Summary from previous lecture

- Laplace transform

$$\mathcal{L}\left[f(t)\right] \equiv F(s) = \int_{0-}^{+\infty} f(t)\mathrm{e}^{-st}\mathrm{d}t.$$

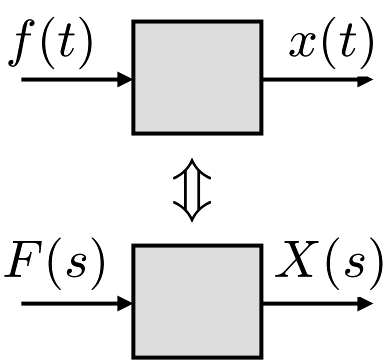
$$\mathcal{L}\left[\dot{f}(t)\right] = sF(s) - f(0-).$$

$$\mathcal{L}\left[u(t)\right] \equiv U(s) = \frac{1}{s}.$$

$$\mathcal{L}\left[\int_{0-}^{t} f(\xi)\mathrm{d}\xi\right] = \frac{F(s)}{s}.$$

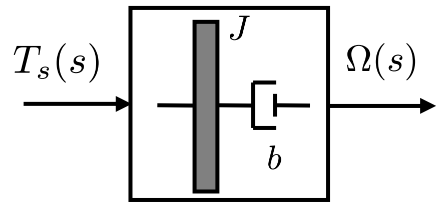
$$\mathcal{L}\left[\mathrm{e}^{-at}\right] = \frac{1}{s+a}.$$

- Transfer functions and impedances



$$\mathrm{TF}(s) = \frac{X(s)}{F(s)}$$

$$Z(s) = \frac{F(s)}{X(s)}$$



$$\mathrm{TF}(s) := \frac{\Omega(s)}{T_s(s)} = \frac{1}{Js+b}.$$

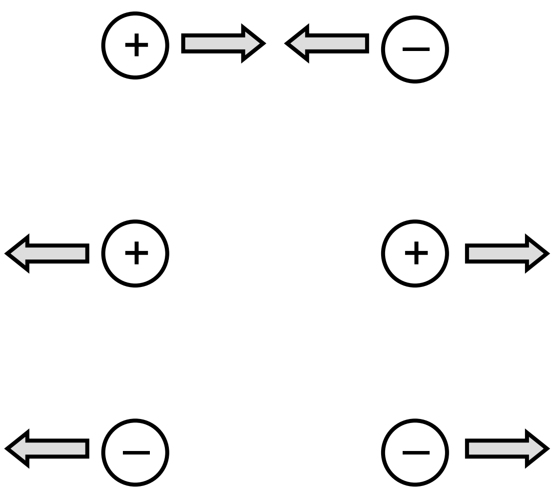
$$Z_J = Js; \quad Z_b = b; \quad \mathrm{TF}(s) = \frac{1}{Z_J + Z_b}$$

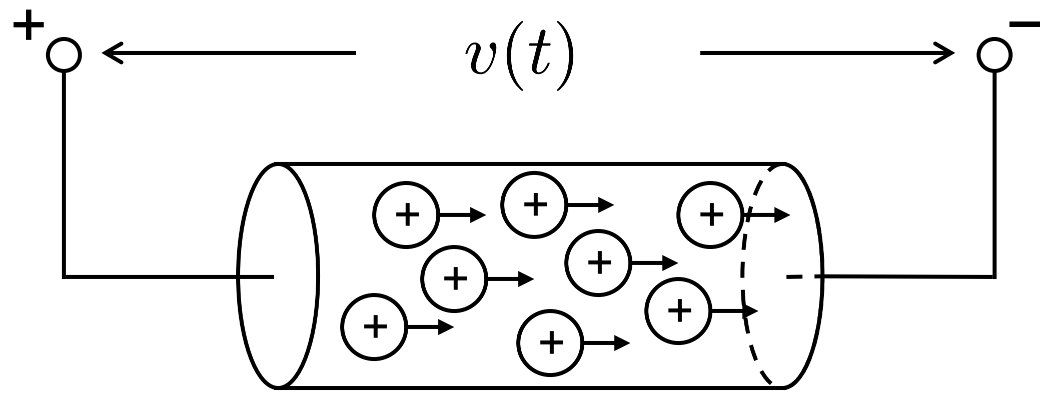
Goals for today

- Dynamical variables in electrical systems:
  - charge,
  - current,
  - voltage.
- Electrical elements:
  - resistors,
  - capacitors,
  - inductors,
  - amplifiers.
- Transfer Functions of electrical systems (networks)
- **Next lecture (Friday):**
  - DC motor (electro-mechanical element) model
  - DC motor Transfer Function

