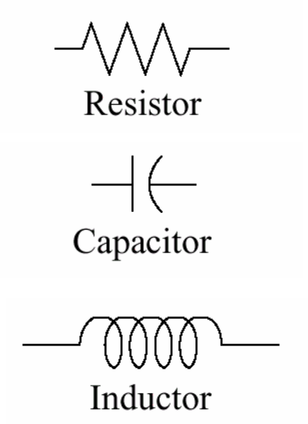
Summary from previous lecture

- Electrical dynamical variables and elements

$$\text{Charge } q(t),\ Q(s).$$
$$\text{Current } i(t) = \dot{q}(t),\ I(s) = sQ(s).$$
$$\text{Voltage } v(t),\ V(s) = Z(s)I(s).$$

$$\text{Resistor } v(t) = Ri(t),\ Z_R(s) = R.$$
$$\text{Capacitor } i(t) = C\dot{v}(t),\ Z_C(s) = 1/Cs.$$
$$\text{Inductor } v(t) = L\mathrm{d}i(t)/\mathrm{d}t,\ Z_L(s) = Ls.$$



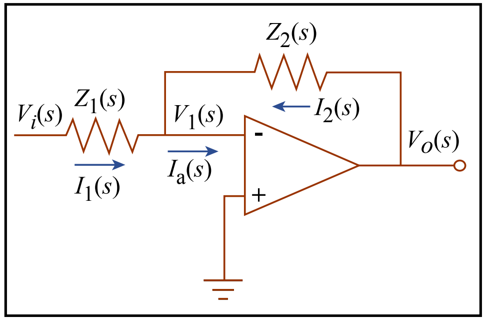
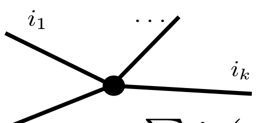


Figure by MIT OpenCourseWare.

Op–Amp in feedback configuration:

$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)}.$$

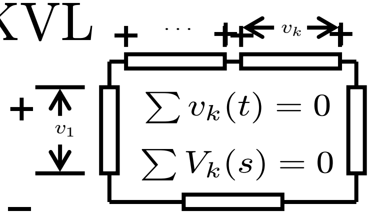
- Electrical networks



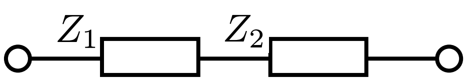
KCL

$$\sum i_k(t) = 0$$
$$\sum I_k(s) = 0$$

KVL

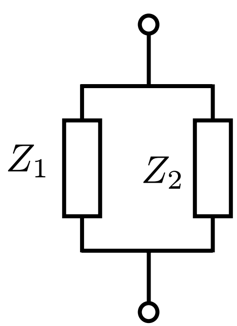


$$\sum v_k(t) = 0$$
$$\sum V_k(s) = 0$$



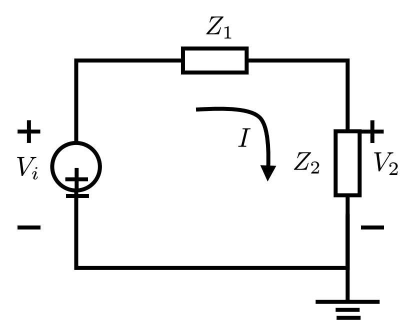
Impedances in series

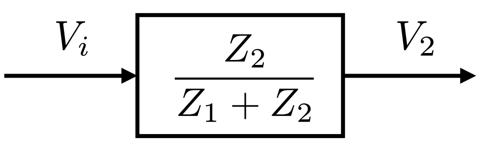
$$Z = Z_1 + Z_2 \quad \frac{1}{G} = \frac{1}{G_1} + \frac{1}{G_2}.$$