Electrical dynamical variables: charge, current, voltage

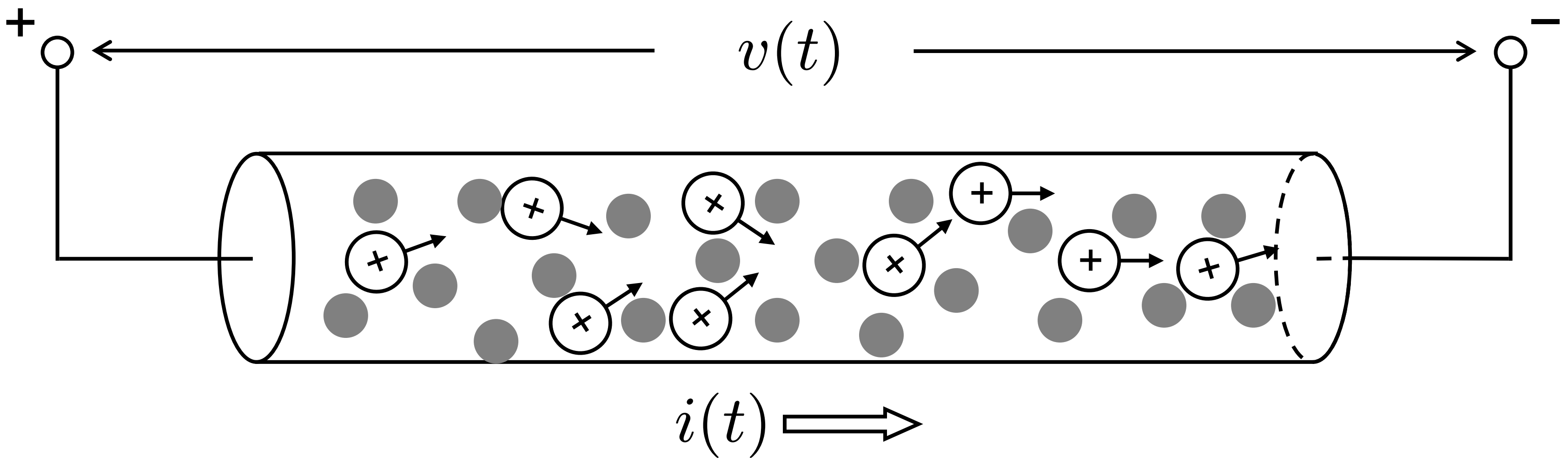
charge q	Coulomb [Cb]
charge flow $\equiv$ current $i(t)$	Ampére [A]=[Cb]/[sec]
voltage (*aka* potential) $v(t)$	Volt [V]





$$i(t) := \frac{\mathrm{d}q(t)}{\mathrm{d}t}$$

Electrical resistance

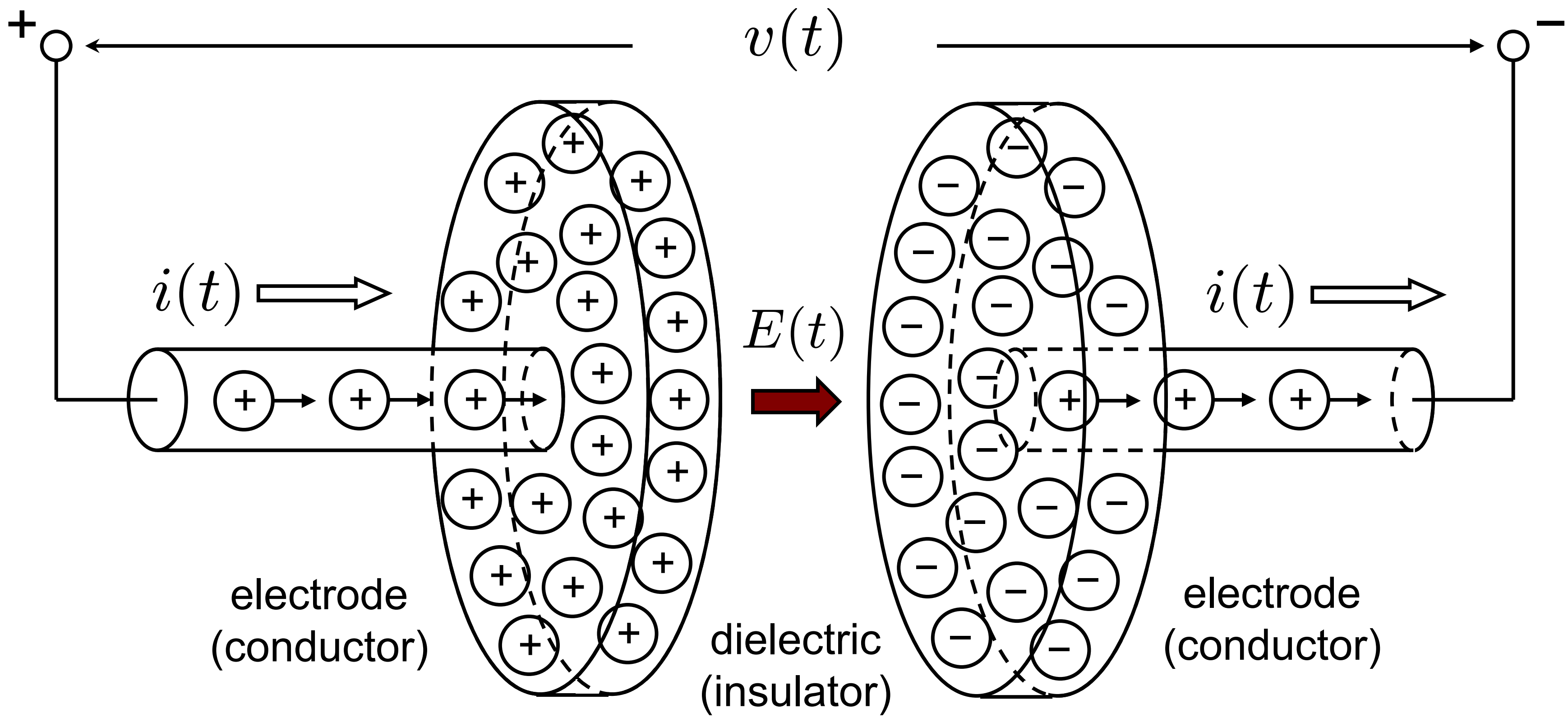


- Collisions between the mobile charges and the material fabric (ions, generally disordered) lead to <u>energy dissipation</u> (loss). As result, energy must be expended to generate current along the resistor; i.e., the current flow requires application of potential across the resistor

$$v(t) = Ri(t) \Rightarrow V(s) = RI(s) \Rightarrow \frac{V(s)}{I(s)} = R \equiv Z_R$$

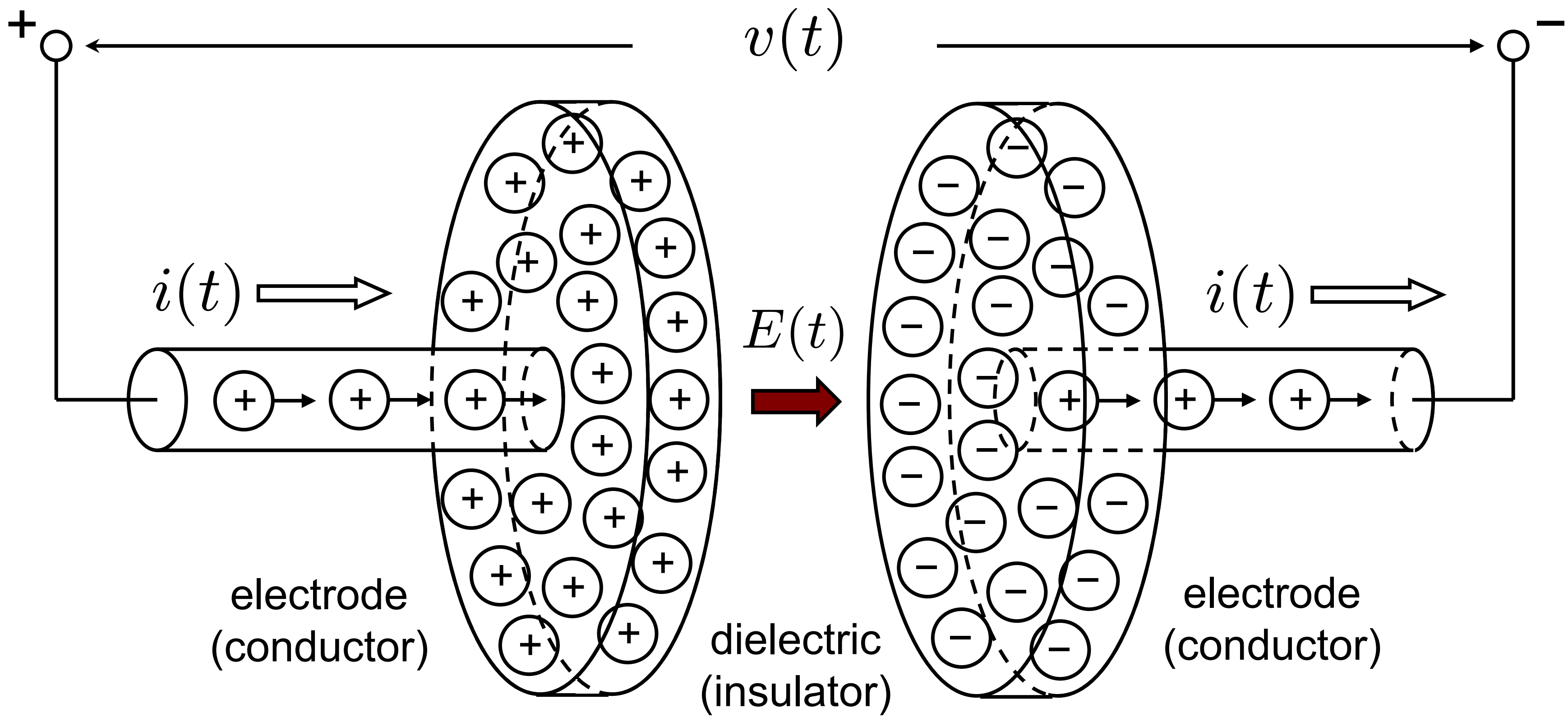
- The quantity Z_R=R is called the <u>resistance</u> (unit: Ohms, or Ω)
- The quantity G_R=1/R is called the <u>conductance</u> (unit: Mhos or Ω^{-1})

Capacitance



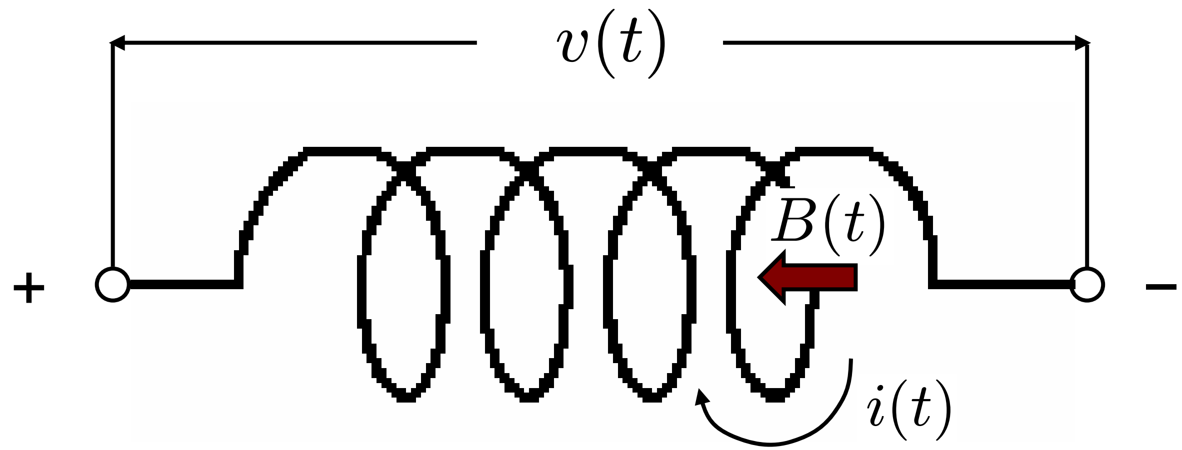
- Since similar charges repel, the potential *v* is necessary to prevent the charges from flowing away from the electrodes (discharge)
- Each change in potential $v(t+\Delta t)=v(t)+\Delta v$ results in change of the energy stored in the capacitor, in the form of charges moving to/away from the electrodes (↔ change in electric field)