Impedances in parallel

$$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2} \quad G = G_1 + G_2.$$



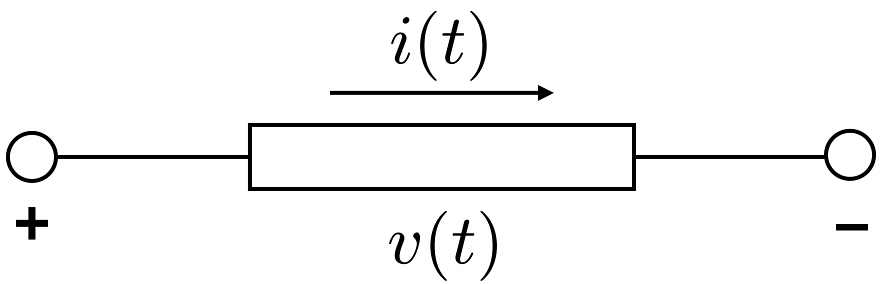


Voltage divider

Goals for today

- The DC motor:
  - basic physics & modeling,
  - equation of motion,
  - transfer function.
- **Next week:**
  - Properties of 1^{st} and 2^{nd} order systems
  - Working in the *s*-domain: poles, zeros, and their significance

Power dissipation in electrical systems



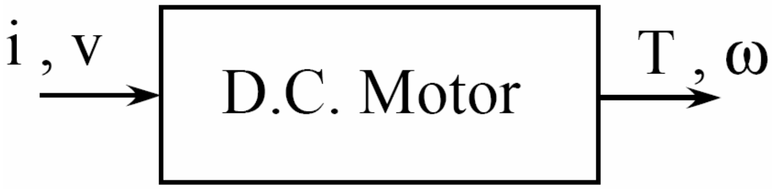
Instantaneous power dissipation

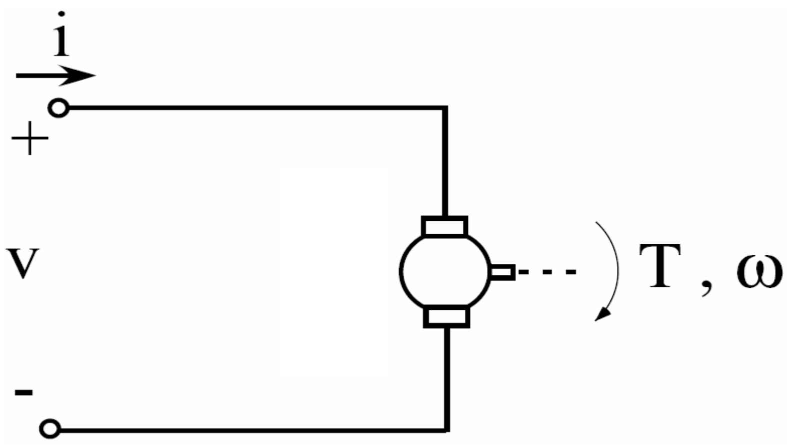
$$P(t) = i(t) \cdot v(t).$$

Unit of power: 1 Watt=1 A $\cdot$ 1 V.

<u>NOTE:</u> $P(s) \neq I(s) \cdot V(s)$. **Why?**

DC Motor as a system





$$P_{in} = P_{out}$$

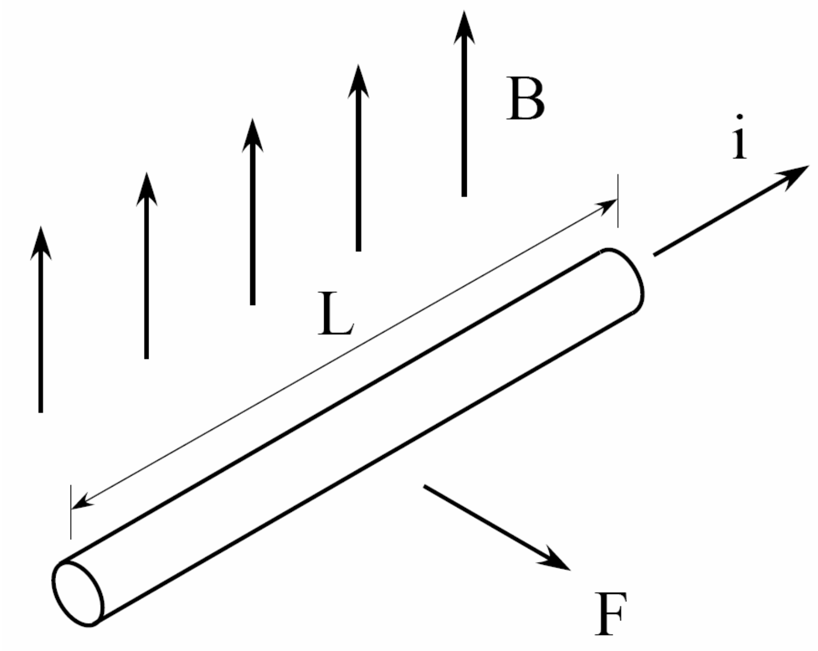
$$i(t) \ast v(t) = T(t) \ast \omega(t)$$

Transducer:

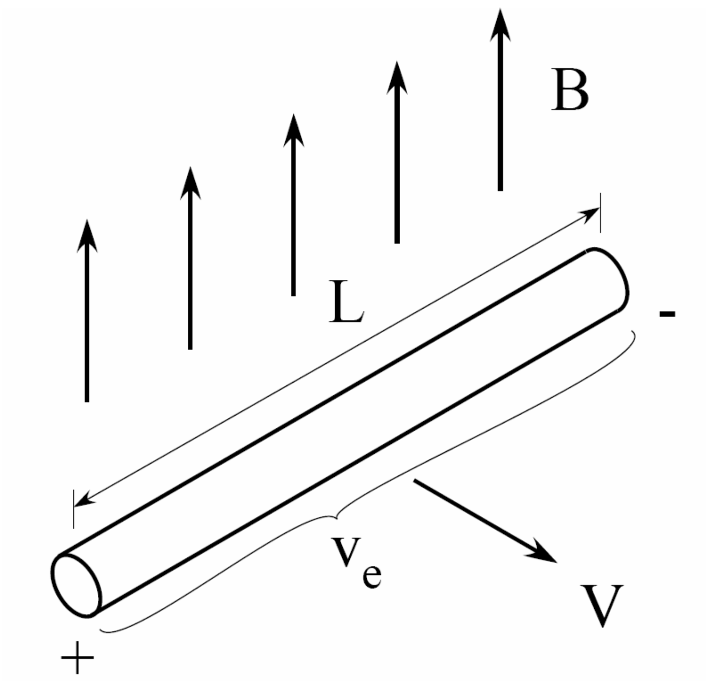
Converts energy from one domain (electrical)
to another (mechanical)

Physical laws applicable to the DC motor

Lorentz law:
magnetic field applies force to a current
(Lorentz force)



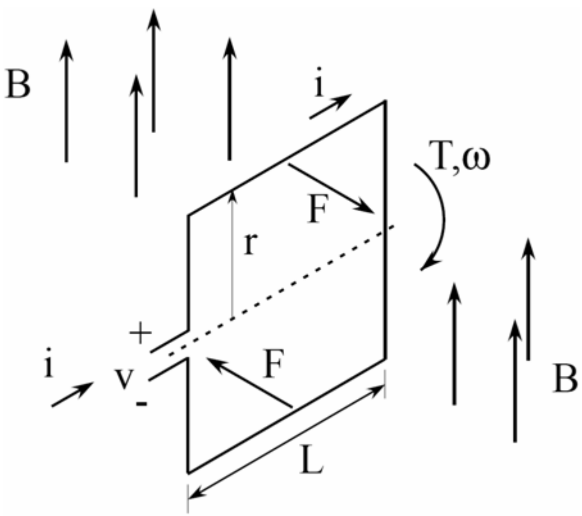
Faraday law:
moving in a magnetic field results
in potential (back EMF)