Capacitance



- Capacitance *C*: $q(t) = Cv(t) \Rightarrow \frac{\mathrm{d}q(t)}{\mathrm{d}t} \equiv i(t) = C\frac{\mathrm{d}v(t)}{\mathrm{d}t}$
- in Laplace domain: $I(s) = CsV(s) \Rightarrow \frac{V(s)}{I(s)} \equiv Z_C(s) = \frac{1}{Cs}$

Inductance



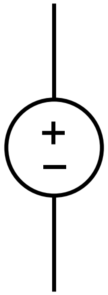
- Current flow *i* around a loop results in magnetic field *B* pointing normal to the loop plane. The magnetic field counteracts changes in current; therefore, to effect a change in current $i(t+\Delta t)=i(t)+\Delta i$ a potential *v* must be applied (*i.e.*, energy expended)
- Inductance *L*: $v(t) = L\,\dfrac{\mathrm{d}i(t)}{\mathrm{d}t}$
- in Laplace domain: $V(s) = LsI(s) \Rightarrow \dfrac{V(s)}{I(s)} \equiv Z_L(s) = Ls$

Summary: passive electrical elements; Sources

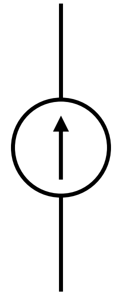
Table removed due to copyright restrictions.

Please see: Table 2.3 in Nise, Norman S. *Control Systems Engineering*. 4th ed. Hoboken, NJ: John Wiley, 2004.

Electrical inputs: voltage source, current source

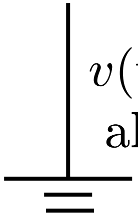


Voltage source:
$v(t)$ independent
of current through.



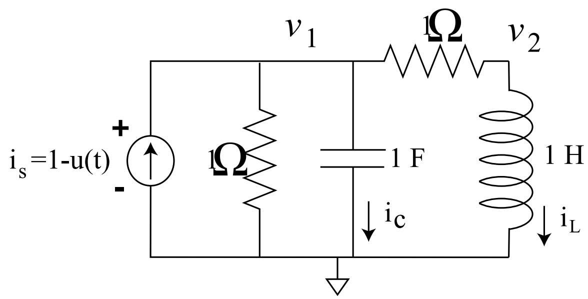
Current source:
$i(t)$ independent
of voltage across.

Ground:
potential reference



$v(t) = 0$
always

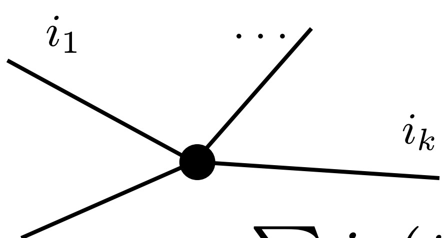
Combining electrical elements: networks



Courtesy of Prof. David Trumper. Used with permission.

Network analysis relies on two physical principles

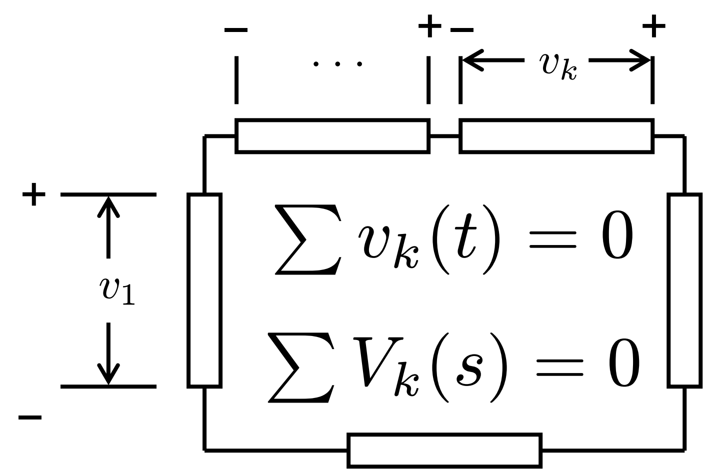
- Kirchhoff Current Law (KCL)
  - charge conservation



$$\sum i_k(t) = 0$$

$$\sum I_k(s) = 0$$

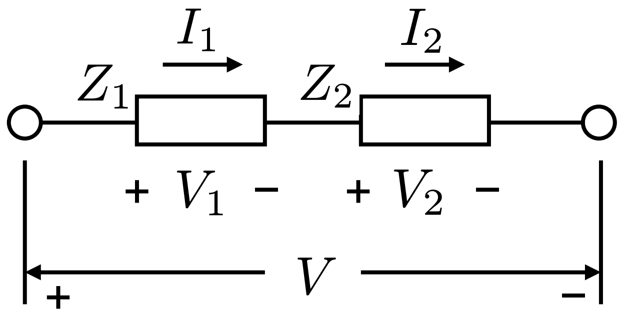
- Kirchhoff Voltage Law (KVL)
  - energy conservation



$$\sum v_k(t) = 0$$

$$\sum V_k(s) = 0$$

Impedances in series and in parallel



Impedances **in series**

KCL: $I_1 = I_2 \equiv I$.

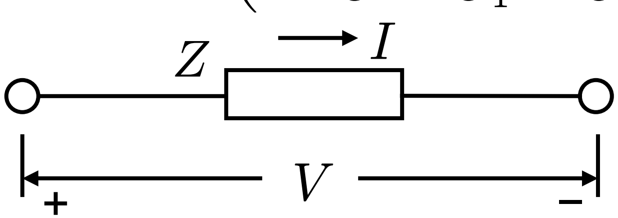
KVL: $V = V_1 + V_2$.