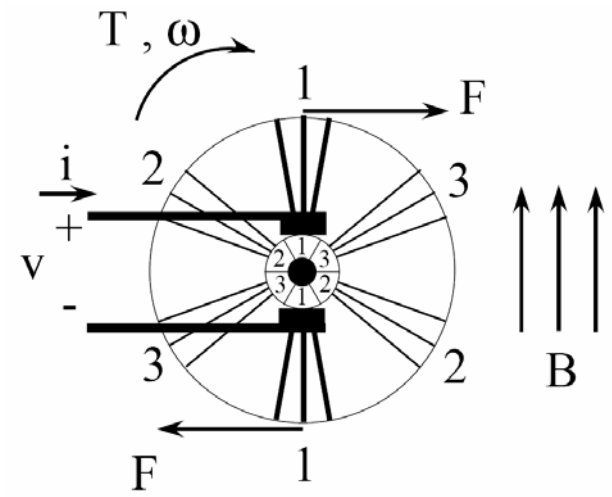


$$F = (\mathbf{i} \times \mathbf{B}) \cdot l = iBl \qquad (\mathbf{i} \perp \mathbf{B})$$

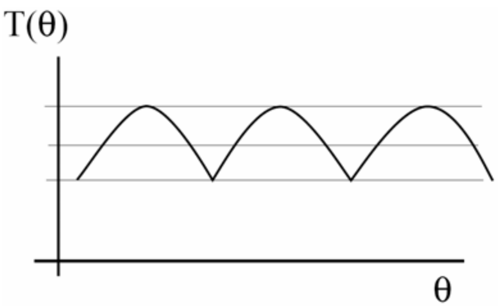
$$v_e = \mathbf{V} \times \mathbf{B} \cdot l = V B\, l \qquad (\mathbf{V} \perp \mathbf{B})$$

DC motor: principle and simplified equations of motion





multiple windings *N*:
continuity of torque



$$T = 2Fr = 2(iBNl)r \qquad \text{(Lorentz law)}$$

$$v_e = 2VBNl = 2(\omega r)BNl \qquad \text{(Faraday law)}$$

or

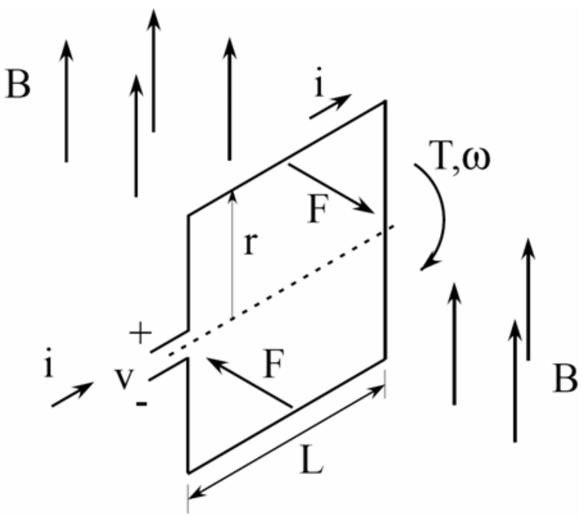
$$T = K_m i$$

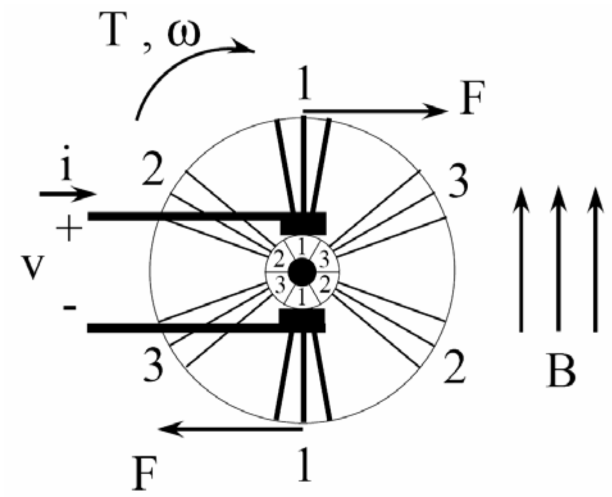
$$v_e = K_v \omega$$

where

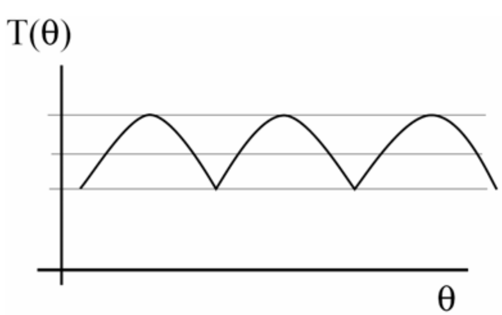
- $K_m \equiv 2BNlr$ torque constant
- $K_v \equiv 2BNlr$ back-emf constant

DC motor: equations of motion in matrix form





multiple windings *N*:
continuity of torque

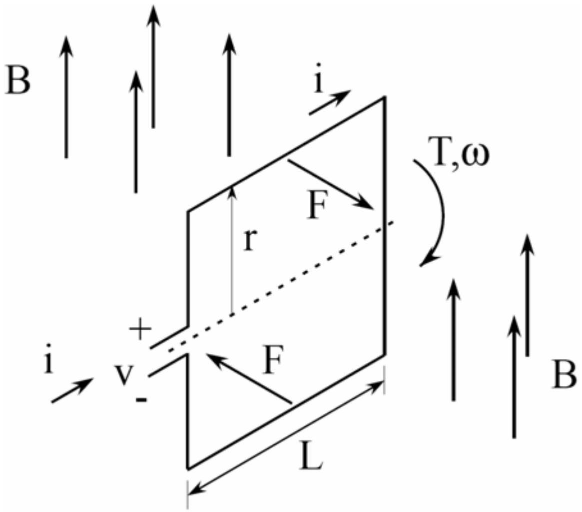


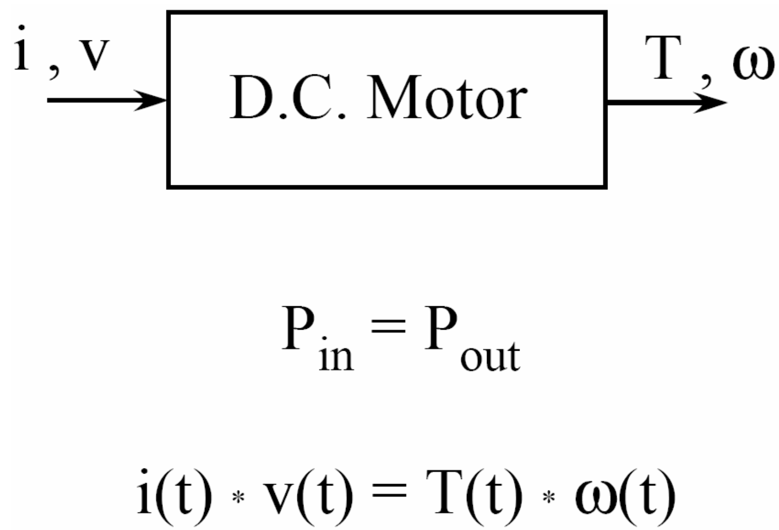
$$\begin{bmatrix} v_e \\ i \end{bmatrix} = \begin{bmatrix} 2BNlr & 0 \\ 0 & \frac{1}{2BNlr} \end{bmatrix} \begin{bmatrix} \omega \\ T \end{bmatrix}$$

or

$$\begin{bmatrix} v_e \\ i \end{bmatrix} = \begin{bmatrix} K_v & 0 \\ 0 & \frac{1}{K_m} \end{bmatrix} \begin{bmatrix} \omega \\ T \end{bmatrix}$$

DC motor: why is $K_m=K_v$?