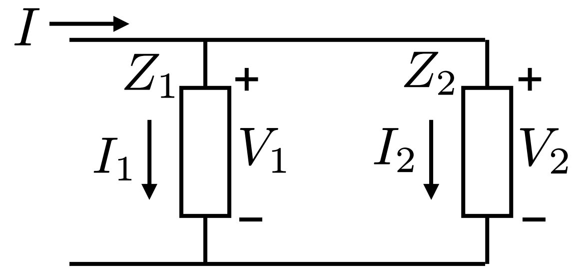
From definition of impedances:

$$Z_1 = \frac{V_1}{I_1}; \qquad Z_2 = \frac{V_2}{I_2}.$$

Therefore, equivalent circuit has

$$Z = Z_1 + Z_2 \left(\Leftrightarrow \frac{1}{G} = \frac{1}{G_1} + \frac{1}{G_2}. \right)$$





Impedances **in parallel**

KCL: $I = I_1 + I_2$.

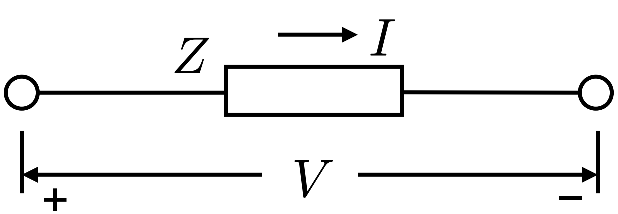
KVL: $V_1 + V_2 \equiv V$.

From definition of impedances:

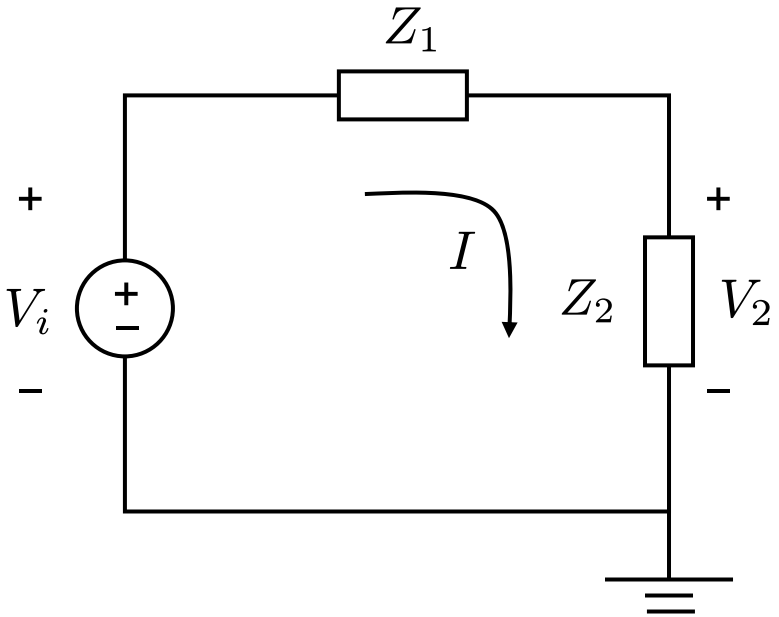
$$Z_1 = \frac{V_1}{I_1}; \qquad Z_2 = \frac{V_2}{I_2}.$$

Therefore, equivalent circuit has

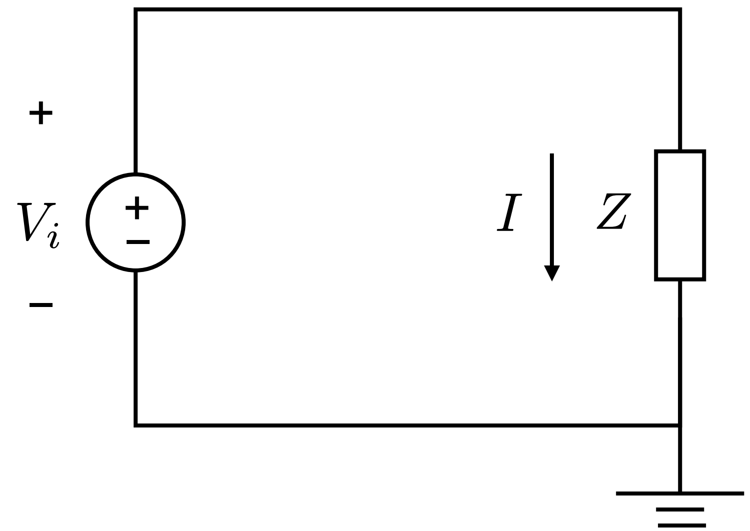
$$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2} \left(\Leftrightarrow G = G_1 + G_2. \right)$$



The voltage divider



Equivalent circuit for computing the current I.



Since the two impedances are in series, they combine to an equivalent impedance

$$Z = Z_1 + Z_2.$$

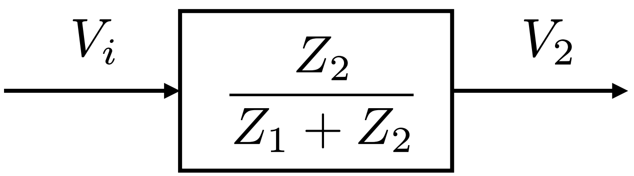
The current flowing through the combined impedance is

$$I = \frac{V}{Z}.$$

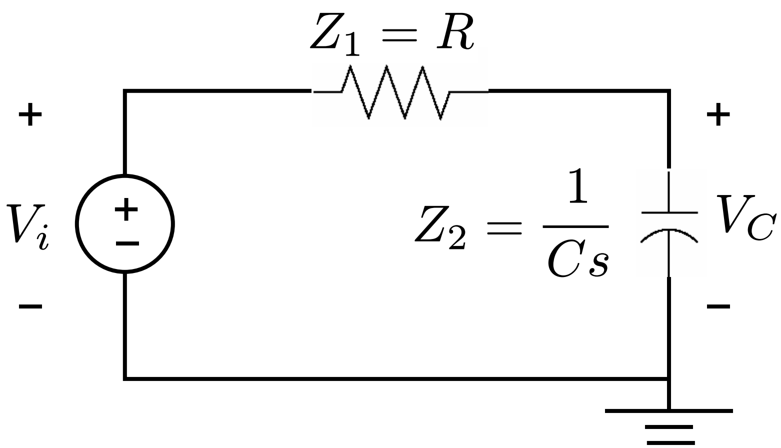
Therefore, the voltage drop across Z_2 is

$$V_2 = Z_2 I = Z_2 \frac{V}{Z} \Rightarrow \frac{V_2}{V_i} = \frac{Z_2}{Z_1 + Z_2}.$$

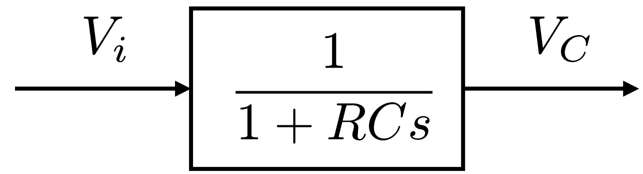
Block diagram & Transfer Function



Example: the *RC* circuit



Block diagram & Transfer Function



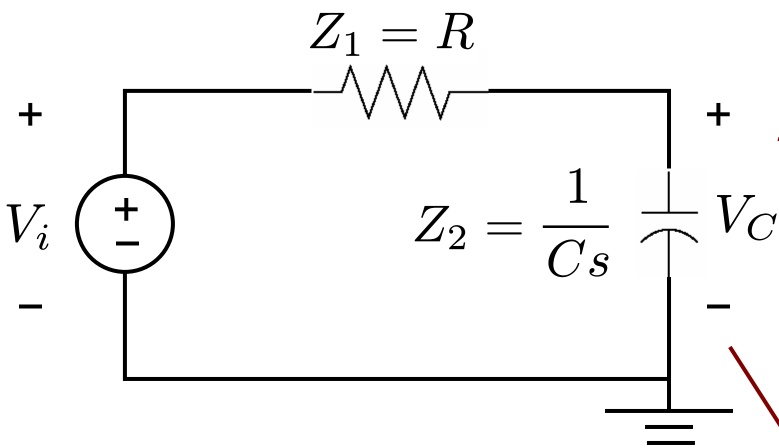
We recognize the voltage divider configuration, with the voltage across the capacitor as output. The transfer function is obtained as

$$\mathrm{TF}(s) = \frac{V_C(s)}{V_i(s)} = \frac{1/Cs}{R + 1/Cs} = \frac{1}{1+RCs} = \frac{1}{1+\tau s},$$

where $\tau \equiv RC$. Further, we note the similarity to the transfer function of the rotational mechanical system consisting of a motor, inertia J and viscous friction coefficient b that we saw in Lecture 3. [The transfer function was $1/b(1+\tau s)$, *i.e.* identical within a multiplicative constant, and the time constant τ was defined as J/b.] We can use the analogy to establish properties of the *RC* system without re–deriving them: e.g., the response to a step input $V_i = V_0 u(t)$ (step response) is

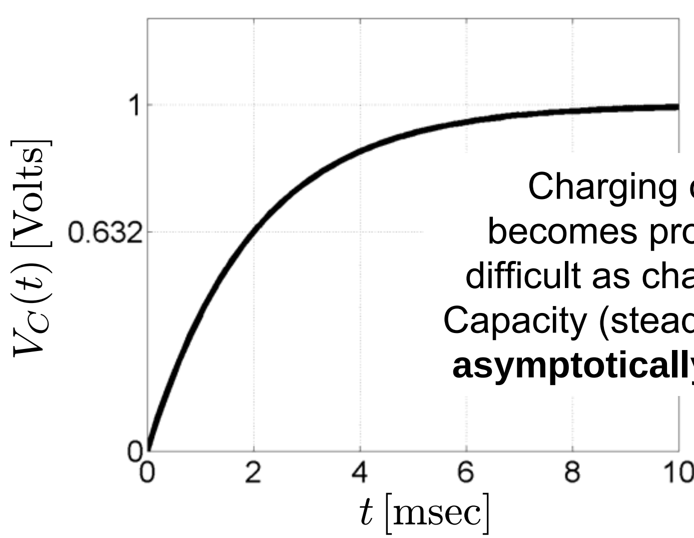
$$V_C(t) = V_0\left(1 - \mathrm{e}^{-t/\tau}\right)u(t), \qquad \text{where now } \tau = RC.$$