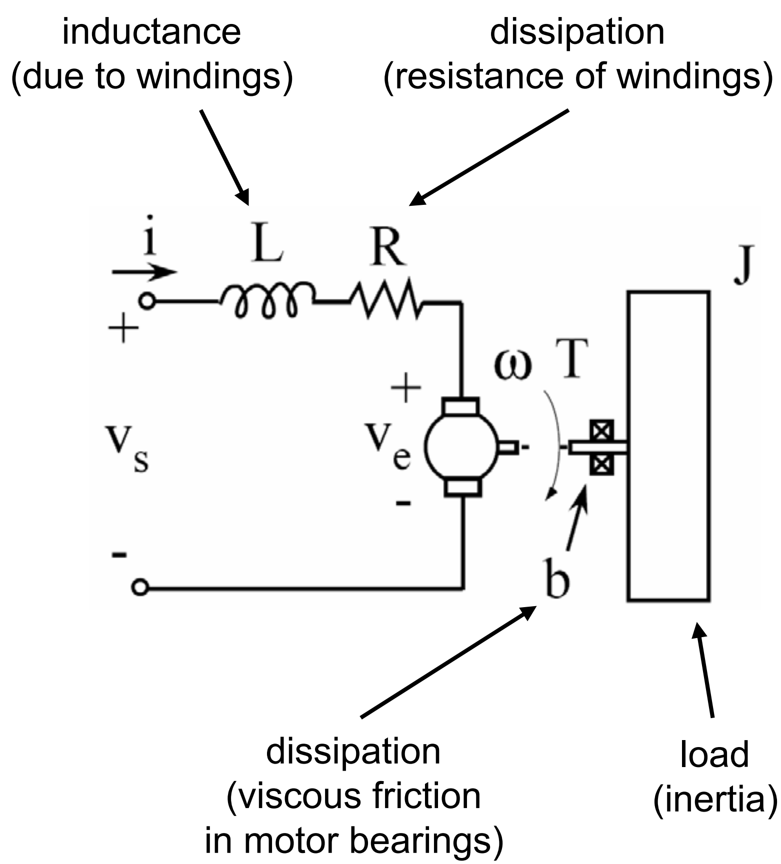
$$P_{in} = P_{out} \qquad \text{(power conservation)}$$

$$\Rightarrow i v_e = T\omega$$

$$\Rightarrow K_v i\omega = K_m i\omega$$

$$\Rightarrow K_v = K_m \qquad \text{QED.}$$

DC motor with mechanical load and realistic electrical properties (*R*, *L*)