Interpretation of the *RC* step response

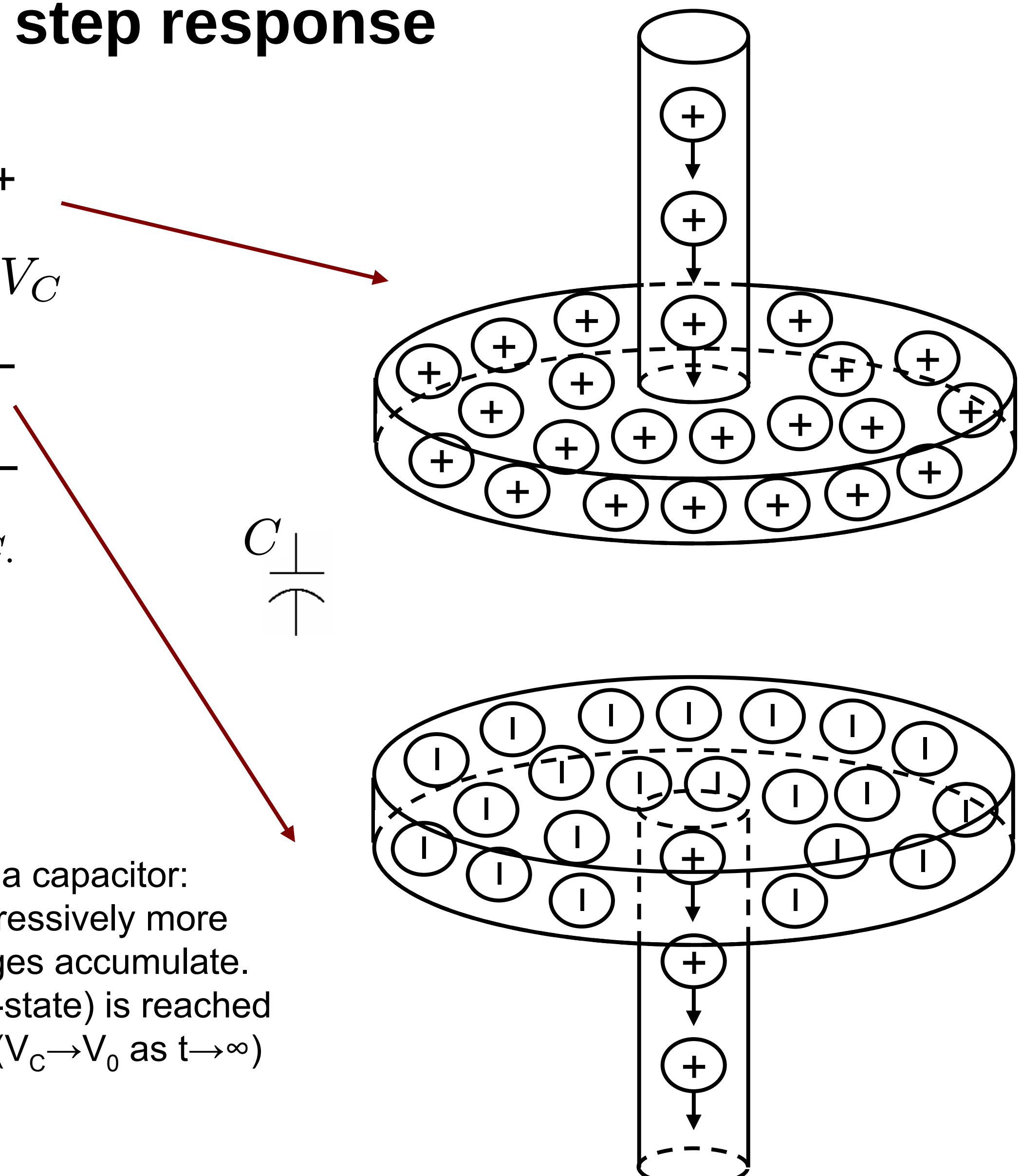


$$V_C(t) = V_0\left(1 - e^{-t/\tau}\right)u(t), \quad \tau = RC.$$

$V_0 = 1$ Volt $\quad R = 2k\Omega \quad C = 1\mu F$



Charging of a capacitor:
becomes progressively more
difficult as charges accumulate.
Capacity (steady-state) is reached
asymptotically ($V_C \rightarrow V_0$ as $t \rightarrow \infty$)



Example: RLC circuit with voltage source

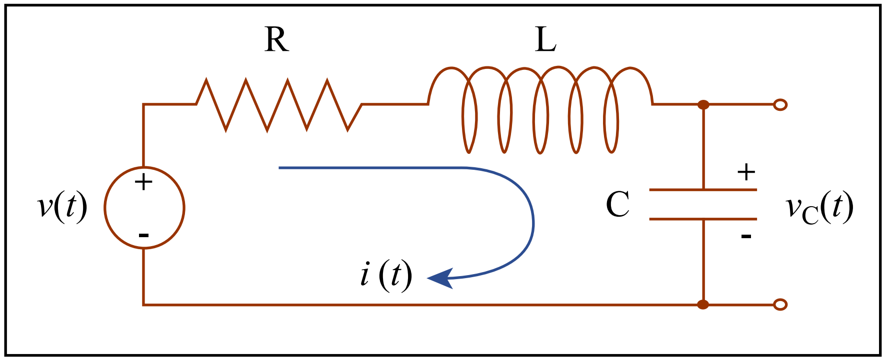


Figure by MIT OpenCourseWare.

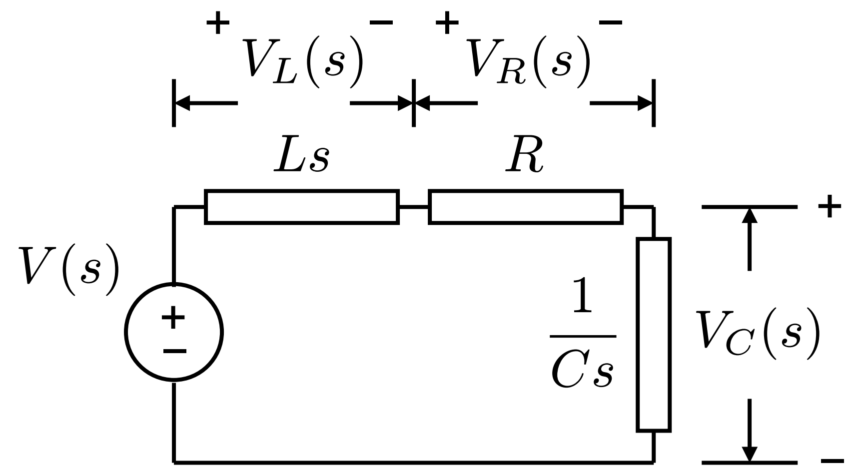


Figure 2.3

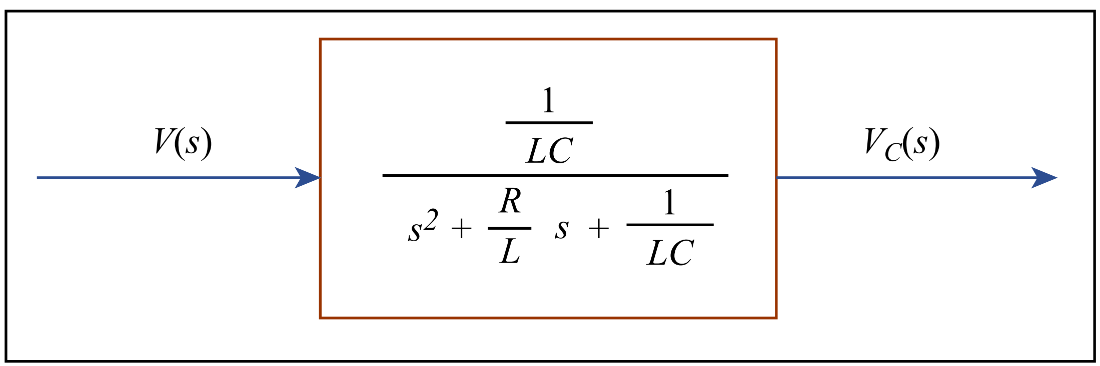


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Figure 2.4

Example: two-loop network

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Please see: Fig. 2.6 and 2.7 in Nise, Norman S. *Control Systems Engineering*. 4th ed. Hoboken, NJ: John Wiley, 2004.

The operational amplifier (op-amp)

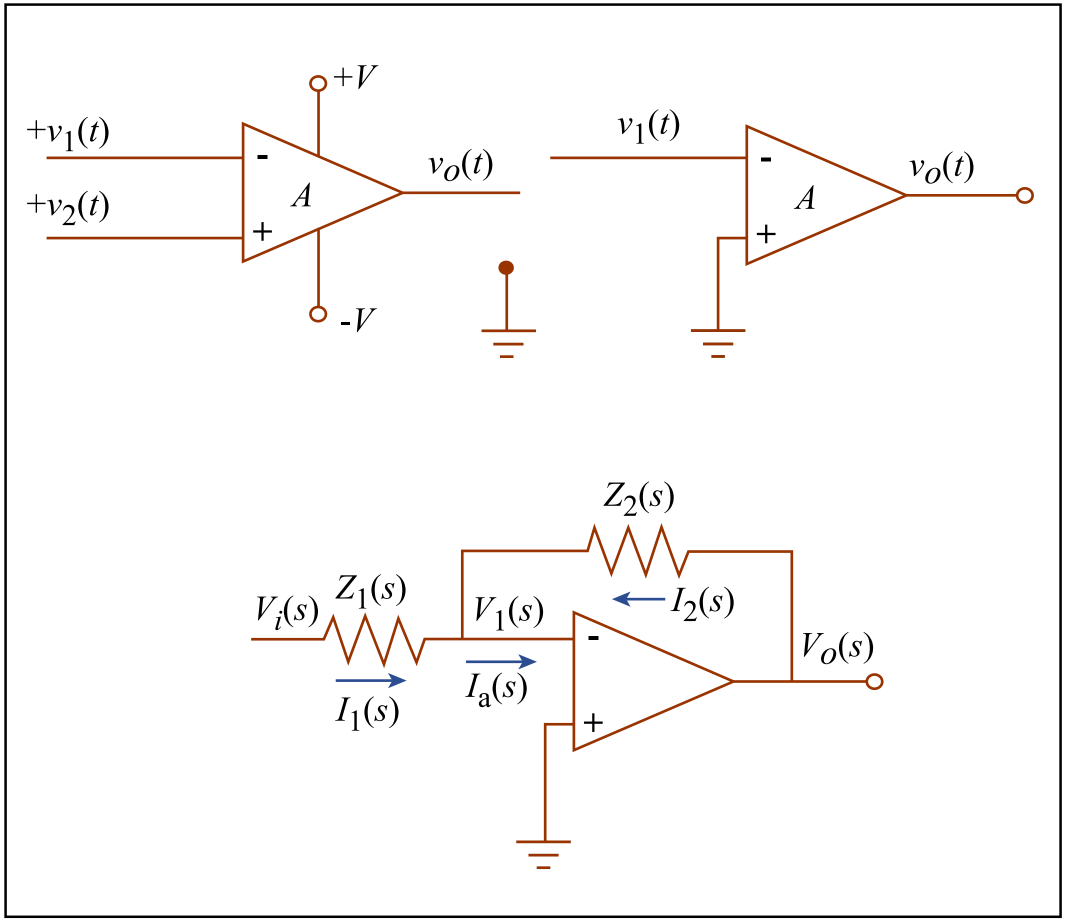

Figure 2.10

Figure by MIT OpenCourseWare.

(a) Generally, $v_o = A\,(v_2 - v_1)$, where A is the amplifier **gain.**

(b) When v_2 is grounded, as is often the case in practice, then $v_o = -Av_1$. (Inverting amplifier.)

(c) Often, A is large enough that we can approximate $A \to \infty$. Rather than connecting the input directly, the op–amp should then instead be used in the **feedback** configuration of Fig. (c). We have:

$$V_1 = 0; \qquad I_a = 0$$

(because V_o must remain finite) therefore

$$I_1 + I_2 = 0;$$

$$V_i - V_1 = V_i = I_1 Z_1;$$

$$V_o - V_1 = V_o = I_2 Z_2.$$

Combining, we obtain

$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)}.$$