Equation of motion – Electrical

$$\text{KCL:} \quad v_s - v_L - v_R - v_e = 0$$

$$\Rightarrow v_s - L\frac{di}{dt} - Ri - K_v\omega = 0$$

Equation of motion – Mechanical

$$\text{Torque Balance:} \quad T = T_b + T_J$$

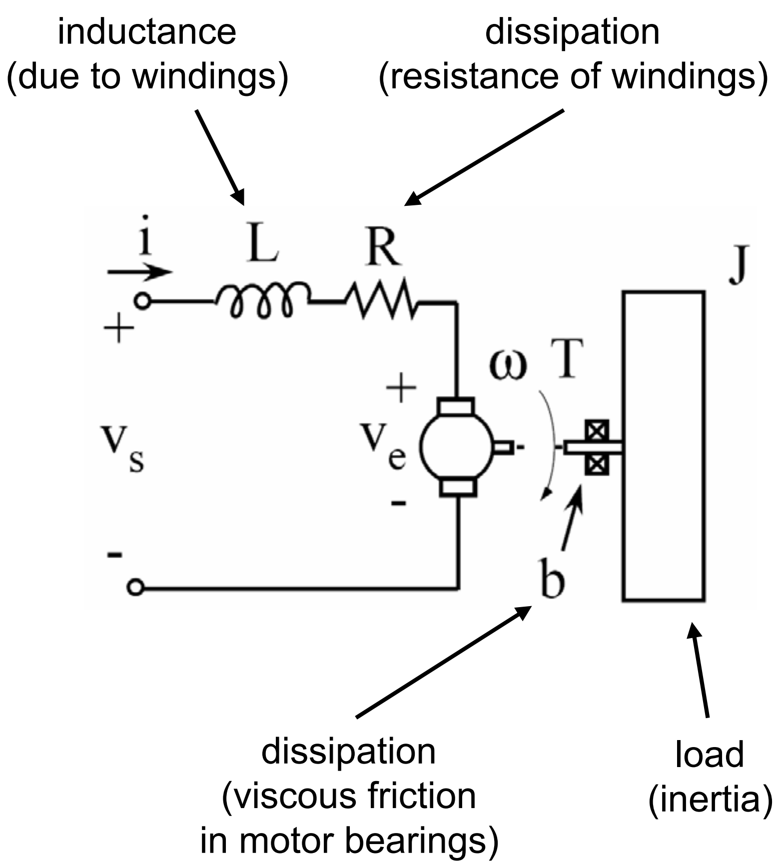
$$\Rightarrow K_m i - b\omega = J\frac{d\omega}{dt}$$

Combined equations of motion

$$L\frac{di}{dt} + Ri + K_v\omega = v_s$$

$$J\frac{d\omega}{dt} + b\omega = K_m i$$

DC motor with mechanical load and realistic electrical properties (*R*, *L*)



Equation of motion – Electrical

$$\text{KCL:} \quad V_s(s) - V_L(s) - V_R(s) - V_e(s) = 0$$

$$V_s(s) - LsI(s) - RI(s) - K_v\Omega(s) = 0$$

Equation of motion – Mechanical

$$\text{Torque Balance:} \quad T(s) = T_b(s) + T_J(s)$$

$$K_m I(s) - b\Omega(s) = Js\Omega(s)$$

Combined equations of motion

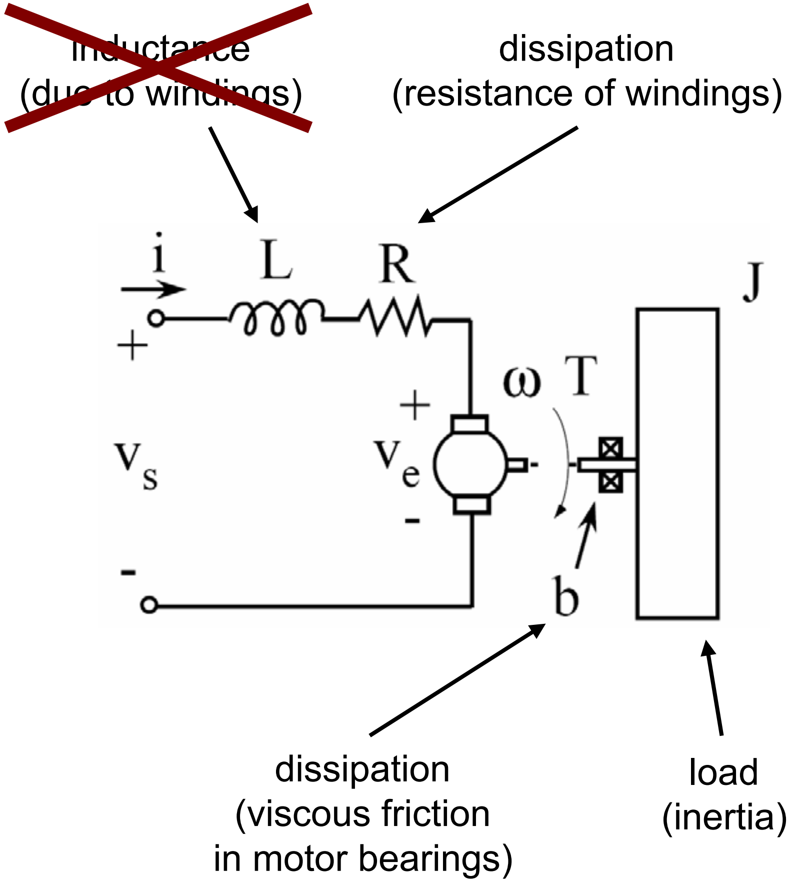
$$LsI(s) + RI(s) + K_v\Omega(s) = V_s(s)$$

$$Js\Omega(s) + b\Omega(s) = K_m I(s)$$

$$\Rightarrow \left[(Ls+R)\left(\frac{Js+b}{K_m}\right) + K_v\right]\Omega(s) = V_s(s)$$

$$\Rightarrow \left[\frac{LJ}{R}s^2 + \left(\frac{Lb}{R} + J\right)s + \left(b + \frac{K_m K_v}{R}\right)\right]\Omega(s) = \frac{K_m}{R}V_s(s)$$

DC motor with mechanical load and realistic electrical properties (*R*, ~~*L*~~)



Neglecting the impedance

$$L \approx 0$$

$$\Rightarrow \left[Js + \left(b + \frac{K_m K_v}{R} \right) \right] \Omega(s) = \frac{K_m}{R} V_s(s)$$

This is our familiar 1^{st}–order system!

If we are given step input $v_s(t) = V_0 u(t)$
$\Rightarrow$ we already know the step response

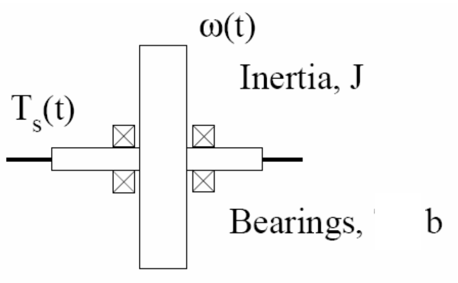
$$\omega(t) = \frac{K_m}{R} V_0 \left(1 - \mathrm{e}^{-t/\tau} \right) u(t),$$

where now the time constant is

$$\tau = \frac{J}{\left(b + \dfrac{K_m K_v}{R} \right)}.$$

Review: step response of 1st order systems we've seen

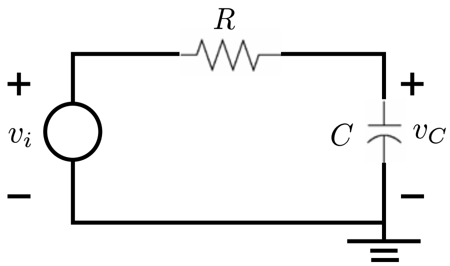
- Inertia with bearings (viscous friction)



Step input $T_s(t) = T_0 u(t) \Rightarrow$ Step response

$$\omega(t) = \frac{T_0}{b}\left(1 - \mathrm{e}^{-t/\tau}\right), \quad \text{where} \quad \tau = \frac{J}{b}.$$

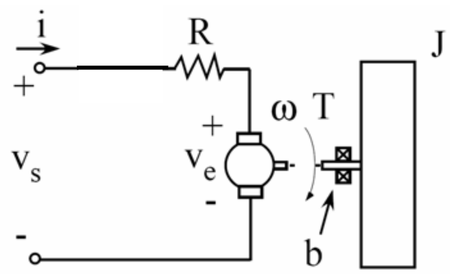
- RC circuit (charging of a capacitor)



Step input $v_i(t) = V_0 u(t) \Rightarrow$ Step response

$$v_C(t) = V_0\left(1 - \mathrm{e}^{-t/\tau}\right), \quad \text{where} \quad \tau = RC.$$

- DC motor with inertia load, bearings and negligible inductance



Step input $v_s(t) = V_0 u(t) \Rightarrow$ Step response

$$\omega(t) = \frac{K_m}{R} V_0 \left(1 - \mathrm{e}^{-t/\tau}\right),$$

$$\text{where} \quad \tau = \frac{J}{\left(b + \dfrac{K_m K_v}{R}\right)}.$$